Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 853607, 23 pages
doi:10.1155/2009/853607

Research Article

Stagnation Zones for $\mathcal{A}$-Harmonic Functions on Canonical Domains

Vladimir M. Miklyukov,[1] Antti Rasila,[2] and Matti Vuorinen[3]

[1] *Department of Mathematics, Volgograd State University, Universitetskii prospect 100, Volgograd 400062, Russia*
[2] *Department of Mathematics and Systems Analysis, Aalto University, P.O. Box 11100, FI-00076 Aalto, Finland*
[3] *Department of Mathematics, University of Turku, 20014 Turku, Finland*

Correspondence should be addressed to Antti Rasila, antti.rasila@iki.fi

Received 1 July 2009; Revised 31 October 2009; Accepted 15 November 2009

Recommended by Sandro Salsa

We study stagnation zones of $\mathcal{A}$-harmonic functions on canonical domains in the Euclidean n-dimensional space. Phragmén-Lindelöf type theorems are proved.

1. Introduction

In this article we investigate solutions of the $\mathcal{A}$-Laplace equation on canonical domains in the n-dimensional Euclidean space.

Suppose that D is a domain in $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a function. For $s > 0$, a subset $\Delta \subset D$ is called *s-zone* (*stagnation zone with the deviation* s) of f if there exists a constant C such that the difference between C and the function f is smaller than s on Δ. We may, for example, consider difference in the sense of the sup norm

$$\|f(x) - C\|_{C(\Delta)} = \sup_{x\in\Delta}|f(x) - C| < s, \tag{1.1}$$

the L^p-norm

$$\|f(x) - C\|_{L^p(\Delta)} = \left(\int_\Delta |f(x) - C|^p d\mathcal{H}^n\right)^{1/p} < s, \tag{1.2}$$

or the Sobolev norm

$$\|f(x) - C\|_{W_p^1(\Delta)} = \left(\int_\Delta |\nabla f(x)|^p \, d\mathcal{H}^n\right)^{1/p} < s, \tag{1.3}$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure in $\mathbb{R}^n$.

For discussion about the history of the question, recent results and applications the reader is referred to [1, 2].

Some estimates of stagnation zone sizes for solutions of the $\mathcal{A}$-Laplace equation on locally Lipschitz surfaces and behavior of solutions in stagnation zones were given in [3]. In this paper we consider solutions of the $\mathcal{A}$-Laplace equation in subdomains of $\mathbb{R}^n$ of a special form, canonical domains. In two-dimensional case, such domains are sectors and strips. In higher dimensions, they are conical and cylindrical regions. The special form of domains allows us to obtain more precise results.

Below we study stagnation zones of generalized solutions of the $\mathcal{A}$-Laplace equation

$$\operatorname{div} \mathcal{A}(x, \nabla f) = 0, \tag{1.4}$$

(see [4]) with boundary conditions of types (see Definitions 1.1 and 1.2 below)

$$\begin{aligned} \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle &= 0, \quad x \in \partial D \setminus G, \\ f \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle &= 0, \quad x \in \partial D \setminus G \end{aligned} \tag{1.5}$$

on canonical domains in the Euclidean n-dimensional space, where G is a closed subset of ∂D. We will prove Phragmén-Lindelöf type theorems for solutions of the $\mathcal{A}$-Laplace equation with such boundary conditions.

1.1. Canonical Domains

Let $n \geq 2$. Fix an integer k, $1 \leq k \leq n$, and set

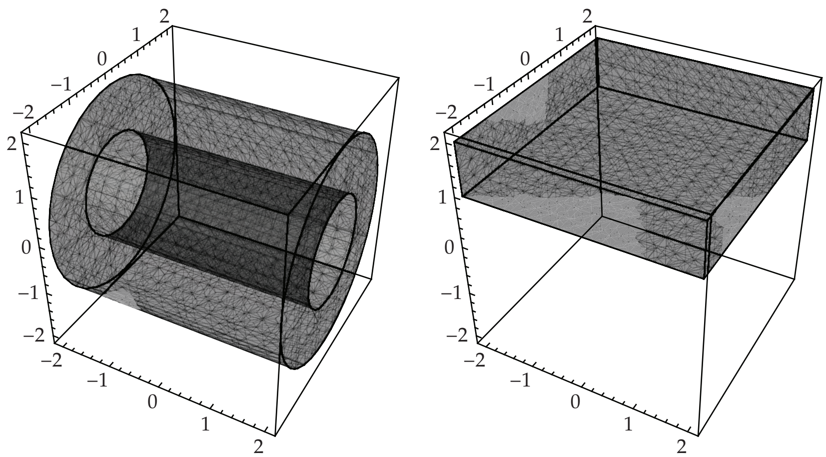
$$d_k(x) = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}. \tag{1.6}$$

We call the set

$$B_k(t) = \{x \in \mathbb{R}^n : d_k(x) < t\} \tag{1.7}$$

a k-ball and

$$\Sigma_k(t) = \{x \in \mathbb{R}^n : d_k(x) = t\} \tag{1.8}$$



(a) (b)

Figure 1: $D^1_{1,2}$ (a) and $D^2_{1,2}$ in $\mathbb{R}^3$.

a k-sphere in $\mathbb{R}^n$. In particular, the symbol $\Sigma_k(0)$ denotes the k-sphere with the radius 0, that is, the set

$$\Sigma_k(0) = \{x = (x_1, \ldots, x_k, \ldots, x_n) : x_1 = \cdots = x_k = 0\}. \tag{1.9}$$

For every $0 < k < n$, we set

$$\begin{gathered} p_k(x) = \left(\sum_{j=k+1}^{n} x_j^2 \right)^{1/2}, \\ \Sigma_k^*(t) = \left\{ x \in \mathbb{R}^n : p_k(x) = t \right\}, \quad t \geq 0. \end{gathered} \tag{1.10}$$

Let $0 < \alpha < \beta < \infty$ be fixed, and let (see Figure 1)

$$D^k_{\alpha,\beta} = \left\{ x \in \mathbb{R}^n \ : \alpha < p_k(x) < \beta \right\}. \tag{1.11}$$

For $k = n - 1$, we also assume that $x_n > 0$. Then for $k = n - 1$, the $D^{n-1}_{\alpha,\beta}$ is the a layer between two parallel hyperplanes, and for $1 \leq k < n - 1$ the boundary of the domain $D^k_{\alpha,\beta}$ consists of two coaxial cylindrical surfaces. The intersections $\Sigma_k(t) \cap D^k_{\alpha,\beta}$ are precompact for all $t > 0$. Thus, the functions $d_k(x)$ are *exhaustion functions* for $D^k_{\alpha,\beta}$.

