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Research Article

Multiplicity of Positive and Nodal Solutions for Nonhomogeneous Elliptic Problems in Unbounded Cylinder Domains

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We show that if $a(x)$ and $f(x)$ satisfy some suitable conditions, then the Dirichlet problem $-\Delta u + u = a(x)|u|^{p-2}u + f(x)$ in Ω has a solution that changes sign in Ω, in addition to two positive solutions where Ω is an unbounded cylinder domain in $\mathbb{R}^N$.

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1. Introduction

Throughout this paper, let $x = (y, z)$ be the generic point of $\mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, where

$$N = m + n \geq 3, \quad m \geq 2, \; n \geq 1, \; 2 < p < \frac{2N}{N-2}. \tag{1.1}$$

In this paper, we study the multiplicity results of both positive and nodal solutions for the nonhomogeneous elliptic problems

$$-\Delta u + u = a(x)|u|^{p-2}u + f(x) \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{1.2}$$

where $0 \in \omega \subseteq \mathbb{R}^m$ is a bounded smooth domain, $\Omega = \omega \times \mathbb{R}^n$ is a smooth unbounded cylinder domain in $\mathbb{R}^N$.

It is assumed that $a(x)$ and $f(x)$ satisfy the following assumptions:

$(a1)$ $a(x)$ is continuous and $a(x) \in (0,1]$ on $\overline{\Omega}$, and

$$\lim_{|z|\to\infty} a(x) = 1 \quad \text{uniformly for } y \in \overline{\omega}; \tag{1.3}$$

$(f1)$ $f(x) \geq 0, f(x) \not\equiv 0, f(x) \in H^{-1}(\Omega)$;

$(f2)$ $\gamma_f > 0$ in which we defined

$$\begin{aligned}\gamma_f = \inf\Bigg\{ &\left[\frac{1}{p-1}\right]^{(p-1)/(p-2)} (p-2)\|u\|^{2(p-1)/(p-2)} \\ &-\int_\Omega f u dx : \int_\Omega a(x)|u|^p dx = 1\Bigg\};\end{aligned} \tag{1.4}$$

$(f3)$ there exist positive constants C_0, ϵ_0, R_0 such that

$$f(x) \leq C_0 \exp\left(-\sqrt{1+\mu_1+\epsilon_0}|z|\right) \quad \text{for } |z| \geq R_0, \text{uniformly for } y \in \overline{\omega}, \tag{1.5}$$

where μ_1 is the first positive eigenvalue of the Dirichlet problem $-\Delta$ in ω.

For the homogeneous case, that is, $f(x) \equiv 0$, Zhu [1] has established the existence of a positive solution and a nodal solution of problem (1.2) in $H^1(\mathbb{R}^N)$ provided $a(x)$ satisfies $a(x) \geq 1$ in $\mathbb{R}^N$ and $a(x) - 1 \geq C/|x|^l$ as $|x| \to \infty$ for some positive constants C and l. More recently, Hsu [2] extended the results of Zhu [1] with $\mathbb{R}^N$ to an unbounded cylinder Ω. Let us recall that, by a nodal solution we mean the solution of problem (1.2) with change of sign.

For the nonhomogeneous case ($f(x) \not\equiv 0$), Adachi and Tanaka [3] have showed that problem (1.2) has at least four positive solutions in $H^1(\mathbb{R}^N)$ for $a(x)$ and $f(x)$ satisfy some suitable conditions, but we place particular emphasis on the existence of nodal solutions. More recently, Chen [4] considered the multiplicity results of both positive and nodal solutions of problem (1.2) in $H^1(\mathbb{R}^N)$. She has showed that problem (1.2) has at least two positive solutions and one nodal solution in $H^1(\mathbb{R}^N)$ when $a(x)$ and $f(x)$ satisfy some suitable assumptions.

In the present paper, motivated by [4] we extend and improve the paper by Chen [4]. We will deal with unbounded cylinder domains instead of the entire space and also obtain the same results as in [4]. Our arguments are similar to those in [5, 6], which are based on Ekeland's variational principle [7].

Now, we state our main results.

Theorem 1.1. *Assume $(a1), (f1), (f2)$ hold and $a(x)$ satisfies assumption $(a2)$.*

(a2) *there exist positive constants C, δ_0, R such that*

$$a(x) \geq 1 - C\ \exp\left(-\sqrt{1+\mu_1+\delta_0}|z|\right) \quad \text{for } |z| \geq R, \quad \text{uniformly for } y \in \overline{\omega}. \tag{1.6}$$

Then problem (1.2) *has at least two positive solutions* u_0 *and* u_1 *in* $H_0^1(\Omega)$. *Furthermore,* u_0 *and* u_1 *satisfy* $0 < u_0 < u_1$, *and* u_0 *is a local minimizer of* I *where* I *is the energy functional of problem* (1.2).

Theorem 1.2. *Assume* $(a1), (f1), (f2), (f3)$ *hold and* $a(x)$ *satisfies assumption* $(a3)$.

