Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 513186, 9 pages
doi:10.1155/2010/513186

Research Article

Extension Theorem for Complex Clifford Algebras-Valued Functions on Fractal Domains

Ricardo Abreu-Blaya,[1] Juan Bory-Reyes,[2] and Paul Bosch[3]

[1] *Departamento de Matemática, Universidad de Holguín, Holguín 80100, Cuba*
[2] *Departamento de Matemática, Universidad de Oriente, Santiago de Cuba 90500, Cuba*
[3] *Facultad de Ingeniería, Universidad Diego Portales, Santiago de Chile 8370179, Chile*

Correspondence should be addressed to Paul Bosch, paul.bosch@udp.cl

Received 1 December 2009; Accepted 20 March 2010

Academic Editor: Gary Lieberman

Copyright © 2010 Ricardo Abreu-Blaya et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Monogenic extension theorem of complex Clifford algebras-valued functions over a bounded domain with fractal boundary is obtained. The paper is dealing with the class of Hölder continuous functions. Applications to holomorphic functions theory of several complex variables as well as to that of the so-called biregular functions will be deduced directly from the isotonic approach.

1. Introduction

It is well known that methods of Clifford analysis, which is a successful generalization to higher dimension of the theory of holomorphic functions in the complex plane, are a powerful tool for study boundary value problems of mathematical physics over bounded domains with sufficiently smooth boundaries; see [1–3].

One of the most important parts of this development is the particular feature of the existence of a Cauchy type integral whose properties are similar to its famous complex prototype. However, if domains with boundaries of highly less smoothness (even nonrectifiable or fractal) are allowed, then customary definition of the Cauchy integral falls, but the boundary value problems keep their interest and applicability. A natural question arises as follows.

Can we describe the class of complex Clifford algebras-valued functions from Hölder continuous space extending monogenically from the fractal boundary of a domain through the whole domain?

In [4] for the quaternionic case and in [5–7] for general complex Clifford algebra valued functions some preliminaries results are given. However, in all these cases the condition ensures that extendability is given in terms of box dimension and Hölder exponent of the functions space considered.

In this paper we will show that there is a rich source of material on the roughness of the boundaries permitted for a positive answer of the question which has not yet been exploited, and indeed hardly touched.

At the end, applications to holomorphic functions theory of several complex variables as well as to the so-called biregular functions (to be defined later) will be deduced directly from the isotonic approach.

The above motivation of our work is of more or less theoretical mathematical nature but it is not difficult to give arguments based on an ample gamma of applications.

Indeed, the M. S. Zhdanov book cited in [8] is a translation from Russian and the original title means literally "The analogues of the Cauchy-type integral in the Theory of Geophysics Fields". In this book is considered, as the author writes, one of the most interesting questions of the Potential plane field theory, a possibility of an analytic extension of the field into the domain occupied by sources.

He gives representations of both a gravitational and a constant magnetic field as such analogues in order to solve now the spatial problems of the separation of field as well as analytic extension through the surface and into the domain with sources.

Our results can be applied to the study of the above problems in the more general context of domains with fractal boundaries, but the detailed discussion of this technical point is beyond the scope of this paper.

2. Preliminaries

Let $(e_1, \ldots, e_m)$ be an orthonormal basis of the Euclidean space $\mathbb{R}^m$.

The complex Clifford algebra, denoted by $\mathbb{C}_m$, is generated additively by elements of the form

$$e_A = e_{j_1} \cdots e_{j_k}, \tag{2.1}$$

where $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$ is such that $j_1 < \cdots < j_k$, and so the complex dimension of $\mathbb{C}_m$ is 2^m. For $A = \emptyset$, $e_\emptyset = 1$ is the identity element.

For $a, b \in \mathbb{C}_m$, the conjugation and the main involution are defined, respectively, as

$$\begin{gathered}\overline{a} = \sum_A \overline{a}_A \overline{e}_A, \quad \overline{e}_A = (-1)^{k(k+1)/2} e_A, \quad |A| = k, \text{ satisfying } \overline{ab} = \overline{b}\,\overline{a}, \\ \widetilde{a} = \sum_A a_A \widetilde{e}_A, \quad \widetilde{e}_A = (-1)^k e_A, \quad |A| = k, \text{ satisfying } \widetilde{ab} = \widetilde{a}\widetilde{b}.\end{gathered} \tag{2.2}$$

If we identify the vectors $(x_1, \ldots, x_m)$ of $\mathbb{R}^m$ with the real Clifford vectors $\underline{x} = \sum_{j=1}^m e_j x_j$, then $\mathbb{R}^m$ may be considered as a subspace of $\mathbb{C}_m$.