1.2. Structure Conditions

Let D be a subdomain of $\mathbb{R}^n$ and let

$$\mathcal{A}(x,\xi) : \overline{D} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{1.12}$$

be a vector function such that for a.e. $x \in \overline{D}$ the function

$$\mathcal{A}(x,\xi) : \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{1.13}$$

is defined and is continuous with respect to ξ. We assume that the function

$$x \longmapsto \mathcal{A}(x,\xi) \tag{1.14}$$

is measurable in the Lebesgue sense for all $\xi \in \mathbb{R}^n$ and

$$\mathcal{A}(x,\lambda\xi) = \lambda\ |\lambda|^{p-2}\mathcal{A}(x,\xi), \quad \lambda \in \mathbb{R} \setminus \{0\},\ p \geq 1. \tag{1.15}$$

Suppose that for a.e. $x \in D$ and for all $\xi \in \mathbb{R}^n$ the following properties hold:

$$\nu_1\ |\xi|^p \leq \langle \xi, \mathcal{A}(x,\xi) \rangle, \qquad |\mathcal{A}(x,\xi)|\ \leq \nu_2\ |\xi|^{p-1}, \tag{1.16}$$

with $p \geq 1$ and some constants $\nu_1, \nu_2 > 0$. We consider the equation

$$\operatorname{div} \mathcal{A}(x, \nabla f) = 0. \tag{1.17}$$

An important special case of (1.17) is the Laplace equation

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0. \tag{1.18}$$

As in [4, Chapter 6], we call continuous weak solutions of (1.17) $\mathcal{A}$*-harmonic* functions. However we should note that our definition of generalized solutions is slightly different from the definition given in [4, page 56].

1.3. Frequencies

Fix $t \geq 0$ and $p \geq 1$. Let O be an open subset of $\Sigma_k^*(t)$ (with respect to the relative topology of $\Sigma_k^*(t)$), and let $\mathcal{D}$ be a nonempty closed subset of ∂O. We set

$$\lambda_{p,\mathcal{D}}(O) = \inf_u \frac{\int_O |\nabla u|^p d\mathcal{H}^{n-1}}{\int_O u^p d\mathcal{H}^{n-1}}, \tag{1.19}$$

where $u \in \mathrm{Lip}_{\mathrm{loc}}(O) \cap C^0(\overline{O})$ with $u|_{\not{D}} = 0$. If $\not{D} = \partial O$, then we call $\lambda_p(O) \equiv \lambda_{p,\not{D}}(O)$ the *first frequency* of the order $p \geq 1$ of the set O. If $\not{D} \neq \partial O$, then the quantity $\lambda_{p,\not{D}}(O)$ is the *third frequency*.

The *second frequency* is the following quantity:

$$\mu_p(O) = \sup_C \inf_u \frac{\int_O |\nabla u|^p d\mathcal{H}^{n-1}}{\int_O (u - C)^p d\mathcal{H}^{n-1}}, \tag{1.20}$$

where the supremum is taken over all constants C and $u \in \mathrm{Lip}_{\mathrm{loc}}(O) \cap C^0(\overline{O})$. See also Pólya and Szegö [5] as well as Lax [6].

1.4. Generalized Boundary Conditions

Suppose that D is a proper subdomain of $\mathbb{R}^n$. Let $\varphi : D \to \mathbb{R}$ be a locally Lipschitz function. We denote by $D_b(\varphi)$ the set of all points $x \in D$ at which φ does not have the differential. Let $U \subset D$ be a subset and let $\partial' U = \partial U \setminus \partial D$ be its boundary with respect to D. If $\partial' U$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable, then it has locally finite perimeter in the sense of De Giorgi, and therefore a unit normal vector $\overline{\mathbf{n}}$ exists $\mathcal{H}^{n-1}$-almost everywhere on ∂U [7, Sections 3.2.14, 3.2.15].

Let $D \subset \mathbb{R}^n$ be a domain and let $G \subset \partial D$ be a subset of the boundary of D. Define the concept of a generalized solution of (1.17) with zero boundary conditions on $\partial D \setminus G$. A subset $U \subset D$ is called *admissible*, if $\overline{U} \cap \overline{G} = \emptyset$ and U have a $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary with respect to D.

Suppose that D is unbounded. Let $G \subset \partial D$ be a set closed in $\mathbb{R}^n \cup \{\infty\}$. We denote by (G, D) the collection of all subdomains $U \subset D$ with $\partial U \subset (D \cup (\partial D \setminus G))$ and $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundaries $\partial' U = \partial U \setminus \partial D$.

Definition 1.1. We say that a locally Lipschitz function $f : D \to \mathbb{R}$ is a generalized solution of (1.17) with the boundary condition

$$\langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle = 0, \quad x \in \partial D \setminus G, \tag{1.21}$$

if for every subdomain $U \in (G, D)$,

$$\mathcal{H}^{n-1}\left[\partial' U \cap D_b(f)\right] = 0, \tag{1.22}$$

and for every locally Lipschitz function $\varphi : \overline{U} \setminus G \to \mathbb{R}$ the following property holds:

$$\int_{\partial' U} \varphi \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = \int_U \langle \mathcal{A}(x, \nabla f), \nabla \varphi \rangle d\mathcal{H}^n. \tag{1.23}$$

Here $\overline{\mathbf{n}}$ is the unit normal vector of $\partial' U$ and $d\mathcal{H}^n$ is the volume element on $\mathbb{R}^n$.

Definition 1.2. We say that a locally Lipschitz function $f : D \to \mathbb{R}$ is a generalized solution of (1.17) with the boundary condition

$$f \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle = 0, \quad x \in \partial D \setminus G, \tag{1.24}$$

if for every subdomain $U \in (G, D)$ with (1.22) and for every locally Lipschitz function $\varphi : \overline{U} \setminus G \to \mathbb{R}$ the following property holds:

$$\int_{\partial' U} \varphi f \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = \int_U \langle \mathcal{A}(x, \nabla f), \nabla(\varphi\, f) \rangle d\mathcal{H}^n. \tag{1.25}$$

In the case of a smooth boundary ∂D, and $f \in C^2(D)$, the relation (1.23) implies (1.17) with (1.21) everywhere on $\partial D \setminus G$. This requirement (1.25) implies (1.17) with (1.24) on $\partial D \setminus G$. See [8, Section 9.2.1].