(a3) *there exist positive constants* $\overline{C}, \overline{R}$, *and* $\overline{\delta}_0 < 1 + \mu_1$ *such that*

$$a(x) \geq 1 + \overline{C} \exp\left(-\sqrt{1 + \mu_1 - \overline{\delta}_0}|z|\right) \quad \text{for } |z| \geq \overline{R}, \quad \text{uniformly for } y \in \overline{\omega}. \tag{1.7}$$

Then problem (1.2) *has a nodal solution in* $H_0^1(\Omega)$ *in addition to two positive solutions* u_0 *and* u_1.

For the case $\Omega = \mathbb{R}^N$, we also have obtained the same results as in Theorems 1.1 and 1.2.

Theorem 1.3. *Assume* $(a1), (f1), (f2)$ *hold and* $a(x)$ *satisfies assumption* $(a2)$.

(a2) *there exist positive constants* C, δ_0, R *such that*

$$a(x) \geq 1 - C \exp\left(-\sqrt{1 + \delta_0}|x|\right) \quad \text{for } |x| \geq R. \tag{1.8}$$

Then problem (1.2) *has at least two positive solutions* u_0 *and* u_1 *in* $H^1(\mathbb{R}^N)$. *Furthermore,* u_0 *and* u_1 *satisfy* $0 < u_0 < u_1$, *and* u_0 *is a local minimizer of* I *where* I *is the energy functional of problem* (1.2).

Theorem 1.4. *Assume* $(a1), (f1), (f2), (f3)$ *hold and* $a(x)$ *satisfies assumption* $(a3)$ *below.*

(a3) *there exist positive constants* $\overline{C}, \overline{R}$ *and* $\overline{\delta}_0 < 1$ *such that*

$$a(x) \geq 1 + \overline{C} \exp\left(-\sqrt{1 - \overline{\delta}_0}|x|\right) \quad \text{for } |x| \geq \overline{R}. \tag{1.9}$$

Then problem (1.2) *has a nodal solution in* $H^1(\mathbb{R}^N)$ *in addition to two positive solutions* u_0 *and* u_1.

Among the other interesting problems which are similar to problem (1.2), Bahri and Berestycki [8] and Struwe [9] have investigated the following equation:

$$-\Delta u = |u|^{p-2}u + f(x) \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{1.10}$$

where $2 < p < 2N/(N-2)$, $f \in L^2(\Omega)$, and Ω is a bounded domain in $\mathbb{R}^N$. They found that (1.10) possesses infinitely many solutions. More recently, Tarantello [5] proved that if $p = 2N/(N-2)$ is the critical Sobolev exponent and $f \in H^{-1}$ satisfying suitable conditions, then (1.10) admits two solutions. For the case when Ω is an unbounded domain, Cao and Zhou [10], Cîrstea and Rădulescu [11], and Ghergu and Rădulescu [12] have been investigated the analogue equation (1.10) involving a subcritical exponent in $\mathbb{R}^N$. Furthermore, Rădulescu and Smets [13] proved existence results for nonautonomous perturbations of critical singular elliptic boundary value problems on infinite cones.

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we will prove Theorem 1.1. In Section 4, we establish the existence of nodal solutions.

2. Preliminaries

In this paper, we always assume that Ω is an unbounded cylinder domain or $\mathbb{R}^N (N \geq 3)$. Let $\Omega_R = \{x \in \Omega : |z| < R\}$ for $R > 0$, and let ϕ be the first positive eigenfunction of the Dirichlet problem $-\Delta$ in ω with eigenvalue μ_1, unless otherwise specified. We denote by C and C_i $(i = 1, 2, \ldots)$ universal constants, maybe the constants here should be allowed to depend on N and p, unless some statement is given. Now we begin our discussion by giving some definitions and some known results.

We define

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right) dx\right)^{1/2},$$
$$\|u\|_q = \left(\int_\Omega |u|^q dx\right)^{1/q}, \quad 1 \leq q < \infty, \tag{2.1}$$
$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|.$$

Let $H_0^1(\Omega)$ be the Sobolev space of the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$ with the dual space $H^{-1}(\Omega)$, $H^1(\mathbb{R}^N) = H_0^1(\mathbb{R}^N)$ and denote $\langle \cdot, \cdot \rangle$ the usual scalar product in $H_0^1(\Omega)$. The energy functional of problem (1.2) is given by

$$I(u) = \frac{1}{2}\int \left(|\nabla u|^2 + u^2\right) - \frac{1}{p}\int a(x)|u|^p - \int fu, \tag{2.2}$$

here and from now on, we omit "dx" and "Ω" in all the integration if there is no other indication. It is well known that I is of C^1 in $H_0^1(\Omega)$ and the solutions of problem (1.2) are the critical points of the energy functional I (see Rabinowitz [14]).

As the energy functional I is not bounded on $H_0^1(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N} = \left\{u \in H_0^1(\Omega) \setminus \{0\} : \langle I'(u), u \rangle = 0\right\}. \tag{2.3}$$

Thus, $u \in \mathcal{N}$ if and only if

$$\langle I'(u), u \rangle = \|u\|^2 - \int a(x)|u|^p - \int fu = 0. \tag{2.4}$$

Easy computation shows that I is bounded from below in the set $\mathcal{N}$. Note that $\mathcal{N}$ contains every nonzero solution of (1.2).

