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Research Article

Existence and Uniqueness of Positive Solution for a Singular Nonlinear Second-Order m-Point Boundary Value Problem

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The existence and uniqueness of positive solution is obtained for the singular second-order m-point boundary value problem $u''(t) + f(t, u(t)) = 0$ for $t \in (0, 1)$, $u(0) = 0$, $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$, where $m \geq 3$, $\alpha_i > 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ are constants, and $f(t, u)$ can have singularities for $t = 0$ and/or $t = 1$ and for $u = 0$. The main tool is the perturbation technique and Schauder fixed point theorem.

1. Introduction

In this paper, we investigate the existence and uniqueness of positive solution for the singular second-order differential equation

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1) \tag{1.1}$$

with the m-point boundary conditions

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.2}$$

where $m \geq 3$, $\alpha_i > 0$ $\;(i = 1, 2, \ldots, m-2)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ are constants, and $f(t, u)$ can have singularities for $t = 0$ and/or $t = 1$ and for $u = 0$.

Multipoint boundary value problems for second-order ordinary differential equations arise in many areas of applied mathematics and physics; see [1–3] and references therein. The study of three-point boundary value problems for nonlinear second-order ordinary differential equations was initiated by Lomtatidze [4, 5]. Since then, the nonlinear second-order multipoint boundary value problems have been studied by many authors; see [1–3, 6–29] and references therein. Most of all the works in the above mentioned references are nonsingular multipoint boundary value problems; see [1–3, 10–17, 20–23, 25, 26, 28, 29], but the works on the singularities have been quite rarely seen; see [4–8, 18, 19, 24, 27].

Recently, Du and Zhao [7], by constructing lower and upper solutions and together with the maximal principle, proved the existence and uniqueness of positive solutions for the following singular second-order m-point boundary value problem:

$$\begin{aligned} &u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ &u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{aligned} \tag{1.3}$$

where $m \geq 3, 0 < \alpha_i < 1$ $(i = 1, 2, \ldots, m-2)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ are constants, $\sum_{i=1}^{m-2} \alpha_i < 1$, $f(t,u)$ is singular at $t = 0$, $t = 1$ and $u = 0$, under conditions that

(H_1) $f(t,u) \in C((0,1) \times (0,+\infty), [0,+\infty))$, and $f(t,u)$ is decreasing in u;

(H_2) $f(t,\lambda) \not\equiv 0$, $\int_0^1 t(1-t) f(t, \lambda t(1-t))dt < +\infty$, for all $\lambda > 0$.

The purpose of this paper is to establish existence and uniqueness result of positive solution to SBVP(1.1), (1.2) under conditions that are weaker than conditions in [7] and hence improve the result in [7] by using perturbation technique and Schauder fixed point theorem [30].

Throughout this paper, we make the following assumptions:

(C_0) $\alpha_i > 0$, $i = 1, 2, \ldots, m-2$ and $\sum_{i=1}^{m-2} \alpha_i \leq 1$;

(C_1) $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous and nonincreasing in u for each fixed $t \in (0,1)$;

(C_2) $0 < \int_0^1 s(1-s) f(s, u_0) ds < +\infty$ for each constant $u_0 \in (0,+\infty)$.

2. Preliminary

We consider the perturbation problems that are given by

$$\begin{aligned} &u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ &u(0) = h, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) + \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) h, \end{aligned} \tag{2.1$_h$}$$

where h is any nonnegative constant.

Definition 2.1. For each fixed constant $h \geq 0$, a function $u(t)$ is said to be a positive solution of BVP$(2.1)_h$ if $u \in C[0,1] \cap C^2(0,1)$ with $u(t) > 0$ on $(0,1]$ such that $u''(t) + f(t,u(t)) = 0$ holds for all $t \in (0,1)$ and $u(0) = h$, $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) + (1 - \sum_{i=1}^{m-2} \alpha_i) h$.

Lemma 2.2. *Assume that conditions* (C_1) *and* (C_2) *are satisfied. Then, for each fixed constant* $u_0 > 0$,

$$\lim_{t\to 0^+} t\int_t^{\eta_1} f(s,u_0)ds = 0, \tag{2.2}$$

$$\lim_{t\to 1^-} (1-t)\int_{\eta_{m-2}}^{t} f(s,u_0)ds = 0. \tag{2.3}$$

Proof. We only prove (2.2). And (2.3) can be proved similarly.