The surface integrals exist by (1.22). Indeed, this assumption guarantees that $\nabla f(x)$ exists $\mathcal{H}^{n-1}$ a.e. on $\partial' U$. The assumption that $U \in (G, D)$ implies existence of a normal vector $\overline{\mathbf{n}}$ for $\mathcal{H}^{n-1}$ a.e. points on $\partial' U$ [7, Chapter 2, Section 3.2]. Thus, the scalar product $\langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle$ is defined and is finite a.e. on $\partial' U$.

2. Saint-Venant's Principle

In this section, we will prove the Saint-Venant principle for solutions of the $\mathcal{A}$-Laplace equation. The Saint-Venant principle states that strains in a body produced by application of a force onto a small part of its surface are of negligible magnitude at distances that are large compared to the diameter of the part where the force is applied. This well known result in elasticity theory is often stated and used in a loose form. For mathematical investigation of the results of this type, see, for example, [9].

In this paper the inequalities of the form (2.5), (2.4) are called the Saint-Venant principle (see also [9, 10]). Here we consider only the case of canonical domains. We plan to consider the general case in another article.

Let $0 < k < n$. Fix a domain D_0 in $\mathbb{R}^k$ with compact and smooth boundary, and write

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \ldots, x_k) \in D_0\}. \tag{2.1}$$

We write $\mathcal{P} = \{x \in \partial D : p_k(x) = \alpha\}$, $\mathcal{Q} = \{x \in \partial D : p_k(x) = \beta\}$, and $G = \mathcal{P} \cup \mathcal{Q}$. Let $t, \tau \in (\alpha, \beta)$, $t < \tau$ and

$$\Delta^k(t, \tau) = \left\{x \in D : t < p_k(x) < \tau\right\}. \tag{2.2}$$

For $s \geq 0$, we set

$$\sigma^k(s) = \left\{x \in \Delta^k(0, \infty) : p_k(x) = s\right\}. \tag{2.3}$$

Theorem 2.1. *Let $\alpha < \tau' < \tau'' < \beta$, and let $0 < k < n$. If $f : D \to \mathbb{R}$ is a generalized solution of* (1.17) *with the generalized boundary condition* (1.21) *on $\partial D \setminus G$, then the inequality*

$$I_1(t,\tau') + \frac{C_1(t)}{\nu_1} \le \left(I_1(t,\tau'') + \frac{C_1(t)}{\nu_1} \right) \exp\left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \mu_p\left(\sigma^k(\tau)\right) d\tau \right] \tag{2.4}$$

holds for all $t \in (\alpha, \tau']$.

If $f : D \to \mathbb{R}$ is a generalized solution of (1.17) *with the generalized boundary condition* (1.24)*, then*

$$I_1(t,\tau') + \frac{C_2(t)}{\nu_1} \le \left(I_1(t,\tau'') + \frac{C_2(t)}{\nu_1} \right) \exp\left[-\frac{\nu_1}{\nu_2} \int_{\tau'}^{\tau''} \lambda_{p,Z_f(\tau)}^{1/p}\left(\sigma^k(\tau)\right) d\tau \right] \tag{2.5}$$

holds for all $t \in (\alpha, \tau']$. Here

$$I_1(t,\tau) = \int_{\Delta^k(t,\tau)} |\nabla f|^p d\mathcal{H}^n, \tag{2.6}$$

$$Z_f(\tau) = \left\{ x \in \Sigma_k^*(\tau) \cap \partial D : \lim_{y \to x} f(y) = 0 \right\}. \tag{2.7}$$

Proof.

Case A. At first we consider the case in which f is a generalized solution of (1.17) with the generalized boundary condition (1.24) on $\partial D \setminus G$. It is easy to see that a.e. on $D_{\alpha,\beta}^k$,

$$|\nabla p_k(x)| = 1. \tag{2.8}$$

The domain $\Delta^k(t,\tau)$ belongs to (G,D). Let $\varphi : \overline{U} \setminus G \to \mathbb{R}$ be a locally Lipschitz function. By (1.25) we have

$$\int_{\partial' \Delta^k(t,\tau)} \varphi f \langle \mathcal{A}(x,\nabla f), \overline{\mathbf{n}} \rangle d\mathcal{H}^{n-1} = \int_{\Delta^k(t,\tau)} \langle \mathcal{A}(x,\nabla f), \nabla(\varphi\, f) \rangle d\mathcal{H}^n. \tag{2.9}$$

But

$$\partial' \Delta^k(t,\tau) = \sigma^k(t) \cup \sigma^k(\tau). \tag{2.10}$$

For $\varphi \equiv 1$, we have by (1.16) and (1.25)

$$\begin{aligned} \nu_1 I_1(t,\tau) &\le \int_{\Delta^k(t,\tau)} \langle \mathcal{A}(x,\nabla f), \nabla f \rangle d\mathcal{H}^n \\ &= \int_{\sigma^k(\tau)} f \langle \mathcal{A}(x,\nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1} - \int_{\sigma^k(t)} f \langle \mathcal{A}(x,\nabla f), \nabla p_k(x) \rangle d\mathcal{H}^{n-1}, \end{aligned} \tag{2.11}$$

since $\overline{\mathbf{n}} = \nabla p_k(x)$ for $x \in \sigma^k(\tau)$ and $\overline{\mathbf{n}} = -\nabla p_k(x)$ for $x \in \sigma^k(t)$. We obtain

$$\nu_1 I_1(t,\tau) + C_2(t) \leq \int_{\sigma^k(\tau)} f\langle \mathcal{A}(x,\nabla f), \nabla p_k(x)\rangle d\mathcal{H}^{n-1}, \tag{2.12}$$

where

$$C_2(t) = \int_{\sigma^k(t)} f\langle \mathcal{A}(x,\nabla f), \nabla p_k(x)\rangle d\mathcal{H}^{n-1}. \tag{2.13}$$

Note that we may also choose

$$\widetilde{C}_2(\tau) = -\int_{\sigma^k(\tau)} f\langle \mathcal{A}(x,\nabla f), \nabla p_k(x)\rangle d\mathcal{H}^{n-1} \tag{2.14}$$

to obtain an inequality similar to (2.12).