The product of two Clifford vectors splits up into two parts:

$$\underline{x}\underline{y} = -\left\langle \underline{x}, \underline{y} \right\rangle + \underline{x} \wedge \underline{y}, \tag{2.3}$$

where

$$\left\langle \underline{x}, \underline{y} \right\rangle = \sum_{j=1}^{m} x_j y_j,$$
$$\underline{x} \wedge \underline{y} = \sum_{j<k} e_j e_k \left(x_j y_k - x_k y_j \right). \tag{2.4}$$

Generally speaking, we will consider $\mathbb{C}_m$-valued functions u on $\mathbb{R}^m$ of the form

$$u = \sum_{A} u_A e_A, \tag{2.5}$$

where u_A are $\mathbb{C}$-valued functions. Notions of continuity and differentiability of u are introduced by means of the corresponding notions for its complex components u_A.

In particular, for bounded set $\mathbf{E} \subset \mathbb{R}^m$, the class of continuous functions which satisfy the Hölder condition of order $\alpha (0 < \alpha \leq 1)$ in $\mathbf{E}$ will be denoted by $C^{0,\alpha}(\mathbf{E})$.

Let us introduce the so-called Dirac operator given by

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}. \tag{2.6}$$

It is a first-order elliptic operator whose fundamental solution is given by

$$\underline{E}\left(\underline{x}\right) = \frac{1}{\omega_m} \frac{\overline{\underline{x}}}{\left|\underline{x}\right|^m} \quad \underline{x} \in \mathbb{R}^m \setminus \{0\}, \tag{2.7}$$

where ω_m is the area of the unit sphere in $\mathbb{R}^m$.

If Ω is open in $\mathbb{R}^m$ and $u \in C^1(\Omega)$, then u is said to be monogenic if $\partial_{\underline{x}} u = 0$ in Ω. Denote by $\mathcal{M}(\Omega)$ the set of all monogenic functions in Ω. The best general reference here is [9].

We recall (see [10]) that a Whitney extension of $u \in C^{0,\alpha}(\mathbf{E})$, $\mathbf{E}$ being compact in $\mathbb{R}^m$, is a compactly supported function $\mathcal{E}_0 u \in C^\infty(\mathbb{R}^m \setminus \mathbf{E}) \cap C^{0,\alpha}(\mathbb{R}^m)$ such that $\mathcal{E}_0 u|_{\mathbf{E}} = u$ and

$$\left|\partial_{\underline{x}} \mathcal{E}_0 u\left(\underline{x}\right)\right| \leq c \operatorname{dist}\left(\underline{x}, \mathbf{E}\right)^{\alpha-1} \quad \text{for } \underline{x} \in \mathbb{R}^m \setminus \mathbf{E}. \tag{2.8}$$

Here and in the sequel, we will denote by c certain generic positive constant not necessarily the same in different occurrences.

The following assumption will be needed through the paper. Let Ω be a Jordan domain, that is, a bounded oriented connected open subset of $\mathbb{R}^m$ whose boundary Γ is a compact topological surface. By Ω^* we denote the complement domain of $\Omega \cup \Gamma$.

By definition (see [11]) the box dimension of Γ, denoted by $\dim \Gamma$, is equal to $\limsup_{\varepsilon \to 0} (\log N_\Gamma(\varepsilon) / -\log \varepsilon)$, where $N_\Gamma(\varepsilon)$ stands for the least number of ε-balls needed to cover Γ.

The limit above is unchanged if $N_\Gamma(\varepsilon)$ is thinking as the number of k-cubes with $2^{-k} \le \varepsilon < 2^{-k+1}$ intersecting Γ. A cube Q is called a k-cube if it is of the form: $[l_1 2^{-k}, (l_1+1)2^{-k}] \times \cdots \times [l_m 2^{-k}, (l_m+1)2^{-k}]$, where $k, l_1, \ldots, l_m$ are integers.