Similarly to the method used in Tarantello [5], we split $\mathcal{N}$ into three parts:

$$\mathcal{N}^{+} = \left\{ u \in \mathcal{N} : \|u\|^2 - (p-1) \int a(x)|u|^p > 0 \right\},$$
$$\mathcal{N}^{0} = \left\{ u \in \mathcal{N} : \|u\|^2 - (p-1) \int a(x)|u|^p = 0 \right\}, \tag{2.5}$$
$$\mathcal{N}^{-} = \left\{ u \in \mathcal{N} : \|u\|^2 - (p-1) \int a(x)|u|^p < 0 \right\}.$$

Let us introduce the problem at infinity associated with problem (1.2) as

$$-\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \quad u \in H_0^1(\Omega), u > 0 \text{ in } \Omega. \tag{2.6}$$

We state here some known results for problem (2.6). First of all, we recall that by Esteban [15] and Lien et al. [16], problem (2.6) has a ground state solution w such that

$$S^{\infty} = I^{\infty}(w) = \sup_{t \geq 0} I^{\infty}(tw) = \left(\frac{1}{2} - \frac{1}{p}\right) S^{p/(p-2)}, \tag{2.7}$$

where $I^{\infty}(u) = (1/2)\|u\|^2 - (1/p)\int |u|^p$, $S^{\infty} = \inf\{I^{\infty}(u) : u \in H_0^1(\Omega), u \not\equiv 0, (I^{\infty})'(u) = 0\}$ and

$$S = \inf\left\{ \int \left(|\nabla u|^2 + u^2\right) : u \in H_0^1(\Omega), \int |u|^p = 1 \right\}. \tag{2.8}$$

Furthermore, from Hsu [2] we can deduce that for any $\epsilon \in (0, 1+\mu_1)$ there exist positive constants $C_\epsilon, \widetilde{C_\epsilon}$ such that, for all $x = (y, z) \in \Omega$,

$$\widetilde{C_\epsilon} \phi(y) \exp\left(-\sqrt{1 + \mu_1 + \epsilon}|z|\right) \leq w(x) \leq C_\epsilon \phi(y) \exp\left(-\sqrt{1 + \mu_1 - \epsilon}|z|\right). \tag{2.9}$$

We also quote the following lemma (see Hsu [17] or K.-J. Chen et al. [18] for the proof) about the decay of positive solution of problem (1.2) which we will use later.

Lemma 2.1. *Assume* $(a1)$, $(f1)$ *and* $(f3)$ *hold. If* $u \in H_0^1(\Omega)$ *is a positive solution of problem* (1.2)*, then*

(i) $u \in L^q(\Omega)$ *for all* $q \in [2, \infty)$;

(ii) $u(y, z) \to 0$ *as* $|z| \to 0$ *uniformly for* $y \in \omega$ *and* $u \in C^{1,\alpha}(\overline{\Omega})$ *for any* $0 < \alpha < 1$;

(iii) *for any* $\epsilon \in (0, 1+\mu_1)$, *there exist positive constants* $c_\epsilon, \widetilde{c_\epsilon}$ *such that, for all* $x = (y, z) \in \Omega$,

$$\widetilde{c_\epsilon} \phi(y) \exp\left(-\sqrt{1 + \mu_1 + \epsilon}|z|\right) \leq u(x) \leq c_\epsilon \phi(y) \exp\left(-\sqrt{1 + \mu_1 + \epsilon}|z|\right). \tag{2.10}$$

We end this preliminaries by the following definition.

Definition 2.2. Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E, \mathbb{R})$.

(i) $\{u_n\}$ is a $(PS)_c$-sequence in E for I if $I(u_n) = c + o(1)$ and $I'(u_n) = o(1)$ strongly in E^{-1} as $n \to \infty$.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{u_n\}$ in E for I has a convergent subsequence.

3. Proof of Theorem 1.1

In this section, we will establish the existence of two positive solutions of problem (1.2).

First, we quote some lemmas for later use (see the proof of Tarantello [5] or Chen [4, Lemmas 2.2, 2.3, and 2.4]).

Lemma 3.1. *Assume* $(a1)$ *and* $(f1)$ *hold, then for every* $u \in H_0^1(\Omega), u \not\equiv 0$, *there exists a unique* $t^- = t^-(u) > 0$ *such that* $t^- u \in \mathcal{N}^-$. *In particular, we have*

$$t^- > \left(\frac{\|u\|^2}{(p-1)\int a(x)|u|^p} \right)^{1/(p-2)} = t_{\max} \tag{3.1}$$

and $I(t^- u) = \max_{t \geq t_{\max}} I(tu)$. *Moreover, if* $\int fu > 0$, *then there exists a unique* $t^+ = t^+(u) > 0$ *such that* $t^+ u \in \mathcal{N}^+$. *In particular,*

$$t^+ < t_{\max}, \tag{3.2}$$

$I(t^+ u) = \min_{0 \leq t \leq t_{\max}} I(tu)$ *and* $I(t^- u) = \max_{t \geq 0} I(tu)$.