For each fixed constant $u_0 > 0$, let

$$v(t) = t\int_t^{\eta_1} f(s,u_0)ds \quad \text{for } t \in (0,\eta_1]. \tag{2.4}$$

Then from the conditions (C_1) and (C_2), we have

$$\begin{gathered} 0 \le v(t) \le \int_t^{\eta_1} sf(s,u_0)ds \le \int_0^{\eta_1} sf(s,u_0)ds < +\infty \quad \text{for } t \in (0,\eta_1], \\ v'(t) = \int_t^{\eta_1} f(s,u_0)ds - tf(t,u_0) \quad \text{for } t \in (0,\eta_1]. \end{gathered} \tag{2.5}$$

Hence from the conditions (C_1) and (C_2), we have

$$\int_0^{\eta_1} |v'(t)|dt \le \int_0^{\eta_1} dt\int_t^{\eta_1} f(s,u_0)ds + \int_0^{\eta_1} tf(t,u_0)dt = 2\int_0^{\eta_1} tf(t,u_0)dt < +\infty. \tag{2.6}$$

This implies that $v'(t) \in L^1(0,\eta_1)$, and hence for each $t \in [0,\eta_1]$,

$$\int_0^t v'(\tau)d\tau = \int_0^t d\tau\int_\tau^{\eta_1} f(s,u_0)ds - \int_0^t \tau f(\tau,u_0)d\tau = t\int_t^{\eta_1} f(s,u_0)ds = v(t). \tag{2.7}$$

Thus, it follows from the absolute continuity of integral that $\lim_{t\to 0^+} v(t) = 0$, that is,

$$\lim_{t\to 0^+} t\int_t^{\eta_1} f(s,u_0)ds = 0. \tag{2.8}$$

This completes the proof of the lemma. □

In the following discussion $G(t,s)$ denotes Green's function for Dirichlet problem:

$$\begin{gathered} -u''(t) = 0, \quad t \in [0,1], \\ u(0) = u(1) = 0. \end{gathered} \tag{2.9}$$

Then Green's function $G(t,s)$ can be expressed as follows:

$$G(t,s)=\begin{cases}(1-t)s, & 0\le s\le t\le 1,\\ (1-s)t, & 0\le t\le s\le 1.\end{cases} \tag{2.10}$$

It is easy to see that Green's function $G(t,s)$ has the following simple properties:

(i) $0\le t(1-t)s(1-s)\le G(t,s)\le s(1-s)$ for $(t,s)\in[0,1]\times[0,1]$;

(ii) $G(t,s)>0$ for $(t,s)\in(0,1)\times(0,1)$;

(iii) $G(0,s)=G(1,s)=0$ for $s\in[0,1]$.

By direct calculation, we can easily obtain the following result.

Lemma 2.3. *Assume that conditions* (C_0), (C_1), *and* (C_2) *are satisfied. Then,* $u(t)$ *is a positive solution of BVP*$(2.1)_h$ $(h>0)$ *if and only if* $u\in C[0,1]$ *is a solution of the following integral equation:*

$$u(t)=\int_0^1 G(t,s)f(s,u(s))ds+\frac{t}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds+h, \tag*{$(2.11)_h$}$$

such that $u(t)>h>0$ *on* $(0,1]$.

Lemma 2.4. *Assume that conditions* (C_0), (C_1), *and* (C_2) *are satisfied. Suppose also that* $u\in C[0,1]$ *is a solution of the following integral equation:*

$$u(t)=\int_0^1 G(t,s)f(s,u(s))ds+\frac{t}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds, \tag{2.12}$$

such that $u(t)>0$ *on* $(0,1]$. *Then,* $u(t)$ *is a positive solution of SBVP*(1.1), (1.2).

Proof. Since $u\in C[0,1]$ is a solution of (2.12) with $u(t)>0$ on $(0,1]$, then for each $t\in(0,1)$,

$$\int_0^t s(1-t)f(s,u(s))ds<+\infty,\qquad \int_t^1 t(1-s)f(s,u(s))ds<+\infty. \tag{2.13}$$

So for each $t\in(0,1)$, we have

$$\int_0^t sf(s,u(s))ds<+\infty,\qquad \int_t^1 (1-s)f(s,u(s))ds<+\infty. \tag{2.14}$$