Fix $d \in (m-1, m]$, assuming that the improper integral $\int_0^1 N_\Gamma(x) x^{d-1} dx$ converges. Note that this is in agreement with [12] for Γ to be d-summable.

Observe that a d-summable surface has box dimension $\dim \Gamma \le d$. Meanwhile, if Γ has box dimension less than d, then Γ is d-summable.

3. Extension Theorems

We begin this section with a basic result on the usual Cliffordian Théodoresco operator defined by

$$\mathcal{T}_\Omega u(\underline{x}) = -\int_\Omega \underline{E}\left(\underline{y}-\underline{x}\right) u(y) dy. \tag{3.1}$$

If $u \in C^{0,\nu}(\Gamma)$ such that $\nu > d/m$, which we may assume, then it follows that $m < (m-d)/(1-\nu)$ and we may choose p such that $m < p < (m-d)/(1-\nu)$. If for such p we can prove that $\partial_{\underline{x}} \mathcal{E}_0 u \in L^p(\Omega)$ then by in [3, Proposition 8.1] it follows that $\mathcal{T}_\Omega \partial_{\underline{x}} \mathcal{E}_0 u$ represents a continuous function in $\mathbb{R}^m$. Moreover, $\mathcal{T}_\Omega \partial_{\underline{x}} \mathcal{E}_0 u \in C^{0,\mu}(\mathbb{R}^m)$ for any $\mu < (m\nu - d)/(m-d)$, which is due to the fact that $\mathcal{T}_\Omega \partial_{\underline{x}} \mathcal{E}_0 u \in C^{0,(p-m)/p}(\mathbb{R}^m)$.

In the remainder of this section we assume that $\nu > d/m$.

3.1. Monogenic Extension Theorem

Theorem 3.1. *If $u \in C^{0,\nu}(\Gamma)$ is the trace of $U \in C^{0,\nu}(\Omega \cup \Gamma) \cap \mathcal{M}(\Omega)$, then*

$$\mathcal{T}_\Omega \partial_{\underline{x}} \mathcal{E}_0 u|_\Gamma = 0. \tag{3.2}$$

Conversely, assuming that (3.2) *holds, then u is the trace of $U \in C^{0,\mu}(\Omega \cup \Gamma) \cap \mathcal{M}(\Omega)$, for any $\mu < (m\nu - d)/(m-d)$.*

Proof. Let $U^* = u - U$ and define

$$\Omega_k = \left\{ \underline{x} \in Q : Q \in \mathcal{W}^j \text{ for some } j \le k \right\} \tag{3.3}$$

and $\Delta_k = \Omega \setminus \Omega_k$.

Note that the boundary of Ω_k, denoted by Γ_k, is actually composed by certain faces (denoted by Σ) of some cubes $Q \in \mathcal{W}^k$. We will denote by ν_Σ, ν_{Γ_k} the outward pointing unit normal to Σ and Γ_k, respectively, in the sense introduced in [13].

Let $\underline{x} \in \Omega$ and let k_0 be so large chosen that $\underline{x} \in \Omega_{k_0}$ and $\mathrm{dist}(\underline{x}, \Gamma_k) \ge |Q_0|$ for $k > k_0$, where Q_0 is a cube of $\mathcal{W}^{k_0}$. Here and below $|Q|$ denotes the diameter of Q as a subset of $\mathbb{R}^m$.

Let $\underline{y} \in \Gamma_k$, $Q \in \mathcal{W}^k$ a cube containing $\underline{y}$, and $\underline{z} \in \Gamma$ such that $|\underline{y} - \underline{z}| = \mathrm{dist}(\underline{y}, \Gamma)$.