Lemma 3.2. *Assume* $(a1), (f1)$ *and* $(f2)$ *hold, then for every* $u \in \mathcal{N} \setminus \{0\}$, *we have*

$$\|u\|^2 - (p-1) \int a(x)|u|^p \neq 0 \quad (\text{i.e., } \mathcal{N}_0 = \{0\}). \tag{3.3}$$

Lemma 3.3. *Assume* $(a1), (f1)$ *and* $(f2)$ *hold, then for every* $u \in \mathcal{N} \setminus \{0\}$, *there exist a* $\epsilon > 0$ *and a* C^1*-map* $t = t(w) > 0, w \in H_0^1(\Omega), \|w\| < \epsilon$ *satisfying that*

$$\begin{gathered} t(0) = 1, \quad t(w)(u-w) \in \mathcal{N}, \quad \text{for } \|w\| < \epsilon, \\ \langle t'(0), w \rangle = \frac{2\int (\nabla u \nabla w + uw) - p\int a(x)|u|^{p-2}uw - \int fw}{\|u\|^2 - (p-1)\int a(x)|u|^p}. \end{gathered} \tag{3.4}$$

Apply Lemmas 3.1, 3.2, 3.3, and Ekeland variational principle [7], and we can establish the existence of the first positive solution.

Proposition 3.4. *Assume* $(a1), (f1)$ *and* $(f2)$ *hold, then the minimization problem* $c_0 = \inf_{\mathcal{N}} I = \inf_{\mathcal{N}^+} I$ *is achieved at a point* $u_0 \in \mathcal{N}^+$ *which is a critical point for* I. *Moreover, if* $f(x) \geq 0$ *and* $f(x) \not\equiv 0$, *then* u_0 *is a positive solution of problem* (1.2) *and* u_0 *is a local minimizer of* I.

Proof. Modifying the proof of Chen [4, Proposition 2.5]. Here we omit it. □

Since $u_0 \in \mathcal{N}^+$ and $c_0 = \inf_{\mathcal{N}} I = \inf_{\mathcal{N}^+} I$, thus, in the search of our second positive solution, it is natural to consider the second minimization problem:

$$c_1 = \inf_{\mathcal{N}^-} I. \tag{3.5}$$

We will establish the existence of the second positive solution of problem (1.2) by proving that I satisfies the $(PS)_{c_1}$-condition.

Proposition 3.5. *Assume $(a1)$, $(f1)$ and $(f2)$ hold, then I satisfies the $(PS)_c$-condition with $c \in (-\infty, c_0 + S^\infty)$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$-sequence for I with $c \in (-\infty, c_0 + S^\infty)$. It is easy to see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, so we can find a $\overline{u} \in H_0^1(\Omega)$ such that $u_n \rightharpoonup \overline{u}$ weakly in $H_0^1(\Omega)$ up to a subsequence and $\overline{u}$ is a critical point of I. Furthermore, we may assume $u_n \to \overline{u}$ a.e. in Ω, $u_n \to \overline{u}$ strongly in $L^s_{\text{loc}}(\Omega)$ for all $1 \leq s < 2N/(N-2)$. Hence we have that $I'(\overline{u}) = 0$ and

$$\int f u_n = \int f \overline{u} + o(1). \tag{3.6}$$

Set $v_n = u_n - \overline{u}$. Then by (3.6) and Brézis and Lieb lemma (see [19]), we obtain

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\int a(x)|u_n|^p - \int f u_n \\ &= I(\overline{u}) + \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int a(x)|v_n|^p + o(1). \end{aligned} \tag{3.7}$$

Moreover, by Vitali's lemma and $I'(\overline{u}) = 0$,

$$\begin{aligned} o(1) &= \langle I'(u_n), u_n\rangle \\ &= \|\overline{u}\|^2 - \int a(x)|\overline{u}|^p - \int f\overline{u} + \|v_n\|^2 - \int a(x)|v_n|^p + o(1) \\ &= \langle I'(\overline{u}), \overline{u}\rangle + \|v_n\|^2 - \int a(x)|v_n|^p + o(1) \\ &= \|v_n\|^2 - \int a(x)|v_n|^p + o(1). \end{aligned} \tag{3.8}$$

In view of assumptions $I(u_n) = c + o(1)$, and (3.7), (3.8), $\overline{u} \in \mathcal{N}$ and by Lemma 3.2, we obtain

$$c \geq c_0 + \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int a(x)|v_n|^p + o(1), \tag{3.9}$$

$$\|v_n\|^2 - \int a(x)|v_n|^p = o(1). \tag{3.10}$$

Hence, we may assume that

$$\|v_n\|^2 \longrightarrow b, \qquad \int a(x)|v_n|^p \longrightarrow b. \tag{3.11}$$