For convenience, let $c=:(1/(1-\sum_{i=1}^{m-2}\alpha_i\eta_i))\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds$. Take $t\in(0,1)$ and Δt such that $t+\Delta t\in(0,1)$, then from the definition of derivative, the mean value theorem of integral, and the absolute continuity of integral, we have

$$
\begin{aligned}
\lim_{\Delta t\to 0}&\frac{u(t+\Delta t)-u(t)}{\Delta t}\\
&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_0^{t+\Delta t}s(1-t-\Delta t)f(s,u(s))ds+\int_{t+\Delta t}^1(1-s)(t+\Delta t)f(s,u(s))ds\right.\\
&\qquad\left.-\int_0^t s(1-t)f(s,u(s))ds-\int_t^1 t(1-s)f(s,u(s))ds\right)+c\\
&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(-\int_0^t s\Delta t f(s,u(s))ds+\int_t^{t+\Delta t}s(1-t-\Delta t)f(s,u(s))ds\right.\\
&\qquad\left.+\int_{t+\Delta t}^1(1-s)\Delta t f(s,u(s))ds-\int_t^{t+\Delta t}t(1-s)f(s,u(s))ds\right)+c\\
&=-\int_0^t sf(s,u(s))ds+t(1-t)f(t,u(t))+\int_t^1(1-s)f(s,u(s))ds-t(1-t)f(t,u(t))+c\\
&=-\int_0^t sf(s,u(s))ds+\int_t^1(1-s)f(s,u(s))ds+c.
\end{aligned}
\tag{2.15}
$$

Hence

$$
u'(t)=-\int_0^t sf(s,u(s))ds+\int_t^1(1-s)f(s,u(s))ds+c\quad \text{for } t\in(0,1). \tag{2.16}
$$

Consequently $u'\in C(0,1)$.

Again, from the definition of derivative and the mean value theorem of integrals, we have

$$
\begin{aligned}
\lim_{\Delta t\to 0}&\frac{u'(t+\Delta t)-u'(t)}{\Delta t}\\
&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(-\int_0^{t+\Delta t}sf(s,u(s))ds+\int_{t+\Delta t}^1(1-s)f(s,u(s))ds\right.\\
&\qquad\left.+\int_0^t sf(s,u(s))ds-\int_t^1(1-s)f(s,u(s))ds\right)\\
&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(-\int_t^{t+1}sf(s,u(s))ds-\int_t^{t+\Delta t}(1-s)f(s,u(s))ds\right)\\
&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(-\int_t^{t+\Delta t}f(s,u(s))ds\right)\\
&=-f(t,u(t))\quad \text{for } t\in(0,1).
\end{aligned}
\tag{2.17}
$$

Hence $u''(t)=-f(t,u(t))$ for $t\in(0,1)$. In particular, $u''\in C(0,1)$.

On the other hand, from (2.12), we have $u(0)=0$ and

$$\begin{aligned}\sum_{i=1}^{m-2}\alpha_i u(\eta_i) &= \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{\eta_i}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds\right)\\ &= \sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{\sum_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds\\ &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{\mathrm{m}-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds\\ &= u(1).\end{aligned} \tag{2.18}$$

In summary, $u(t)$ is a positive solution of SBVP(1.1), (1.2). This completes the proof of the lemma. □

Remark 2.5. Assume that all conditions in Lemma 2.4 hold. Then

(1) if $f\in C([0,1)\times[0,+\infty),[0,+\infty))$, we have

$$u\in C[0,1]\cap C^1[0,1)\cap C^2(0,1); \tag{2.19}$$

(2) if $f\in C((0,1]\times(0,+\infty),[0,+\infty))$, we get

$$u\in C[0,1]\cap C^1(0,1]\cap C^2(0,1). \tag{2.20}$$

Lemma 2.6. *Assume that conditions* (C_0), (C_1), *and* (C_2) *are satisfied. Then, for each constant* $h>0$, *BVP*$(2.1)_h$ *has a unique solution* $u(t;h)$ *with* $u(t;h)\geq h$ *on* $[0,1]$.

Proof. We begin by defining an operator T in D_h by

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s))ds+\frac{t}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds+h, \tag{2.21}$$

where $D_h := \{u\in C[0,1] : u(t)\geq h \text{ on } [0,1]\}$ is a convex closed set. Then from Lemma 2.2 and the condition (C_2), we have $Tu\in C[0,1]$ and Tu satisfies

$$\begin{gathered}(Tu)''(t)+f(t,u(t))=0,\quad t\in(0,1),\\ (Tu)(0)=h,\qquad (Tu)(1)=\sum_{i=1}^{m-2}\alpha_i(Tu)(\eta_i)+\left(1-\sum_{i=1}^{m-2}\alpha_i\right)h.\end{gathered} \tag{2.22}$$

We now apply Schauder fixed point theorem [30] to obtain the existence of a fixed point for T. To do this, it suffices to verify that T is continuous in D_h and $\overline{T(D_h)}$ is a compact set.