Next we will estimate the right side of (2.12). By (1.16) and the Hölder inequality,

$$\begin{aligned} &\left|\int_{\sigma^k(\tau)} f\langle \mathcal{A}(x,\nabla f), \nabla p_k(x)\rangle d\mathcal{H}^{n-1}\right| \\ &\quad\leq \int_{\sigma^k(\tau)} |f||\mathcal{A}(x,\nabla f)| d\mathcal{H}^{n-1} \leq \nu_2 \int_{\sigma^k(\tau)} |f||\nabla f|^{p-1} d\mathcal{H}^{n-1} \\ &\quad\leq \nu_2 \left(\int_{\sigma^k(\tau)} |f|^p d\mathcal{H}^{n-1}\right)^{1/p} \left(\int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}\right)^{(p-1)/p}. \end{aligned} \tag{2.15}$$

By using (1.19), we may write

$$\int_{\sigma^k(\tau)} |f|^p d\mathcal{H}^{n-1} \leq \lambda^{-1}_{p,Z_f(\tau)}\left(\sigma^k(\tau)\right) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}, \tag{2.16}$$

$$\left|\int_{\sigma^k(\tau)} f\langle \mathcal{A}(x,\nabla f), \nabla p_k(x)\rangle d\mathcal{H}^{n-1}\right| \leq \nu_2\, \lambda^{-1/p}_{p,Z_f(\tau)}\left(\sigma^k(\tau)\right) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}. \tag{2.17}$$

By (2.12) and the Fubini theorem,

$$\begin{gathered} \nu_1 I_1(t,\tau) + C_2(t) \leq \nu_2 \lambda^{-1/p}_{p,Z_f(\tau)}\left(\sigma^k(\tau)\right) \frac{dI_1}{d\tau}(t,\tau), \\ \frac{\nu_1}{\nu_2}\lambda^{1/p}_{p,Z_f(\tau)}\left(\sigma^k(\tau)\right) \leq \frac{(dI_1/d\tau)(t,\tau)}{(I_1(t,\tau) + C_2(t)/\nu_1)}. \end{gathered} \tag{2.18}$$

By integrating this differential inequality, we have

$$\exp\left[\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\lambda_{p,Z_f(\tau)}^{1/p}\left(\sigma^k(\tau)\right)d\tau\right] \leq \frac{I_1(t,\tau'')+C_2(t)/\nu_1}{I_1(t,\tau')+C_2(t)/\nu_1} \tag{2.19}$$

for arbitrary $\tau',\tau'' \in (\alpha,\beta)$ with $\tau' < \tau''$. We have shown that

$$I_1(t,\tau')+\frac{C_2(t)}{\nu_1} \leq \left(I_1(t,\tau'')+\frac{C_2(t)}{\nu_1}\right)\exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\lambda_{p,Z_f(\tau)}^{1/p}\left(\sigma_p^k(\tau)\right)d\tau\right]. \tag{2.20}$$

Case B. Now we assume that f is a generalized solution of (1.17) with the boundary condition (1.21) on $\partial D \setminus G$. Fix $t < \tau$. By choosing $\varphi \equiv 1$ in (1.23), we see that

$$\int_{\sigma^k(t)\cup\sigma^k(\tau)}\langle \mathcal{A}(x,\nabla f),\overline{\mathbf{n}}\rangle d\mathcal{H}^{n-1} = 0. \tag{2.21}$$

For an arbitrary constant C, we get from this and (1.23)

$$\int_{\sigma^k(t)\cup\sigma^k(\tau)}(f-C)\langle \mathcal{A}(x,\nabla f),\overline{\mathbf{n}}\rangle d\mathcal{H}^{n-1} = \int_{\Delta^k(t,\tau)}\langle \mathcal{A}(x,\nabla f),\nabla f\rangle d\mathcal{H}^{n}. \tag{2.22}$$

Thus

$$\int_{\Delta^k(t,\tau)}\langle \mathcal{A}(x,\nabla f),\nabla f\rangle d\mathcal{H}^{n} \leq C_1(t) + \int_{\sigma^k(\tau)}|f-C|\left|\mathcal{A}(x,\nabla p_k(x))\right|d\mathcal{H}^{n-1}, \tag{2.23}$$

where

$$C_1(t) = \int_{\sigma^k(t)}|f-C|\left|\mathcal{A}(x,\nabla p_k(x))\right|d\mathcal{H}^{n-1} \tag{2.24}$$

or

$$\nu_1 I_1(t,\tau) + C_1(t) \leq \nu_2\int_{\sigma^k(\tau)}|f-C||\nabla f|^{p-1}d\mathcal{H}^{n-1}. \tag{2.25}$$

As above, we obtain

$$\int_{\sigma^k(\tau)}|f-C||\nabla f|^{p-1}d\mathcal{H}^{n-1} \leq \left(\int_{\sigma^k(\tau)}|f-C|^p d\mathcal{H}^{n-1}\right)^{1/p}\left(\int_{\sigma^k(\tau)}|\nabla f|^p d\mathcal{H}^{n-1}\right)^{(p-1)/p}. \tag{2.26}$$

By using (1.20), we get

$$\left(\int_{\sigma^k(\tau)} |f - C_3|^p d\mathcal{H}^{n-1}\right)^{1/p} \le \mu_p^{-1/p}\left(\sigma^k(\tau)\right) \left(\int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}\right)^{1/p}, \tag{2.27}$$

where $C_3 = C_3(f)$ is the constant from (1.20). Then by (2.26) and (2.27),

$$\int_{\sigma^k(\tau)} |f - C_3| |\nabla f|^{p-1} d\mathcal{H}^{n-1} \le \mu_p^{-1}\left(\sigma^k(\tau)\right) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1}, \tag{2.28}$$

and by (2.25) we have

$$\nu_1 I_1(t,\tau) + C_1(t) \le \nu_2 \mu_p^{-1}\left(\sigma^k(\tau)\right) \int_{\sigma^k(\tau)} |\nabla f|^p d\mathcal{H}^{n-1} \tag{2.29}$$

or

$$\nu_1 I_1(t,\tau) + C_1(t) \le \nu_2 \mu_p^{-1}\left(\sigma^k(\tau)\right) \frac{dI_1}{dt}(t,\tau). \tag{2.30}$$

By integrating this inequality, we have shown that

$$I_1(t,\tau') + \frac{C_1(t)}{\nu_1} \le \left(I_1(t,\tau'') + \frac{C_1(t)}{\nu_1}\right) \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \mu_p\left(\sigma^k(\tau)\right) d\tau\right]. \tag{2.31}$$

□

3. Stagnation Zones

Next we apply the Saint-Venant principle to obtain information about stagnation zones of generalized solutions of (1.17). We first consider zones with respect to the Sobolev norm. Other results of this type follow immediately from well-known imbedding theorems.