Since $U^* \in C^{0,\nu}(\Gamma)$, $U^*|_\Gamma = 0$, we have

$$\left|U^*\left(\underline{y}\right)\right| = \left|U^*\left(\underline{y}\right) - U^*(\underline{z})\right| \leq c\left|\underline{y}-\underline{z}\right|^\nu \leq c|Q|^\nu. \tag{3.4}$$

Let Σ be an $(m-1)$-dimensional face of Γ_k and $Q \in \mathcal{W}^k$ the k-cube containing Σ; then if $k > k_0$, we have

$$\left|\int_\Sigma \underline{E}\left(\underline{y}-\underline{x}\right)\nu_\Sigma U^*\left(\underline{y}\right)d\underline{y}\right| \leq \frac{1}{|Q_0|^{m-1}}\int_\Sigma \left|U^*\left(\underline{y}\right)\right|d\underline{y} \leq \frac{c}{|Q_0|^{m-1}}|Q|^{\nu-1+m}. \tag{3.5}$$

Each face of Γ_k is one of those $2m$ of some $Q \in \mathcal{W}^k$. Therefore, for $k > k_0$

$$\left|\int_{\Gamma_k} \underline{E}\left(\underline{y}-\underline{x}\right)\nu_\Sigma U^*\left(\underline{y}\right)d\underline{y}\right| \leq \frac{c}{|Q_0|^{m-1}}\sum_{Q\in\mathcal{W}^k}|Q|^{\nu-1+m}. \tag{3.6}$$

Since $\nu - 1 + m > \nu m > d$, we get

$$\lim_{k\to\infty}\int_{\Gamma_k} \underline{E}\left(\underline{y}-\underline{x}\right)\nu_{\Gamma_k}\left(\underline{y}\right)U^*\left(\underline{y}\right)d\underline{y} = 0. \tag{3.7}$$

By Stokes formula we have

$$\begin{aligned}\int_\Omega \underline{E}\left(\underline{y}-\underline{x}\right)\partial_{\underline{x}}U^*\left(\underline{y}\right)d\underline{y} &= \lim_{k\to\infty}\left(\int_{\Delta_k}+\int_{\Omega_k}\right)\underline{E}\left(\underline{y}-\underline{x}\right)\partial_{\underline{x}}U^*\left(\underline{y}\right)d\underline{y}\\ &= \lim_{k\to\infty}\left(\int_{\Delta_k}\underline{E}\left(\underline{y}-\underline{x}\right)\partial_{\underline{x}}U^*\left(\underline{y}\right)d\underline{y} - \int_{\Gamma_k}\underline{E}\left(\underline{y}-\underline{x}\right)\nu_{\Gamma_k}\left(\underline{y}\right)U^*\left(\underline{y}\right)d\underline{y}\right) = 0.\end{aligned} \tag{3.8}$$

Therefore

$$\mathcal{T}_\Omega\partial_{\underline{x}}\mathcal{E}_0 u|_\Gamma = \mathcal{T}_\Omega\partial_{\underline{x}}U|_\Gamma = 0. \tag{3.9}$$

The same conclusion can be drawn for $\underline{x} \in \mathbb{R}^m \setminus \overline{\Omega}$. The only point now is to note that $\operatorname{dist}(\underline{x},\Gamma_k) \geq \operatorname{dist}(\underline{x},\Gamma)$ for $\underline{x} \in \mathbb{R}^m \setminus \overline{\Omega}$.

Finally, due to the fact that

$$\begin{aligned}\int_\Omega |\partial_{\underline{x}}\mathcal{E}_0 u(y)|^p dy &= \sum_{Q\in\mathcal{W}}\int_Q |\partial_{\underline{x}}\mathcal{E}_0 u(y)|^p dy \leq c\sum_{Q\in\mathcal{W}}\int_Q (\operatorname{dist}(y,\Gamma))^{p(\nu-1)}dy\\ &\leq c\sum_{Q\in\mathcal{W}}|Q|^{p(\nu-1)}|Q|^n = c\sum_{Q\in\mathcal{W}}|Q|^{m-p(1-\nu)} < +\infty,\end{aligned} \tag{3.10}$$

we prove that $\partial_{\underline{x}}\mathcal{E}_0 u \in L^p(\Omega)$, and the second assertion follows directly by taking $U = \mathcal{E}_0 u + \mathcal{T}_\Omega\partial_{\underline{x}}\mathcal{E}_0 u$.

The finiteness of the last sum follows from the d-summability of Γ together with the fact that $m - p(1-\nu) > d$. □

For Ω^* the following analogous result can be obtained.