Take $u_0 \in D_h$, and let $\{u_k\}_{k=1}^{\infty} \subset D_h$ such that

$$\|u_k - u_0\|_{C[0,1]} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.23}$$

Then for each $t \in (0,1)$,

$$f(t, u_k(t)) \longrightarrow f(t, u_0(t)) \quad \text{as } k \longrightarrow \infty. \tag{2.24}$$

From the definition of T, we have

$$(Tu_k)(t) = \int_0^1 G(t,s) f(s, u_k(s)) ds + \frac{t}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s) f(s, u_k(s)) ds + h. \tag{2.25}$$

Also, from the conditions (C_1) and (C_2), we have

$$\begin{gathered} f(t, u_0(t)) + f(t, u_k(t)) \leq 2 f(t, h) \quad \text{for } t \in (0,1), \\ \int_0^1 s(1-s) f(s,h) ds < +\infty. \end{gathered} \tag{2.26}$$

Thus by Lebesgue-dominated convergence theorem, we have

$$\begin{aligned} \max_{t\in[0,1]} |(Tu_k)(t) - (Tu_0)(t)| \leq{} & \int_0^1 G(s,s) \left| f(s, u_k(s)) - f(s, u_0(s)) \right| ds \\ & + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_0^1 G(s,s) \left| f(s, u_k(s)) - f(s, u_0(s)) \right| ds \\ ={} & \left(1 + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \right) \int_0^1 s(1-s) \left| f(s, u_k(s)) - f(s, u_0(s)) \right| ds \\ \longrightarrow{} & 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{2.27}$$

Therefore, $T : D_h \to D_h$ is continuous.

Next we need to show that $T(D_h)$ is a relatively compact subset of $C[0,1]$.

(1) From the definition of T and the conditions (C_1) and (C_2), for each $u \in D_h$ we have

$$0 < h \leq (Tu)(t) \leq (Th)(t) \quad \text{for } t \in [0,1]. \tag{2.28}$$

This implies that $T(D_h)$ is uniformly bounded.

(2) For each $u \in D_h$, since

$$(Tu)'(t) = -\int_0^t sf(s,u(s))ds + \int_t^1 (1-s)f(s,u(s))ds + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,u(s))ds \quad \text{for } t\in[0,1], \tag{2.29}$$

then

$$\left|(Tu)'(t)\right| \le \int_0^t sf(s,h)ds + \int_t^1 (1-s)f(s,h)ds + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,h)ds =: M(t) \quad \text{for } t\in[0,1]. \tag{2.30}$$

Obviously $M(t)\ge 0$ on $[0,1]$, and

$$\begin{aligned}\int_0^1 M(t)dt &= 2\int_0^1 s(1-s)f(s,h)ds + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 G(\eta_i,s)f(s,h)ds\\ &\le 2\int_0^1 s(1-s)f(s,h)ds + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^1 s(1-s)f(s,h)ds\\ &= \left(2+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}\right)\int_0^1 s(1-s)f(s,h)ds < +\infty.\end{aligned} \tag{2.31}$$

Thus $M\in L^1(0,1)$. From the absolute continuity of integral, we have that for each number $\varepsilon>0$, there is a positive number $\delta>0$ such that for all $t_1,t_2\in[0,1]$, if $|t_1-t_2|<\delta$, then $|\int_{t_1}^{t_2}M(t)dt|<\varepsilon$. It follows that for all $t_1,t_2\in[0,1]$ with $|t_1-t_2|<\delta$, we have

$$|(Tu)(t_2)-(Tu)(t_1)| = \left|\int_{t_1}^{t_2}(Tu)'(t)dt\right| \le \left|\int_{t_1}^{t_2}\left|(Tu)'(t)\right|dt\right| \le \left|\int_{t_1}^{t_2}M(t)dt\right| < \varepsilon. \tag{2.32}$$

Therefore $T(D_h)$ is equicontinuous on $[0,1]$. It follows from Ascoli-Arzela theorem that $T(D_h)$ is a relatively compact subset of $C[0,1]$. Consequently, by Schauder fixed point theorem [30], T has a fixed point $u(t;h)\in D_h$. Obviously, $u(t;h)>h>0$ on $(0,1]$. Hence from Lemma 2.3, $u(t;h)$ is a solution of BVP $(2.1)_h$.