By the definition of S, we have $\|v_n\|^2 \geq S\|v_n\|_p^2$, combining with (3.11) and $\|a\|_\infty = 1$, and we get that $b \geq Sb^{2/p}$. Either $b = 0$ or $b \geq S^{p/(p-2)}$. If $b = 0$, the proof is complete. Assume that $b \geq S^{p/(p-2)}$, from (2.7), (3.9), and (3.11), we get

$$c \geq c_0 + \left(\frac{1}{2} - \frac{1}{p}\right)b \geq c_0 + \left(\frac{1}{2} - \frac{1}{p}\right)S^{p/(p-2)} \geq c_0 + S^\infty, \tag{3.12}$$

which is a contradiction. Therefore, $b = 0$ and we conclude that $u_n \rightarrow \overline{u}$ strongly in $H_0^1(\Omega)$. □

Let $e_N = (0, 0, \ldots, 0, 1) \in \mathbb{R}^N$, let $e_n = (0, 0, \ldots, 0, 1) \in \mathbb{R}^n$, and let $k > 0$ be a constant, we denote $w_k(x) = w(x - ke_N)$ and $u_k(x) = u_0(x + ke_N)$ for $x \in \Omega$ where w is the ground state solution of problem (2.6) and u_0 is the first positive solution of problem (1.2).

Proposition 3.6. *Assume $(a1)$, $(a2)$ and $(f1)$ hold, then there exists $k_0 \geq 1$ such that*

$$I(u_0 + tw_{k_0}) < c_0 + S^\infty, \quad \forall\, t > 0. \tag{3.13}$$

The following estimates are important to find a path which lies below the first level of the break down of the $(PS)_c$ condition. Here we use an interaction phenomenon between u_0 and w_{k_0}.

To give a proof of Proposition 3.6, we need to establish some lemmas.

Lemma 3.7. *Let $B_1 = \{x = (y, z) \in \Omega : y \in \omega_0, |z| \leq 1\}$, and $\omega_0 \subset\subset \omega$ is a domain in $\mathbb{R}^m$. Then for any $\epsilon \in (0, 1 + \mu_1)$, there exists a positive constant $C_1(\epsilon)$ such that*

$$\int_{B_1} u_k(x) \geq C_1 e^{-\sqrt{1+\mu_1+\epsilon}k}, \quad \forall\, k \geq 1. \tag{3.14}$$

Proof. From (2.10), we have for $k \geq 1$,

$$\begin{aligned}\int_{B_1} u_k(x) &= \int_{B_1} u(x+ke_N)\\ &\geq \int_{B_1} \tilde{c}_\epsilon \phi(y) e^{-\sqrt{1+\mu_1+\epsilon}|z+ke_N|}\\ &\geq \tilde{c}_\epsilon e^{-\sqrt{1+\mu_1+\epsilon}(k+1)} \int_{B_1} \phi(y)\\ &\geq C_1 e^{-\sqrt{1+\mu_1+\epsilon}k}.\end{aligned} \tag{3.15}$$

□

Lemma 3.8. *Let Θ be a domain in $\mathbb{R}^n$, and let $z = (z_1, z_2, \ldots, z_n)$ be a vector in $\mathbb{R}^n$. If $g : \Theta \to \mathbb{R}$ satisfies*

$$\int_\Theta \left| g(z) e^{\sigma|z|} \right| dz < \infty \text{ for some } \sigma > 0, \tag{3.16}$$

then

$$\left(\int_\Theta g(z) e^{-\sigma|z+ke_n|} dz\right) e^{\sigma k} = \int_\Theta g(z) e^{-\sigma z_n} dz + o(1) \quad \text{as } k \longrightarrow \infty, \tag{3.17}$$

or

$$\left(\int_\Theta g(z) e^{-\sigma|z-ke_n|} dz\right) e^{\sigma k} = \int_\Theta g(z) e^{\sigma z_n} dz + o(1) \quad \text{as } k \longrightarrow \infty. \tag{3.18}$$

Proof. We know $\sigma|ke_n| \leq \sigma|z| + \sigma|z+ke_n|$, then

$$\left| g(z) e^{-\sigma|z+ke_n|} e^{\sigma|ke_n|} \right| \leq \left| g(z) e^{\sigma|z|} \right|. \tag{3.19}$$

Since $-\sigma|z+ke_n| + \sigma|ke_n| = -\sigma(\langle z, ke_n\rangle / |ke_n|) + o(1) = -\sigma z_n + o(1)$ as $k \to \infty$, the lemma follows from the Lebesgue's dominated convergence theorem. □

Now, we give the proof of Proposition 3.6.

The Proof of Proposition 3.6

Recall $B_1 = \{x = (y, z) \in \Omega \mid y \in \omega_0, |z| \leq 1\}$, where $\omega_0 \subset\subset \omega$ is a domain in $\mathbb{R}^m$. For $k \geq 1$, let

$$\begin{aligned}D_k &= \{x \in \Omega : x - ke_N \in B_1\},\\ r &= \min_{x \in D_k} w_k(x) = \min_{x \in B_1} w(x) > 0.\end{aligned} \tag{3.20}$$