Theorem 3.2. *Let $u \in C^{0,\nu}(\Gamma)$. If u is the trace of $U \in C^{0,\nu}(\Omega^* \cup \Gamma) \cap \mathcal{M}(\Omega^*)$, and $U(\infty) = 0$, then*

$$\mathcal{T}_{\Omega^*} \partial_{\underline{x}} \mathcal{E}_0 u|_\Gamma = -u(\underline{x}). \tag{3.11}$$

Conversely, assuming that (3.11) *holds, then u is the trace of $U \in C^{0,\mu}(\Omega^* \cup \Gamma) \cap \mathcal{M}(\Omega^*)$, for any $\mu < (m\nu - d)/(m-d)$.*

3.2. Isotonic Extension Theorem

For our purpose we will assume that the dimension of the Euclidean space m is even whence we will put $m = 2n$ from now on.

In a series of recent papers, so-called isotonic Clifford analysis has emerged as yet a refinement of the standard case but also has strong connections with the theory of holomorphic functions of several complex variables and biregular ones, even encompassing some of its results; see [14–18].

Put

$$I_j = \frac{1}{2}(1 + i e_j e_{n+j}), \quad j = 1, \ldots, n; \tag{3.12}$$

then a primitive idempotent is given by

$$I = \prod_{j=1}^{n} I_j. \tag{3.13}$$

We have the following conversion relations:

$$e_{n+j} a I = i \tilde{a} e_j I, \tag{3.14}$$

with $a \in \mathbb{C}_n$ (complex Clifford algebra generated by $\{e_1, \ldots, e_n\}$).

Note that for $a, b \in \mathbb{C}_n$ one also has that

$$aI = bI \Longleftrightarrow a = b. \tag{3.15}$$

Let us introduce the following real Clifford vectors and their corresponding Dirac operators:

$$\begin{aligned} \underline{x}_1 &= \sum_{j=1}^{n} e_j x_j, \qquad & \partial_{\underline{x}_1} &= \sum_{j=1}^{n} e_j \partial_{x_j}, \\ \underline{x}_2 &= \sum_{j=1}^{n} e_j x_{n+j}, \qquad & \partial_{\underline{x}_2} &= \sum_{j=1}^{n} e_j \partial_{x_{n+j}}. \end{aligned} \tag{3.16}$$

The function $u : \mathbb{R}^{2n} \to \mathbb{C}_n$ is said to be isotonic in $\Omega \subset \mathbb{R}^{2n}$ if and only if u is continuously differentiable in Ω and moreover satisfies the equation

$$\partial_{\underline{x}}^{\text{isot}} u := \partial_{\underline{x}_1} u + i\tilde{u}\partial_{\underline{x}_2} = 0. \tag{3.17}$$

We will denote by $\mathcal{I}(\Omega)$ the set of all isotonic functions in Ω.

We find ourselves forced to introduce two extra Cauchy kernels, defined by

$$\begin{aligned} \underline{E}_1(\underline{x}) &= \frac{1}{\omega_{2n}} \frac{\overline{\underline{x}_1}}{|\underline{x}|^{2n}} \quad \underline{x} \in \mathbb{R}^{2n} \setminus \{0\}, \\ \underline{E}_2(\underline{x}) &= \frac{1}{\omega_{2n}} \frac{\overline{\underline{x}_2}}{|\underline{x}|^{2n}} \quad \underline{x} \in \mathbb{R}^{2n} \setminus \{0\}. \end{aligned} \tag{3.18}$$

Now we may introduce the isotonic Théodoresco transform of a function u to be

$$\left(\mathcal{T}_\Omega^{\text{isot}} u\right)(\underline{x}) := -\int_\Omega \left[\underline{E}_1\left(\underline{y} - \underline{x}\right) u\left(\underline{y}\right) + i\tilde{u}\left(\underline{y}\right)\underline{E}_2\left(\underline{y} - \underline{x}\right)\right] d\underline{y}. \tag{3.19}$$

It is straightforward to deduce that

$$\mathcal{T}_\Omega(uI) = \left(\mathcal{T}_\Omega^{\text{isot}} u\right) I. \tag{3.20}$$

Theorem 3.3. *If $u \in C^{0,\nu}(\Gamma)$, is the trace of $U \in C^{0,\nu}(\Omega \cup \Gamma) \cap \mathcal{I}(\Omega)$, then*

$$\mathcal{T}_\Omega^{isot} \partial_{\underline{x}}^{isot} \mathcal{E}_0 u|_\Gamma = 0. \tag{3.21}$$

Conversely, assuming that (3.21) holds, then u is the trace of $U \in C^{0,\mu}(\Omega \cup \Gamma) \cap \mathcal{I}(\Omega)$, for any $\mu < (2n\nu - d)/(2n - d)$.