Next, we will show the uniqueness of solution. Let us suppose that $u_1(t;h)$, $u_2(t;h)$ are two different solutions of BVP$(2.1)_h$. Then there exists $t_0\in(0,1]$ such that $u_1(t_0;h)\ne u_2(t_0;h)$. Without loss of generality, assume that $u_1(t_0;h)>u_2(t_0;h)$. Let $w(t):=u_1(t;h)-u_2(t;h)$, then $w(0)=0$, $w(t_0)>0$, and hence there exists $t_1\in[0,t_0)$ such that

$$w(t_1)=0, \quad w(t)>0 \quad \text{for } t\in(t_1,t_0]. \tag{2.33}$$

Further we have $w(t) > 0$ on $(t_1, 1]$. In fact, assume to the contrary that the conclusion is false. Then there exists $t_2 \in (t_0, 1]$ such that $w(t_2) \le 0$. Thus there exists $t_3 \in (t_0, t_2]$ such that

$$w(t_3) = 0, \quad w(t) > 0 \quad \text{for } t \in [t_0, t_3). \tag{2.34}$$

Since $w(t_1) = 0$, $w(t) > 0$ on $(t_1, t_0]$, then

$$w''(t) = -f(t, u_1(t;h)) + f(t, u_2(t;h)) \ge 0 \quad \text{for } t \in [t_1, t_3]. \tag{2.35}$$

It follows from $w(t_1) = w(t_3) = 0$ that $w(t) \le 0$ on $[t_1, t_3]$. This is a contradiction to $w(t) > 0$ on (t_1, t_3).

Now we prove that $w(t) \ge 0$ on $[0, t_1]$. In fact, assume to the contrary that the conclusion is false. Then there exists $t_4 \in (0, t_1)$ such that $w(t_4) < 0$. Since $w(0) = w(t_1) = 0$, then there exist t_5, t_6 with $0 \le t_5 < t_4 < t_6 \le t_1$ such that

$$w(t_5) = w(t_6) = 0, \quad w(t) < 0 \quad \text{for } t \in (t_5, t_6). \tag{2.36}$$

Thus,

$$w''(t) = -f(t, u_1(t;h)) + f(t, u_2(t;h)) \le 0 \quad \text{for } t \in [t_5, t_6]. \tag{2.37}$$

It follows from $w(t_5) = w(t_6)$ that $w(t) \ge 0$ on $[t_5, t_6]$. This is a contradiction to $w(t) < 0$ on (t_5, t_6).

In summary, we have $w(t) \ge 0$ on $[0, t_1]$ and $w(t) > 0$ on $(t_1, 1]$. Thus

$$\begin{aligned} w(t) &= \int_0^1 G(t,s)\left[f(s, u_1(s;h)) - f(s, u_2(s;h))\right] ds \\ &\quad + \frac{t}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s)\left[f(s, u_1(s;h)) - f(s, u_2(s;h))\right] ds \\ &\le 0 \quad \text{for } t \in (0, 1]. \end{aligned} \tag{2.38}$$

This is a contradiction to $w(t) > 0$ on $(t_1, 1]$. This completes the proof of the lemma. □

Lemma 2.7. *Assume that conditions* (C_0), (C_1), *and* (C_2) *are satisfied. Then, the unique solution* $u(t;h)$ *of BVP*$(2.1)_h$ *is nondecreasing in* h.

Proof. Let $0 < h_2 < h_1$, and let $u(t;h_1), u(t;h_2)$ be the solutions of BVP$(2.1)_{h_1}$ and BVP$(2.1)_{h_2}$, respectively. We will show

$$u(t;h_1) \ge u(t;h_2) \quad \text{for } t \in [0, 1]. \tag{2.39}$$

Assume to the contrary that the above inequality is false. Then there exists $t_0 \in (0,1]$ such that $u(t_0;h_1) < u(t_0;h_2)$. Since $u(0;h_1) = h_1 > h_2 = u(0;h_2)$, we have that there exists $t_1 \in (0,t_0)$ such that

$$u(t_1;h_1) = u(t_1;h_2), \quad u(t;h_1) < u(t;h_2) \quad \text{for } t \in (t_1,t_0]. \tag{2.40}$$

Next we prove $u(t;h_1) < u(t;h_2)$ on $(t_0,1]$. In fact, assume to the contrary that the conclusion is false. Then there exists $t_2 \in (t_0,1]$ such that

$$u(t_2;h_1) = u(t_2;h_2), \quad u(t;h_1) < u(t;h_2) \quad \text{for } t \in [t_0,t_2). \tag{2.41}$$