3.1. *Stagnation Zones with Respect to the* W_p^1*-Norm*

We rewrite (2.4) and (2.5) in another form. Let $0 < k < n$ and let $0 < \alpha < \beta$. Fix a domain D_0 in $\mathbb{R}^k$ with compact and smooth boundary, and write

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \ldots, x_k) \in D_0\}. \tag{3.1}$$

We write

$$p_k^*(x) = p_k(x) - \frac{\alpha + \beta}{2}. \tag{3.2}$$

For $x \in D^k_{\alpha,\beta}$ and

$$\beta^* = \frac{\beta - \alpha}{2}, \tag{3.3}$$

we have

$$-\beta^* < p^*_k(x) < \beta^*, \tag{3.4}$$

and we denote

$$D^{*,k}_{\beta^*} = \{x \in \mathbf{R}^n : -\beta^* < p^*_k(x) < \beta^*\}. \tag{3.5}$$

Let $-\beta^* < \tau' \leq \tau'' < \beta^*$. We write

$$\Delta^{*,k}(\tau', \tau'') = \{x \in D : \tau' < p^*_k(x) < \tau''\}, \tag{3.6}$$

$$I_2(\tau', \tau'') = \int_{\Delta^{*,k}(\tau',\tau'')} |\nabla f|^p d\mathscr{H}^n. \tag{3.7}$$

Let $0 < \tau' < \tau'' < \beta^*$. By (2.5) we have, for $t \in (-\tau, \tau)$,

$$I_2(t, \tau') + \frac{C_4(t)}{\nu_1} \leq \left(I_2(t, \tau'') + \frac{C_4(t)}{\nu_1}\right) \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \lambda^{1/p}_{p, Z^*_f(\tau)}\left(\sigma^{*,k}(\tau)\right) d\tau\right], \tag{3.8}$$

where

$$Z^*_f(\tau) = \left\{x \in \partial D : p^*_k(x) = \tau \wedge \lim_{y \to x} f(y) = 0\right\}. \tag{3.9}$$

By choosing the estimate as in (2.14), we also have

$$I_2(-\tau', t) + \frac{\widetilde{C}_4(t)}{\nu_1} \leq \left(I_2(-\tau'', t) + \frac{\widetilde{C}_4(t)}{\nu_1}\right) \exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \lambda^{1/p}_{p, Z^*_f(\tau)}\left(\sigma^{*,k}(\tau)\right) d\tau\right], \tag{3.10}$$

where

$$\sigma^{*,k}(s) = \left\{x \in \Delta^{*,k}(-\infty, \infty) : p^*_k(x) = s\right\}. \tag{3.11}$$

By adding these inequalities and noting that $C_4(t) + \widetilde{C}_4(t) = 0$, we obtain

$$\begin{aligned} &I_2(-\tau', t) + I_2(t, \tau') \\ &\quad \le \left(I_2(-\tau'', t) + I_2(t, \tau'')\right) \\ &\quad\quad \times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \lambda_{p,Z_f^*(\tau)}^{1/p}\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \lambda_{p,Z_f^*(\tau)}^{1/p}\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}. \end{aligned} \tag{3.12}$$

Thus we have the estimate

$$\begin{aligned} I_2(-\tau', \tau') \le I_2(-\tau'', \tau'') \\ \times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \lambda_{p,Z_f^*(\tau)}^{1/p}\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \lambda_{p,Z_f^*(\tau)}^{1/p}\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}. \end{aligned} \tag{3.13}$$

Similarly, from (2.4) we obtain

$$\begin{aligned} I_2(-\tau', \tau') \le I_2(-\tau'', \tau'') \\ \times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}. \end{aligned} \tag{3.14}$$

From this we obtain the following theorem on stagnation W_p^1-zones.

Theorem 3.1. *Let* $0 < k < n$, $\beta > \alpha > 0$, *and let* $-\beta^* < \tau' \le \tau'' < \beta^*$ *where* β^* *is as in* (3.3). *If* f *is a solution of* (1.17) *on* D *with the generalized boundary condition* (1.21) *on* $\partial D \setminus G$, *where* $G = \{x \in \partial D : p_k^*(x) = \pm\beta^*\}$ *and*

$$\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\} < s^{1/p} \tag{3.15}$$

or a solution of (1.17) *on* D *with the generalized boundary condition* (1.24) *on* $\partial D \setminus G$ *and*

$$\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \lambda_{p,Z_f^*(\tau)}^{1/p}\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \lambda_{p,Z_f^*(\tau)}^{1/p}\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\} < s^{1/p}, \tag{3.16}$$

then the subdomain $\Delta^{*,k}(-\tau', \tau')$ *is an* s*-zone with respect to the* W_p^1*-norm, that is,*

$$\int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p \, d\mathscr{H}^n < s, \tag{3.17}$$

where $\Delta^{*,k}$ *is as in* (3.6).

3.2. Stagnation Zones with Respect to the L^p-Norm

Let $\beta > \alpha > 0$, and let $-\beta^* < \tau' \le \tau'' < \beta^*$ where β^* is as in (3.3).

Denote by C_5 the best constant of the imbedding theorem from $W_p^1(D_{\beta^*}^{*,k})$ to $L^p(D_{\beta^*}^{*,k})$ that is in the inequality

$$\|g - C\|_{L^p(D_{\beta^*}^{*,k})} \le C_5 \; \|g\|_{W_p^1(D_{\beta^*}^{*,k})} \tag{3.18}$$

if such constant exists (see Maz'ya [11] or [12]). Then we obtain from (3.13), (3.14)

$$\begin{aligned} &\|f - C\|^p_{L^p(\Delta^{*,k}(-\tau',\tau'))} \\ &\le C_5^p I_2(-\tau'',\tau'') \times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z_f^*(\tau)}\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z_f^*(\tau)}\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\|f - C\|^p_{L^p(\Delta^{*,k}_{\tau'})} \\ &\le C_5^p I_2(-\tau'',\tau'') \times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}. \end{aligned} \tag{3.20}$$

These relations can be used to obtain information about stagnation zones with respect to the L^p-norm. Namely, we have the following.

Theorem 3.2. *Let $0 < k < n$, and let*

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \ldots, x_k) \in D_0\}, \tag{3.21}$$

where D_0 is a domain in $\mathbb{R}^k$ with compact and smooth boundary. If f is a solution of (1.17) *on D, with the generalized boundary condition,* (1.21) *or* (1.24)*, on $\partial D \setminus G$, where $G = \{x \in \partial D : p_k^*(x) = \pm\beta^*\}$, and the right side of,* (3.19) *or* (3.20)*, is smaller than $s > 0$, then the domain $\Delta^{*,k}(-\tau', \tau')$ is a stagnation zone with the deviation s^p in the sense of the L^p-norm on D.*

3.3. Stagnation Zones for Bounded, Uniformly Continuous Functions

Let $\beta > \alpha > 0$, and let $-\beta^* < \tau' \le \tau'' < \beta^*$ where β^* is as in (3.3).