Proof. Let U be an isotonic extension of u to Ω such that $U \in C^{0,\nu}(\overline{\Omega})$. Then, UI is a monogenic extension of uI to Ω, which obviously belongs to $C^{0,\nu}(\overline{\Omega})$. Therefore

$$\mathcal{T}_\Omega^{\text{isot}} [\partial_{\underline{x}} \mathcal{E}_0 uI]|_\Gamma = 0 \tag{3.22}$$

by Theorem 3.1.

We thus get

$$\mathcal{T}_\Omega^{\text{isot}} \left[\partial_{\underline{x}}^{\text{isot}} \mathcal{E}_0 uI\right]|_\Gamma = \mathcal{T}_\Omega [\partial_{\underline{x}} \mathcal{E}_0 uI]|_\Gamma = 0, \tag{3.23}$$

the first equality being a direct consequence of (3.20). According to (3.15) we have (3.21), which is the desired conclusion.

On account of Theorem 3.1 again, the converse assertion follows directly by taking $U = \mathcal{E}_0 u + \mathcal{T}_\Omega^{\text{isot}}[\partial_{\underline{x}}^{\text{isot}} \mathcal{E}_0 u]$, and the proof is complete. □

Remark 3.4. Theorems 3.1 and 3.3 extend the results in [4–7], since the restriction putted there ($\nu > \dim \Gamma/m$) implies that of this paper.

4. Applications

In this last section, we will briefly discuss two particular cases which arise when considering (3.17).

Case 1. It is easily seen that if u takes values in the space of scalars $\mathbb{C}$, then u is isotonic if and only if

$$\left(\partial_{x_j} + i\partial_{x_{n+j}}\right)u = 0, \quad j = 1, \dots, n, \tag{4.1}$$

which means that u is a holomorphic function with respect to the n complex variables $x_j + ix_{n+j}$, $j = 1, \dots, n$.

Case 2. If u, isotonic function, takes values in the real Clifford algebra $\mathbb{R}_{0,n}$, then

$$\begin{aligned} \partial_{\underline{x}_1} u &= 0, \\ \tilde{u}\partial_{\underline{x}_2} &= 0, \end{aligned} \tag{4.2}$$

or, equivalently, by the action of the main involution on the second equation we arrive to the overdetermined system:

$$\begin{aligned} \partial_{\underline{x}_1} u &= 0, \\ u\partial_{\underline{x}_2} &= 0, \end{aligned} \tag{4.3}$$

whose solutions are called biregular functions. For a detailed study we refer the reader to [19–21].

The proof of Theorem 3.3 may readily be adapted to establish analogous results for both holomorphic and biregular functions context. Clearly, we prove that if we replace u by a $\mathbb{C}$-valued, respectively, $\mathbb{R}_{0,n}$-valued function, such that (3.21) holds, then there exists an isotonic extension U, which, by using the classical Dirichlet problem, takes values precisely in $\mathbb{C}$ or $\mathbb{R}_{0,n}$, respectively. On the other direction the proof is immediate. The corresponding statements are left to the reader.

Acknowledgments

The topic covered here has been initiated while the first two authors were visiting IMPA, Rio de Janeiro, in July of 2009. Ricardo Abreu and Juan Bory wish to thank CNPq for financial support. Ricardo Abreu wishes to thank the Faculty of Ingeneering, Universidad Diego Portales, Santiago de Chile, for the kind hospitality during the period in which the final version of the paper was eventually completed. This work has been partially supported by CONICYT (Chile) under FONDECYT Grant 1090063.

References

[1] K. Gürlebeck and W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, vol. 89 of *International Series of Numerical Mathematics*, Birkhäuser, Basel, Switzerland, 1990.