We also remark that for all $s > 0,\ t > 0$,

$$(s+t)^p - s^p - t^p - ps^{p-1}t \geq 0, \tag{3.21}$$

and for any $s_0 > 0$ and $r_0 > 0$ there exists $C_2(s_0, r_0) > 0$ such that for all $s \in [0, r_0],\ t \in [s_0, r_0]$,

$$(s+t)^p - s^p - t^p - ps^{p-1}t \geq C_2(s_0, r_0)st. \tag{3.22}$$

Since I is continuous in $H_0^1(\Omega)$, there exists $t_1 > 0$ such that for all $t \in [0, t_1]$,

$$I(u_0 + tw_k) < I(u_0) + I^{\infty}(w), \quad \forall\ k \geq 0, \tag{3.23}$$

and by the fact that $I(u_0 + tw_k) \to -\infty$ as $t \to \infty$ uniformly in $k \geq 1$, then there exists $t_0 > 0$ such that

$$\sup_{t \geq 0} I(u_0 + tw_k) = \sup_{0 \leq t \leq t_0} I(u_0 + tw_k). \tag{3.24}$$

Thus, we only need to show that there exists a constant $k_0 \geq 1$ such that

$$\sup_{t_1 \leq t \leq t_0} I(u_0 + tw_k) < I(u_0) + I^{\infty}(w), \quad \forall\ k \geq k_0. \tag{3.25}$$

Straightforward computation gives us

$$\begin{aligned}
I(u_0 + tw_k) &= \frac{t^2}{2}\|u_0\|^2 + \frac{t^2}{2}\|w_k\|^2 + \langle u_0, tw_k\rangle - \frac{1}{p}\int a(x)|u_0 + tw_k|^p \\
&\quad - \int f u_0 - t\int f w_k \\
&= I(u_0) + I^{\infty}(tw_k) \\
&\quad - \frac{1}{p}\int \left(a(x)|u_0 + tw_k|^p - a(x)|u_0|^p - a^{\infty}|tw_k|^p\right) + t\int a(x)|u_0|^{p-1}w_k \\
&= I(u_0) + I^{\infty}(tw) \\
&\quad - \frac{1}{p}\int a(x)\Big(|u_0 + tw_k|^p - |u_0|^p - |tw_k|^p - p|u_0|^{p-1}tw_k\Big) \\
&\quad + \frac{1}{p}\int \left(a^{\infty}|tw_k|^p - a(x)|tw_k|^p\right) \\
&\leq c_0 + S^{\infty} - (I) + (II),
\end{aligned} \tag{3.26}$$

where

$$(I) = \frac{1}{p}\int a(x)\Big(|u_0 + tw_k|^p - |u_0|^p - |tw_k|^p - p|u_0|^{p-1}tw_k\Big),$$
$$(II) = \frac{1}{p}\int (a^\infty - a(x))|tw_k|^p. \tag{3.27}$$

Thus, we only need to prove that there exists a constant $k_0 \geq 1$ such that

$$-(I) + (II) < 0, \quad \forall\, t \in [t_1, t_0]. \tag{3.28}$$

Now we estimate (I) and (II). Without loss of generality, we may assume that $\delta_0 < (p^2 - 1)(1 + \mu_1)$. Thus, we can choose $\widetilde{\epsilon}_0$ small enough such that

$$p\sqrt{1 + \mu_1 - \widetilde{\epsilon}_0} > \sqrt{1 + \mu_1 + \delta_0}. \tag{3.29}$$

By (3.21),

$$(I) = \frac{1}{p}\int a(x)\Big(|u_0 + tw_k|^p - |u_0|^p - |tw_k|^p - p|u_0|^{p-1}tw_k\Big)$$
$$\geq \frac{1}{p}\int_{D_k} a(x)\Big(|u_0 + tw_k|^p - |u_0|^p - |tw_k|^p - p|u_0|^{p-1}tw_k\Big). \tag{3.30}$$

Let $a_0 = \inf_{x\in\Omega} a(x) > 0$, $s_0 = t_1 \min_{x\in D_k} w_k(x)$, $r_0 = \max\{\max_{x\in\Omega} u_0(x), t_0 \max_{x\in\Omega} w(x)\} > 0$ and by applying (3.22), we obtain

$$(I) \geq \frac{a_0}{p}\int_{D_k} C_2(s_0, r_0) t u_0 w_k$$
$$\geq \frac{a_0}{p} C_2(s_0, r_0) t_1 \int_{x\in B_1} u_k w \quad \forall\, t \in [t_1, t_0]. \tag{3.31}$$

Let $\epsilon = \delta_0/2$. Then applying (3.14), we have for $A = (a_0/p)C_1(\delta_0/2)C_2(s_0, r_0)t_1(\min_{x\in B_1} w(x))$

$$(I) \geq Ae^{-\sqrt{1+\mu_1+(\delta_0/2)}k}. \tag{3.32}$$

Next from $(a2)$, (2.9), (3.29), and Lemma 3.8, there exists a k_1 such that for any $k \geq k_1$,