As before, denote by C_6 the best constant of the imbedding theorem from $W_p^1(D_{\beta^*}^{*,k})$ to $C(D_{\beta^*}^{*,k})$, that is in the inequality

$$\|g - C\|_{C(D_{\beta^*}^{*,k})} \le C_6 \; \|g\|_{W_p^1(D_{\beta^*}^{*,k})} \tag{3.22}$$

if such constant exists. For example, if the domain $D_{\beta^*}^{*,k}$ is convex, then (3.22) holds for $p > n$ (see Maz'ya [11] or [12, page 85]).

In this case from (3.13), (3.14), we obtain

$$\|f-C\|_{C(\Delta^{*,k}(-\tau',\tau'))} \leq C_6^p I_2(-\tau'',\tau'')\times\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'}\lambda^{1/p}_{p,Z_f^*(\tau)}\left(\sigma^{*,k}(\tau)\right)d\tau\right],\exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\lambda^{1/p}_{p,Z_f^*(\tau)}\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}, \tag{3.23}$$

$$\|f-C\|_{C(\Delta^{*,k}(-\tau',\tau'))} \leq C_6^p I_2(-\tau'',\tau'')\times\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'}\mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right],\exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\mu_p\left(\sigma^{*,k}(\tau)\right)d\tau\right]\right\}. \tag{3.24}$$

These relations can be used to obtain theorems about stagnation zones for bounded, uniformly continuous functions.

Theorem 3.3. *Let $0<k<n$. If f is a solution of (1.17), $p>n$, on D where D is as before with the generalized boundary condition, (1.21) or (1.24), on $\partial D\setminus G$ where $G=\{x\in\partial D : p_k^*(x)=\pm\beta^*\}$ and the right side of, (3.23) or (3.24), is smaller than $s>0$, then the domain $\Delta^{*,k}(-\tau',\tau')$ is a stagnation zone with the deviation s in the sense of the norm $\|\cdot\|_{C^0(\Delta^{*,k}(-\tau',\tau'))}$.*

4. Other Applications

Next we prove Phragmén-Lindelöf type theorems for the solutions of the $\mathcal{A}$-Laplace equation with boundary conditions (1.21) and (1.24).

4.1. Estimates for W_p^1-Norms

Let $\beta>\alpha>0$, and let D_0 be a domain in $\mathbb{R}^k$ with compact and smooth boundary. Write

$$D=D_0\times\mathbb{R}^{n-k}=\{x\in\mathbb{R}^n : (x_1,\ldots,x_k)\in D_0\}. \tag{4.1}$$

Suppose that β^* is as in (3.3). First we will prove some estimates of the W_p^1-norm of a solution. Let f be a solution of (1.17) on $D_{\beta^*}^{*,k}$ with the generalized boundary condition (1.21) on $\partial D\setminus G$. Fix $0<\tau'<\tau''<\beta^*$ and estimate $\|f\|_{W_p^1(\Delta^{*,k}(-\tau',\tau'))}$.

Let $\psi:[\tau',\tau'']\to(0,\infty)$ be a Lipschitz function such that

$$\psi(\tau')=1,\qquad \psi(\tau'')=0. \tag{4.2}$$

We choose

$$\phi(t)=\begin{cases}1 & \text{for } |t|<\tau',\\ \psi(|t|) & \text{for } \tau'\leq|t|\leq\tau''.\end{cases} \tag{4.3}$$

The function $\varphi(x) = \phi(p_k^*(x))$ is admissible in Definition 1.1 for

$$U = \Delta^{*,k}(-\tau'', \tau''). \tag{4.4}$$

As in (2.22), we may by (1.23) write

$$\begin{aligned}&\int_{\sigma^{*,k}(-\tau'')\cup\sigma^{*,k}(\tau'')} \phi^p(p_k^*(x))(f-C)\langle \mathcal{A}(x,\nabla f), \overline{\mathbf{n}}\rangle d\mathcal{H}^{n-1}\\&\quad= \int_{\Delta^{*,k}(-\tau'',\tau'')} \langle \mathcal{A}(x,\nabla f), \nabla(\phi^p(p_k^*(x))(f-C))\rangle d\mathcal{H}^n.\end{aligned} \tag{4.5}$$

By the construction of ϕ, (4.2), and (4.3), the surface integral is equal to zero, and we have

$$\begin{aligned}&\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^p(p_k^*(x))\langle \mathcal{A}(x,\nabla f), \nabla f\rangle d\mathcal{H}^n\\&\quad= -p\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))(f-C)\langle \mathcal{A}(x,\nabla f), \nabla\phi(p_k^*(x))\rangle d\mathcal{H}^n.\end{aligned} \tag{4.6}$$

Thus by (1.16),

$$\begin{aligned}&\nu_1\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^p(p_k^*(x))|\nabla f|^p d\mathcal{H}^n\\&\quad\le p\nu_2\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))|f-C||\nabla f|^{p-1}\ |\nabla\phi(p_k^*(x))| d\mathcal{H}^n.\end{aligned} \tag{4.7}$$

Now we note that

$$|\nabla\phi(p_k^*(x))| = |\phi'(p_k^*(x))|, \tag{4.8}$$

and by the Hölder inequality,

$$\begin{aligned}&\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))|f-C||\nabla f|^{p-1}|\nabla\phi(p_k^*(x))| d\mathcal{H}^n\\&\quad= \int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))|\nabla f|^{p-1}|\phi'(p_k^*(x))||f-C| d\mathcal{H}^n\\&\quad\le \left(\int_{\Delta^{*,k}(-\tau'',\tau'')} \phi^p(p_k^*(x))|\nabla f|^p d\mathcal{H}^n\right)^{(p-1)/p}\left(\int_{\Delta^{*,k}(-\tau'',\tau'')} |\phi'(p_k^*(x))|^p|f-C|^p d\mathcal{H}^n\right)^{1/p}.\end{aligned} \tag{4.9}$$

From this inequality and (4.7), we obtain

$$\nu_1^p \int_{\Delta^{*,k}(-\tau',\tau')} \phi^p(p_k^*(x)) |\nabla f|^p d\mathcal{H}^n \leq p^p \nu_2^p \int_{\Delta^{*,k}(-\tau'',\tau'')} |\phi'(p_k^*(x))|^p |f-C|^p d\mathcal{H}^n. \tag{4.10}$$

Because $\phi(p_k^*(x)) \equiv 1$ on $\Delta^{*,k}(-\tau',\tau')$, we have the following inequality:

$$\nu_1^p \int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq p^p \nu_2^p \int_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f-C|^p d\mathcal{H}^n. \tag{4.11}$$