[2] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, John Wiley & Sons, New York, NY, USA, 1997.

[3] K. Gürlebeck, K. Habetha, and W. Sprössig, *Holomorphic Functions in the Plane and* n*-Dimensional Space*, Birkhäuser, Basel, Switzerland, 2007.

[4] A. Abreu-Blaya, J. Bory-Reyes, and T. Moreno-García, "Holomorphic extension theorems in Lipschitz domains of $\mathbb{C}^2$," *Advances in Applied Clifford Algebras*, vol. 20, no. 1, pp. 1–12, 2010.

[5] A. Abreu-Blaya, J. Bory-Reyes, D. P. Peña, and F. Sommen, "Biregular extendability via isotonic Clifford analysis," *Mathematical Methods in the Applied Sciences*, vol. 33, no. 4, pp. 384–393, 2010.

[6] R. Abreu-Blaya, J. Bory-Reyes, and T. Moreno-García, "Teodorescu transform decomposition of multivector fields on fractal hypersurfaces," in *Wavelets, Multiscale Systems and Hypercomplex Analysis*, vol. 167 of *Operator Theory: Advances and Applications*, pp. 1–16, Birkhäuser, Basel, Switzerland, 2006.

[7] R. Abreu-Blaya, J. Bory-Reyes, and T. Moreno-García, "A biregular extensiontheorem from a fractal surface in $\mathbb{R}^{2n}$," to appear in *Georgian Mathematical Journal*.

[8] M. Shapiro, N. L. Vasilievski, and M. S. Zhdanov, "Space analogs of the Cauchy type integrals and the quaternion theory," in *Integral Transforms in Geophysics*, M. S. Zhdanov, Ed., p. 367, Springer, Heidelberg, Germany, 1988, (Appendix A: 344–350).

[9] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, vol. 76 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1982.

[10] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, no. 30, Princeton University Press, Princeton, NJ, USA, 1970.

[11] K. J. Falconer, *The Geometry of Fractal Sets*, vol. 85 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1986.

[12] J. Harrison and A. Norton, "The Gauss-Green theorem for fractal boundaries," *Duke Mathematical Journal*, vol. 67, no. 3, pp. 575–588, 1992.

[13] H. Federer, *Geometric Measure Theory*, vol. 153 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, New York, NY, USA, 1969.

[14] R. Abreu-Blaya and J. Bory-Reyes, "A Martinelli-Bochner formula on fractal domains," *Archiv der Mathematik*, vol. 92, no. 4, pp. 335–343, 2009.

[15] R. Abreu-Blaya, J. Bory-Reyes, D. Peña Peña, and F. Sommen, "The isotonic Cauchy transform," *Advances in Applied Clifford Algebras*, vol. 17, no. 2, pp. 145–152, 2007.

[16] A. Abreu-Blaya, J. Bory-Reyes, D. P. Peña, and F. Sommen, "Holomorphic extension theorems in Lipschitz domains of $\mathbb{C}^2$," *Advances in Applied Clifford Algebras*, vol. 20, no. 1, pp. 1–12, 2010.

[17] J. Bory-Reyes, D. Peña Peña, and F. Sommen, "A Davydov theorem for the isotonic Cauchy transform," *Journal of Analysis and Applications*, vol. 5, no. 2, pp. 109–121, 2007.

[18] F. Sommen and D. Peña Peña, "Martinelli-Bochner formula using Clifford analysis," *Archiv der Mathematik*, vol. 88, no. 4, pp. 358–363, 2007.

[19] F. Brackx and W. Pincket, "A Bochner-Martinelli formula for the biregular functions of Clifford analysis," *Complex Variables: Theory & Application*, vol. 4, no. 1, pp. 39–48, 1984.

[20] F. Brackx and W. Pincket, "The biregular functions of Clifford analysis: some special topics," in *Clifford Algebras and Their Applications in Mathematical Physics (Canterbury, 1985)*, vol. 183 of *NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences*, pp. 159–166, Reidel, Dordrecht, The Netherlands, 1986.

[21] F. Brackx and W. Pincket, "Two Hartogs theorems for nullsolutions of overdetermined systems in Euclidean space," *Complex Variables: Theory & Application*, vol. 4, no. 3, pp. 205–222, 1985.