Hence

$$u''(t;h_1) - u''(t;h_2) = -f(t,u(t;h_1)) + f(t,u(t;h_2)) \le 0 \quad \text{for } t \in [t_1,t_2]. \tag{2.42}$$

It follows from $u(t_i;h_1) = u(t_i;h_2)$, $i = 1,2$ that $u(t;h_1) \ge u(t;h_2)$ on $[t_1,t_2]$. This is a contradiction to $u(t;h_1) < u(t;h_2)$ on (t_1,t_2). Thus $u(t;h_1) < u(t;h_2)$ on $(t_1,1]$. This implies that

$$u''(t;h_1) - u''(t;h_2) = -f(t,u(t;h_1)) + f(t,u(t;h_2)) \le 0 \quad \text{for } t \in [t_1,1]. \tag{2.43}$$

It follows from $u'(t_1;h_1) - u'(t_1;h_2) \le 0$ that $u'(t;h_1) - u'(t;h_2) \le 0$ on $[t_1,1]$. Hence, from $u(t;h_1) < u(t;h_2)$ on $(t_1,1]$, we have $u'(1;h_1) - u'(1;h_2) < 0$. Thus

$$u(1;h_1) - u(1;h_2) < u(\eta_{m-2};h_1) - u(\eta_{m-2};h_2). \tag{2.44}$$

There are two cases to consider.

Case 1 (see $[t_1 \ge \eta_{m-2}]$). In this case, we have

$$u(\eta_i;h_1) - u(\eta_i;h_2) \ge 0, \quad i = 1,2,\dots,m-2. \tag{2.45}$$

Hence from the boundary conditions of BVP$(2.1)_h$, we have

$$\begin{aligned} u(1;h_1) - u(1;h_2) &= \sum_{i=1}^{m-2}\alpha_i u(\eta_i;h_1) + \left(1 - \sum_{i=1}^{m-2}\alpha_i\right)h_1 \\ &\quad - \sum_{i=1}^{m-2}\alpha_i u(\eta_i;h_2) - \left(1 - \sum_{i=1}^{m-2}\alpha_i\right)h_2 \\ &\ge \sum_{i=1}^{m-2}\alpha_i\left(u(\eta_i;h_1) - u(\eta_i;h_2)\right) \ge 0. \end{aligned} \tag{2.46}$$

This is a contradiction to $u(1;h_1) - u(1;h_2) < 0$.

Case 2 (see $[t_1 < \eta_{m-2}]$). In this case, we have

$$u(1;h_1) - u(1;h_2) < u(\eta_{m-2};h_1) - u(\eta_{m-2};h_2) < 0,$$
$$u(\eta_{m-2};h_1) - u(\eta_{m-2};h_2) \le u(\eta_i;h_1) - u(\eta_i;h_2), \quad i = 1,2,\dots,m-3. \tag{2.47}$$

It follows from (C_0) that

$$u(1;h_1) - u(1;h_2) < \sum_{i=1}^{m-2}\alpha_i\big(u(\eta_{m-2};h_1) - u(\eta_{m-2};h_2)\big) \le \sum_{i=1}^{m-2}\alpha_i\big(u(\eta_i;h_1) - u(\eta_i;h_2)\big). \tag{2.48}$$

This is a contradiction to the boundary conditions of BVP$(2.1)_h$.

In summary, we have $u(t;h_1) \ge u(t;h_2)$ on $[0,1]$. This completes the proof of the lemma. □

3. Main Results

We now state and prove our main results for singular second-order m-point boundary value problem (1.1), (1.2).

Theorem 3.1. *Assume that conditions (C_0), (C_1), and (C_2) are satisfied. Then, SBVP*(1.1), (1.2) *has at most one positive solution.*

Proof. Suppose that $u_1(t)$ and $u_2(t)$ are any two positive solutions of SBVP(1.1), (1.2). We now prove that $u_1(t) \equiv u_2(t)$ on $[0,1]$. To do this, let $v(t) = u_1(t) - u_2(t)$ on $[0,1]$. We will show that $v(t) \equiv 0$ on $[0,1]$. There are three cases to consider.