Next we will find that

$$\min_{\psi} \int_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f-C|^p d\mathcal{H}^n, \tag{4.12}$$

where the minimum is taken over all ψ in (4.3). We have

$$\begin{aligned} &\int_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f-C|^p d\mathcal{H}^n \\ &\quad = \int_{-\tau''}^{-\tau'} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1} \\ &\qquad + \int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\min_{\psi} \int_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p |f(x)-C|^p d\mathcal{H}^n \\ &\quad \leq \min_{\psi} \int_{-\tau''}^{-\tau'} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1} \\ &\qquad + \min_{\psi} \int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1} \equiv A_1 + A_2. \end{aligned} \tag{4.14}$$

Because by the Hölder inequality

$$\begin{aligned} 1 &\leq \left(\int_{\tau'}^{\tau''} |\psi'(\tau)| d\tau \right)^p \leq \left[\int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1} \right] \\ &\quad \times \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1} \right)^{1/(1-p)} \right]^{p-1}, \end{aligned} \tag{4.15}$$

we have

$$\left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \le \int_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}, \tag{4.16}$$

and hence,

$$A_2 \ge \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}. \tag{4.17}$$

It is easy to see that here the equality holds for a special choice of ψ. Thus

$$A_2 = \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}. \tag{4.18}$$

Similarly,

$$A_1 = \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}. \tag{4.19}$$

From (4.14) we obtain

$$\begin{aligned}\min_{\psi} \int_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} &|\psi'(p_k^*(x))|^p |f-C|^p d\mathcal{H}^n \\ &\le \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \\ &\quad + \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}.\end{aligned} \tag{4.20}$$

By using (4.11), we obtain the inequality

$$\begin{aligned}p^{-p}\left(\frac{\nu_1}{\nu_2}\right)^p \int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n &\le \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \\ &\quad + \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p},\end{aligned} \tag{4.21}$$

where C is an arbitrary constant. From this we obtain

$$\int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq C_7 \max\left\{ \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \right\}, \tag{4.22}$$

where $C_7 = 2p^p(\nu_2/\nu_1)^p$.

Similarly, for the solutions of the $\mathcal{A}$-Laplace equation with the boundary condition (1.24), we may prove that

$$p^{-p}\left(\frac{\nu_1}{\nu_2}\right)^p \int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} + \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}. \tag{4.23}$$

It follows that

$$\int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p d\mathcal{H}^n \leq C_7 \max\left\{ \left[\int_{-\tau''}^{-\tau'} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^{*,k}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \right\}. \tag{4.24}$$

4.2. Phragmén-Lindelöf Type Theorems I

We prove Phragmén-Lindelöf type theorems for cylindrical domains. Let $k = n-1$. Fix a domain D_0 in $\mathbb{R}^{n-1}$ with compact and smooth boundary. Consider the domain

$$D = D_0 \times \mathbb{R} = \{x \in \mathbb{R}^n : (x_1, \ldots, x_{n-1}) \in D_0\}. \tag{4.25}$$

Let $f_0 : D \to \mathbb{R}$ be a generalized solution of (1.17) with (1.15) and (1.16) satisfying the boundary condition (1.21) on ∂D.

Fix $\beta > \alpha > 0$, and let β^* be as in (3.3). Let $f(x) = f_0(x - \beta^* e_n)$, where e_n is the nth unit coordinate vector, and let $0 < \tau' < \tau'' < \beta^* < \infty$. By (4.22)

$$\int_{\Delta^{*,k}(-\tau'',\tau'')} |\nabla f|^p d\mathcal{H}^n \le C_7 \max\left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \right\}. \tag{4.26}$$

By using (3.14), we obtain from this the inequality

$$I_2(-\tau',\tau') \le C_7 \max\left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \right\}$$
$$\times \max\left\{ \exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \mu_p\left(\sigma^{*,n-1}(\tau)\right) d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \mu_p\left(\sigma^{*,n-1}(\tau)\right) d\tau\right] \right\}. \tag{4.27}$$

We observe that in this case

$$\mu_p\left(\sigma^{*,n-1}(\tau)\right) \equiv \mu_p\left(\sigma^{n-1}(0)\right), \tag{4.28}$$

and hence,

$$\int_{\tau'}^{\tau''} \mu_p\left(\sigma^{*,n-1}(\tau)\right) d\tau = \mu_p\left(\sigma^{n-1}(0)\right)(\tau'' - \tau'). \tag{4.29}$$

It follows that

$$I_2(-\tau',\tau') \le C_7 \max\left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \right\} \times \exp\left[-\frac{\nu_1}{\nu_2}\,\mu_p\left(\sigma^{n-1}(0)\right)(\tau''-\tau')\right]. \tag{4.30}$$

By letting $\beta,\tau'' \to +\infty$, we obtain the following statement.

Theorem 4.1. *Fix a domain D_0 in $\mathbb{R}^{n-1}$ with compact and smooth boundary. Let*

$$D = D_0\times\mathbb{R} = \{x\in\mathbb{R}^n : (x_1,\dots,x_{n-1})\in D_0\}, \tag{4.31}$$

and let $f : D \to \mathbb{R}$ be a generalized solution of (1.17) *with* (1.15) *and* (1.16) *satisfying the boundary condition* (1.21) *on ∂D. If for a constant C the right side of* (4.30) *goes to* 0 *as $\tau'' \to \infty$, then $f \equiv \mathrm{const}$ on D.*

Similarly for a solution f of (1.17) with (1.15) and (1.16) satisfying the boundary condition (1.24), we may write

$$I_2(-\tau',\tau') \le C_7 \max\left\{ \left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-a}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \right\} \times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z_f^*(\tau)}\left(\sigma^{*,n-1}(\tau)\right)d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z_f^*(\tau)}\left(\sigma^{*,n-1}(\tau)\right)d\tau\right]\right\}. \tag{4.32}$$

However here we do not have any identity similar to (4.28). We have the following.

Theorem 4.2. *Fix a domain D_0 in $\mathbb{R}^{n-1}$ with compact and smooth boundary. Let*

$$D = D_0\times\mathbb{R} = \{x\in\mathbb{R}^n : (x_1,\dots,x_{n-1})\in D_0\}, \tag{4.33}$$

and let $f : D \to \mathbb{R}$ be a generalized solution of (1.17) *with* (1.15) *and* (1.16) *satisfying the boundary condition* (1.24) *on ∂D. If the right side of* (4.32) *tends to* 0 *as $\tau'' \to \infty$, then $f \equiv 0$ on ∂D.*

If $f(x) = 0$ everywhere on ∂D, then an identity similar to (4.28) holds in the following form:

$$\lambda_p^{1/p}\left(\sigma^{*,n-1}(\tau)\right) \equiv \lambda_p^{1/p}\left(\sigma^{n-1}(0)\right). \tag{4.34}$$

As above, we find that

$$\begin{aligned} I_2(-\tau',\tau') \le C_7 \max\Bigg\{ &\left[\int_{-\tau''-1}^{-\tau''} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \\ &\left[\int_{\tau''}^{\tau''+1} d\tau \left(\int_{\sigma^{*,n-1}(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \Bigg\} \\ &\times \exp\left[-\frac{\nu_1}{\nu_2}\lambda_p^{1/p}\left(\sigma^{n-1}(0)\right)(\tau''-\tau')\right]. \end{aligned} \tag{4.35}$$

Thus we obtain the following.