$$\begin{aligned}
(II) &= \frac{1}{p}\int (a^\infty - a(x))|tw_k|^p \\
&= \frac{1}{p}\int_{\Omega_R} (a^\infty - a(x))|tw_k|^p + \frac{1}{p}\int_{\Omega\setminus\Omega_R} (a^\infty - a(x))|tw_k|^p \\
&\leq \frac{t_0^p}{p}(a^\infty + \|a\|_\infty)\int_{\Omega_R} C_{\tilde{\varepsilon}_0}^p \phi^p(y) e^{-p\sqrt{1+\mu_1-\tilde{\varepsilon}_0}|z-ke_n|} \\
&\quad + \frac{t_0^p}{p}\int_{\Omega\setminus\Omega_R} CC_{\tilde{\varepsilon}_0}^p \phi^p(y) e^{-\sqrt{1+\mu_1+\delta_0}|z|} e^{-p\sqrt{1+\mu_1-\tilde{\varepsilon}_0}|z-ke_n|} \\
&\leq C_3 e^{-p\sqrt{1+\mu_1-\tilde{\varepsilon}_0}k} + \frac{t_0^p}{p} CC_{\tilde{\varepsilon}_0}^p \int_\omega \phi^p(y)dy \int_{\mathbb{R}^n} e^{-\sqrt{1+\mu_1+\delta_0}|z+ke_n|} e^{-p\sqrt{1+\mu_1-\tilde{\varepsilon}_0}|z|} dz \\
&\leq C_3 e^{-p\sqrt{1+\mu_1-\tilde{\varepsilon}_0}k} + C_4 e^{-\sqrt{1+\mu_1+\delta_0}k}.
\end{aligned} \tag{3.33}$$

From (3.29), we have for $B = 2\max\{C_3, C_4\}$,

$$(II) \leq Be^{-\sqrt{1+\mu_1+\delta_0}k}. \tag{3.34}$$

Finally, we can choose $k_0 \geq k_1$ large enough such that

$$Be^{-\sqrt{1+\mu_1+\delta_0}k} < Ae^{-\sqrt{1+\mu_1+(\delta_0/2)}k}, \quad \forall\, k \geq k_0. \tag{3.35}$$

Thus from (3.26) and (3.32)–(3.35), we obtain (3.13). This completes the proof of Proposition 3.6.

Proposition 3.9. *For $c_1 = \inf_{\mathcal{N}^-} I$, there exists a $(PS)_{c_1}$-sequence $\{u_n\} \subset \mathcal{N}^-$ for I. In particular, we have $c_1 < c_0 + S^\infty$.*

Proof. Set $\Sigma = \{u \in H_0^1(\Omega) : \|u\| = 1\}$ and define the map $\Psi : \Sigma \to \mathcal{N}^-$ given by $\Psi(u) = t^-(u)u$. Since the continuity of $t^-(u)$ follows immediately from its uniqueness and extremal property, thus Ψ is continuous with continuous inverse given by $\Psi^{-1}(u) = u/\|u\|$. Clearly $\mathcal{N}^-$ disconnecting $H_0^1(\Omega)$ is exactly two components:

$$\begin{aligned}
U_1 &= \left\{u = 0 \text{ or } u : \|u\| < t^-\left(\frac{u}{\|u\|}\right)\right\}, \\
U_2 &= \left\{u : \|u\| > t^-\left(\frac{u}{\|u\|}\right)\right\},
\end{aligned} \tag{3.36}$$

and $\mathcal{N}^+ \subset U_1$.

We will prove that there exists t_0 such that $u_0 + t_0 w_{k_0} \in U_2$. Denote $t_1 = t^-((u_0 + tw_{k_0})/\|u_0 + tw_{k_0}\|)$. Since $t^-((u_0 + tw_{k_0})/\|u_0 + tw_{k_0}\|)((u_0 + tw_{k_0})/\|u_0 + tw_{k_0}\|) \in \mathcal{N}^-$, we have

$$t_1^2 - \frac{t_1^p \int a(x)|u_0 + tw_{k_0}|^p}{\|u_0 + tw_{k_0}\|^p} = \frac{t_1}{\|u_0 + tw_{k_0}\|}\int f(u_0 + tw_{k_0}) \geq 0. \tag{3.37}$$

Thus

$$\begin{aligned} t_1 &\leq \left[\frac{\|u_0 + tw_{k_0}\|}{\left(\int a(x)|u_0 + tw_{k_0}|^p\right)^{1/p}}\right]^{p/(p-2)} = \left[\frac{\|(u_0/t) + w_{k_0}\|}{\left(\int a(x)|(u_0/t) + w_{k_0}|^p\right)^{1/p}}\right]^{p/(p-2)} \\ &\leq \left[\frac{\|(u_0/t) + w_{k_0}\|}{\left(\int a_0|(u_0/t) + w_{k_0}|^p\right)^{1/p}}\right]^{p/(p-2)} \quad \text{where } a_0 = \inf_\Omega a(x) > 0 \\ &\longrightarrow a_0^{1/p-2}\|w_{k_0}\| < \infty \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{3.38}$$

Therefore, there exists $t_2 > 0$ such that $t_1 = t^-((u_0 + tw_{k_0})/\|u_0 + tw_{k_0}\|) < \|w_{k_0}\|$, for $t \geq t_2$. Since $t_0 > t_2 + 1$, then

$$\begin{aligned} \|u_0 + t_0 w_{k_0}\|^2 &= \|u_0\|^2 + t_0^2\|w_{k_0}\|^2 + 2t_0\int(\nabla u_0 \nabla w_{k_0} + u_0 w_{k_0}) \\ &= \|u_0\|^2 + t_0^2\|w_{k_0}\|^2 + 2t_0\int |w_{k_0}|^{p-1}u_0 \\ &> t_0^2\|w_{k_0}\|^2 > \|w_{k_0}\|^2 > t_1^2, \end{aligned} \tag{3.39}$$

hence $u_0 + t_0 w_{k_0} \in U_2$.