Case 1 (see $[v(1) > 0]$). In this case, we have that $v(t) \ge 0$ on $[0,1]$. In fact, assume to the contrary that the conclusion is false. Then, there exists $t_0 \in (0,1)$ such that $v(t_0) < 0$. Since $v(0) = 0$ and $v(1) > 0$, then there exist $t_1, t_2 \in [0,1)$ with $t_1 < t_0 < t_2$ such that

$$v(t) < 0 \quad \text{on } (t_1,t_2), \qquad v(t_1) = v(t_2) = 0. \tag{3.1}$$

Thus

$$v''(t) = u_1''(t) - u_2''(t) = -f(t,u_1(t)) + f(t,u_2(t)) \le 0 \quad \text{for } t \in (t_1,t_2). \tag{3.2}$$

Hence $v(t) \ge 0$ on $[t_1,t_2]$, which is a contradiction to $v(t) < 0$ on (t_1,t_2). Therefore $v(t) \ge 0$ on $[0,1]$. Consequently

$$v''(t) = -f(t,u_1(t)) + f(t,u_2(t)) \ge 0 \quad \text{for } t \in (0,1). \tag{3.3}$$

Thus $v(t)$ is convex on $[0,1]$. Since $v(1) > 0$ and

$$v(1) = u_1(1) - u_2(1) = \sum_{i=1}^{m-2}\alpha_i u_1(\eta_i) - \sum_{i=1}^{m-2}\alpha_i u_2(\eta_i) = \sum_{i=1}^{m-2}\alpha_i v(\eta_i), \tag{3.4}$$

then there exists $i_0 \in \{1,2,\dots,m-2\}$ such that

$$v(\eta_{i_0}) = \max\{v(\eta_i) : i = 1,2,\dots,m-2\} > 0, \tag{3.5}$$

and hence from (C_0) and $0 < \eta_{i_0} < 1$, we have

$$v(1) \leq \sum_{i=1}^{m-2} \alpha_i v(\eta_{i_0}) \leq v(\eta_{i_0}) < \frac{1}{\eta_{i_0}} v(\eta_{i_0}), \tag{3.6}$$

which is a contradiction to that $v(t)$ is convex on $[0,1]$.

Case 2 (see $[v(1) = 0]$). In this case, we have that $v(t) \equiv 0$ on $[0,1]$. In fact, assume to the contrary that the conclusion is false. Then, there exists $t_0 \in (0,1)$ such that $v(t_0) \neq 0$. We may assume without loss of generality that $v(t_0) > 0$. Then from $v(0) = v(1) = 0$, there exist $t_1, t_2 \in [0,1]$ with $t_1 < t_0 < t_2$ such that

$$v(t) > 0 \quad \text{on } (t_1, t_2), \quad v(t_1) = v(t_2) = 0. \tag{3.7}$$

Thus

$$v''(t) = -f(t, u_1(t)) + f(t, u_2(t)) \geq 0 \quad \text{for } t \in (t_1, t_2). \tag{3.8}$$

Since $v(t_1) = v(t_2) = 0$, then

$$v(t) \leq 0 \quad \text{for } t \in (t_1, t_2), \tag{3.9}$$

which is a contradiction to that $v(t) > 0$ on (t_1, t_2).

Case 3 (see $[v(1) < 0]$). In this case, similar to the proof of Case 1 we can easily show that $v(t) \leq 0$ on $[0,1]$. Consequently

$$v''(t) = -f(t, u_1(t)) + f(t, u_2(t)) \leq 0 \quad \text{for } t \in (0,1). \tag{3.10}$$

Thus $v(t)$ is concave on $[0,1]$. Since $v(1) = \sum_{i=1}^{m-2} \alpha_i v(\eta_i) < 0$, then there exists $i_1 \in \{1,2,\dots,m-2\}$ such that $v(\eta_{i_1}) = \min\{v(\eta_i) : i = 1,2,\dots,m-2\} < 0$, and hence from $0 < \eta_{i_1} < 1$, we have

$$v(1) \geq \sum_{i=1}^{m-2} \alpha_i v(\eta_{i_1}) \geq v(\eta_{i_1}) > \frac{1}{\eta_{i_1}} v(\eta_{i_1}), \tag{3.11}$$

which is a contradiction to that $v(t)$ is concave on $[0,1]$.

In summary, $v(t) \equiv 0$ on $[0,1]$, that is, $u_1(t) \equiv u_2(t)$ on $[0,1]$. This completes the proof of the theorem. □

Theorem 3.2. *Assume that conditions* (C_0), (C_1), *and* (C_2) *are satisfied. Then SBVP*(1.1), (1.2) *has exactly one positive solution.*

Proof. The uniqueness of positive solution to SBVP(1.1), (1.2) follows from Theorem 3.1 immediately. Thus we only need to show the existence.