Corollary 4.3. *Fix a domain D_0 in $\mathbb{R}^{n-1}$ with compact and smooth boundary. Let*

$$D = D_0 \times \mathbb{R} = \{x \in \mathbb{R}^n : (x_1, \ldots, x_{n-1}) \in D_0\}, \tag{4.36}$$

and let $f : D \to \mathbb{R}$ be a generalized solution of (1.17) *with* (1.15) *and* (1.16) *satisfying the boundary condition $f = 0$ on ∂D. If the right side of* (4.35) *tends to* 0 *as $\tau'' \to \infty$, then $f \equiv$* const *on ∂D.*

4.3. Phragmén-Lindelöf Type Theorems II

We prove Phragmén-Lindelöf type theorems for canonical domains of an arbitrary form. Let $1 \le k < n-1$. We consider a domain

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \ldots, x_k) \in D_0\}, \tag{4.37}$$

where D_0 is a domain in $\mathbb{R}^k$ with compact and smooth boundary. Let f be a generalized solution of (1.17) with (1.15) and (1.16) satisfying the boundary condition (1.21) on ∂D.

Fix $\tau_0 > 0$. Let $\tau_0 < \tau' < \tau'' < \infty$. By (4.22) we may write

$$\int_{D^k_{0,\tau'}} |\nabla f|^p d\mathcal{H}^n \le C_8 \left[\int_{\tau'}^{\tau''} d\tau \left(\int_{\sigma^k(\tau)} |f(x) - C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \tag{4.38}$$

where $C_8 = C_7/2$. As in (3.14), we obtain from (2.4) the estimate

$$\int_{D^k_{0,\tau_0}} |\nabla f|^p d\mathcal{H}^n \le \int_{D^k_{0,\tau'}} |\nabla f|^p d\mathcal{H}^n \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau_0}^{\tau'} \mu_p\left(\sigma^k(\tau)\right)d\tau\right]. \tag{4.39}$$

By combining these inequalities, we obtain

$$\begin{aligned}\int_{D^k_{0,\tau_0}} |\nabla f|^p d\mathcal{H}^n \le C_8 &\left[\int_{\tau'}^{\tau''} d\tau\left(\int_{\sigma^k(\tau)} |f(x)-C|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \\ &\times \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau_0}^{\tau'} \mu_p\left(\sigma^k(\tau)\right)d\tau\right].\end{aligned} \tag{4.40}$$

The inequality (4.40) holds for arbitrary constant C and every $\tau'' > \tau'$. Thus the following statement holds.

Theorem 4.4. *Let $f : D \to \mathbb{R}$ be a generalized solution of* (1.17) *with* (1.15) *and* (1.16) *satisfying the boundary condition* (1.21) *on ∂D, $1 \le k < n-1$. If for a constant C the right side of* (4.40) *tends to* 0 *as $\tau', \tau'' \to +\infty$, then $f \equiv$ const on D.*

If f satisfies (1.17) with (1.15), (1.16) and the boundary condition (1.24) on ∂D, then we have

$$\begin{aligned}\int_{D^k_{0,\tau_0}} |\nabla f|^p d\mathcal{H}^n \le C_8 &\left[\int_{\tau'}^{\tau''} d\tau\left(\int_{\sigma^k(\tau)} |f(x)|^p d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p} \\ &\times \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau_0}^{\tau'} \lambda^{1/p}_{p,Z_f(\tau)}\left(\sigma^k(\tau)\right)d\tau\right].\end{aligned} \tag{4.41}$$

We obtain the following.

Theorem 4.5. *Fix a domain D_0 in $\mathbb{R}^k$, where $1 \le k < n-1$, with compact and smooth boundary. Let*

$$D = D_0 \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : (x_1, \dots, x_k) \in D_0\}, \tag{4.42}$$

and let $f : D \to \mathbb{R}$ be a generalized solution of (1.17) *with* (1.15) *and* (1.16) *satisfying the boundary condition* (1.24) *on ∂D. If for a constant C the right side of* (4.41) *tends to* 0 *as $\tau', \tau'' \to +\infty$, then $f \equiv 0$ on D.*

References

[1] V. M. Miklyukov, Ed., *Proceedings of the Seminar 'Superslow Processes'*, Issue 1, Volgograd State University, Volgograd, Russia, 2006.
[2] V. M. Miklyukov, Ed., *Proceedings of the Seminar 'Superslow Processes'*, Issue 2, Volgograd State University, Volgograd, Russia, 2007.
[3] V. M. Miklyukov, "Stagnation zones of A-solutions, Memory I.N. Vekua," *Georgian Mathematical Journal*, vol. 14, no. 3, pp. 519–531, 2007.
[4] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford, UK, 1993.
[5] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, NJ, USA, 1951.
[6] P. D. Lax, "A Phragmén-Lindelöf theorem in harmonic analysis and its application to some questions in the theory of elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 10, pp. 361–389, 1957.
[7] H. Federer, *Geometric Measure Theory*, vol. 153 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, New York, NY, USA, 1969.
[8] V. M. Miklyukov, *Introduction to Nonsmooth Analysis*, Izd-vo VolGU, Volgograd, Russia, 2nd edition, 2008.
[9] C. Barbarosie and A.-M. Toader, "Saint-Venant's principle and its connections to shape and topology optimization," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 88, no. 1, pp. 23–32, 2008.
[10] O. A. Oleĭnik and G. A. Yosifian, "Boundary value problems for second order elliptic equations in unbounded domains and Saint-Venant's principle," *Annali della Scuola Normale Superiore, Classe di Scienze*, vol. 4, no. 4, pp. 269–290, 1977.
[11] V. G. Maz'ya, *Sobolev Spaces*, Springer Series in Soviet Mathematics, Springer, Berlin, Germany, 1985.
[12] R. Adams and J. Fournier, *Sobolev Spaces*, vol. 140 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 2nd edition, 2003.