Let $\{h_j\}_{j=1}^{\infty}$ be a decreasing sequence that converges to the number 0. Then from Lemma 2.6, BVP$(2.1)_{h_j}$ has a unique solution $u(t;h_j) := u_j(t)$. From Lemma 2.7 and $(2.11)_h$, we have that for each $j < k$,

$$0 \leq u_j(t) - u_k(t) \leq h_j - h_k \quad \text{for } t \in [0,1]. \tag{3.12}$$

Thus there exists $u \in C[0,1]$ such that

$$\lim_{j\to\infty} u_j(t) = u(t) \geq 0, \quad \text{uniformly on } [0,1]. \tag{3.13}$$

It is easy to see that $u(t)$ satisfies boundary conditions (1.2).

Now we prove that

$$u(t) > 0 \quad \text{for } t \in (0,1]. \tag{3.14}$$

At first, we prove that

$$u(\eta_{i_0}) = \max\{u(\eta_i) : i = 1,2,\ldots,m-2\} > 0, \tag{3.15}$$

where $i_0 \in \{1,2,\ldots,m-2\}$. In fact, assume to the contrary that the conclusion is false. Then

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = 0. \tag{3.16}$$

From the fact that each function in the sequence $\{u_j\}_{j=1}^{\infty}$ is concave, we have that $u(t)$ is concave. It follows from $u(0) = u(\eta_{i_0}) = u(1) = 0$ that $u(t) \equiv 0$ on $[0,1]$. Thus when j is large enough, $u_j(t)$ is small enough such that $u_j(t) \leq h_1$ on $[0,1]$. Hence from condition (C_1), we have

$$\begin{aligned} u_j(\eta_{i_0}) &= \int_0^1 G(\eta_{i_0}, s) f(s, u_j(s))ds \\ &\quad + \frac{\eta_{i_0}}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s) f(s, u_j(s))ds + h_j \\ &> \int_0^1 G(\eta_{i_0}, s) f(s, h_1)ds > 0. \end{aligned} \tag{3.17}$$

Let $j \to \infty$, we have

$$u(\eta_{i_0}) \geq \int_0^1 G(\eta_{i_0}, s) f(s, h_1) ds > 0. \tag{3.18}$$

This is a contradiction to $u(\eta_{i_0}) = 0$. Thus $u(\eta_{i_0}) > 0$, and hence $u(1) > 0$. Since $u(t)$ is concave, then $u(t) > 0$ on $(0, 1]$. Since

$$u_j(t) = \int_0^1 G(t, s) f(s, u_j(s)) ds + \frac{t}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s) f(s, u_j(s)) ds + h_j, \tag{3.19}$$

then passing to the limit, by Monotone convergence theorem [31], we have

$$u(t) = \int_0^1 G(t, s) f(s, u(s)) ds + \frac{t}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s) f(s, u(s)) ds. \tag{3.20}$$

Therefore by Lemma 2.4, $u(t)$ is a positive solution of SBVP(1.1), (1.2). This completes the proof of the theorem. □

Finally, we give an example to which our results can be applicable.

Example 3.3. Consider the singular nonlinear second-order m-point boundary value problem:

$$\begin{aligned} u'' + \frac{1}{t^{\beta_1}(1-t)^{\beta_2} u^{2-\beta_1}} &= 0, \quad t \in (0, 1), \\ u(0) = 0, \qquad u(1) &= \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{aligned} \tag{3.21}$$

where $m \geq 3$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\alpha_i > 0$ $(i = 1, 2, \ldots, m-2)$, $\sum_{i=1}^{m-2} \alpha_i \leq 1$, and $\beta_1, \beta_2 \in (0, 2)$.

Let

$$f(t, u) = \frac{1}{t^{\beta_1}(1-t)^{\beta_2} u^{2-\beta_1}} \quad \text{for } (t, u) \in (0, 1) \times (0, +\infty). \tag{3.22}$$

Obviously, the function $f(t, u)$ is singular at $t = 0, 1$ and $u = 0$. It is easy to verify that $f(t, u)$ satisfies conditions (C_1) and (C_2). So from Theorem 3.2, SBVP(3.21) has exactly one positive solution. However, we note that Theorem 2 in [7] cannot guarantee that SBVP(3.21) has a unique positive solution, since

$$\int_0^1 t(1-t) f(t, \lambda t(1-t)) dt = +\infty \quad \text{for } \lambda > 0. \tag{3.23}$$

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