$\mathcal{N}^-$ disconnects $H_0^1(\Omega)$ in exactly two components, so we can find an $s \in (0,1)$ such that $u_0 + st_0 w_{k_0} \in \mathcal{N}^-$. Therefore $c_1 \leq I(u_0 + st_0 w_{k_0}) < c_0 + S^\infty$, which follows from Proposition 3.6.

Analogously to the proof of Proposition 3.4, by the Ekeland variational principle we can show that there exists a $(PS)_{c_1}$-sequence $\{u_n\} \subset \mathcal{N}^-$ for I. □

Proposition 3.10. *Assume $(a1), (a2), (f1)$ and $(f2)$ hold, then the functional I has a minimizer $u_1 \in \mathcal{N}^-$ which is also a critical point of I and $u_1 > 0$ for $f \geq 0, f \not\equiv 0$.*

Proof. From Propsitions 3.5 and 3.9, we can deduce that $u_n \to u_1$ strongly in $H_0^1(\Omega)$. Consequently, u_1 is a critical point of I, $u_1 \in \mathcal{N}^-$ (since $\mathcal{N}^-$ is closed) and $I(u_1) = c_1$.

By Lemma 3.1, we can choose a number $t^-(|u_1|) > 0$ such that $t^-(|u_1|)|u_1| \in \mathcal{N}^-$. Since $u_1 \in \mathcal{N}^-, t^-(u_1) = 1$. Applying Lemma 3.1 again, we conclude that

$$\begin{gathered} t^-(|u_1|) \geq t_{\max}(|u_1|) = t_{\max}(u_1), \\ c_1 = I(u_1) = \max_{t \geq t_{\max}(u_1)} I(tu_1) \geq I(t^-(|u_1|)u_1) \geq I\left(t^-(|u_1|)|u_1|\right) \geq c_1. \end{gathered} \tag{3.40}$$

Hence $I(t^-(|u_1|)u_1) = c_1$. So we can always take $u_1 \geq 0$. By the maximum principle for weak solutions (see Gilbarg and Trudinger [20]) we can show that if $f \geq 0, f \not\equiv 0$, then $u_1 > 0$ in Ω. □

The proof of Theorem 1.1

By Propositions 3.4 and 3.10, we obtain the conclusion of Theorem 1.1.

4. Existence of Nodal Solution

In this section, we will study the existence of nodal solutions for problem (1.2). To this end, we need to compare some different minimization problems. Define

$$\mathcal{N}_1^- = \{u = u^+ - u^- \in \mathcal{N} : u^+ \in \mathcal{N}^-\},$$
$$\mathcal{N}_2^- = \{u = u^+ - u^- \in \mathcal{N} : -u^- \in \mathcal{N}^-\}. \tag{4.1}$$

Here, we use notation $u^\pm = \max\{\pm u, 0\}$. Set

$$\beta_1 = \inf_{u \in \mathcal{N}_1^-} I(u), \tag{4.2}$$

$$\beta_2 = \inf_{u \in \mathcal{N}_2^-} I(u). \tag{4.3}$$

Then we have

Proposition 4.1. *(a) If $\beta_1 < c_1$, then the minimization problem* (4.2) *attains its infimum at a point which defines a sign changing critical point of I. (b) Analogously, if $\beta_2 < c_1$ the same conclusion holds for the minimization problem* (4.3).

Proof. The proof is almost the same as that in Tarantello [6, Proposition 3.1] . □

The above proposition would yield the conclusion for the main theorem only if the given relations between β_1, β_2, and c_1 could be established. While it is not clear whether or not such inequalities should hold, we will use these values to compare with another minimization problem. Namely, set

$$\mathcal{N}_*^- = \mathcal{N}_1^- \cap \mathcal{N}_2^- = \{u = u^+ - u^- \in \mathcal{N} : u^+, -u^- \in \mathcal{N}^-\} \subset \mathcal{N}^- \tag{4.4}$$

and define

$$c_2 = \inf_{u \in \mathcal{N}_*^-} I(u). \tag{4.5}$$

It is clear that $c_2 \geq c_1$. Since I satisfies $(PS)_c$ condition only locally, we need the following upper bound for c_2. Recall that $e_N = (0, 0, \ldots, 0, 1) \in \mathbb{R}^N$, $e_n = (0, 0, \ldots, 0, 1) \in \mathbb{R}^n$ and $w_k(x) = w(x - ke_N)$ where $k > 1$ and w is the ground state solution of problem (2.6).

Lemma 4.2. *Assume* $(a1)$, $(a3)$ *and* $(f1)$–$(f3)$ *hold. For any fixed* $k > 1$, *there exist* $s > 0,\ t > 0$ *such that*

$$su_1 - tw_k \in \mathcal{N}_*^-, \tag{4.6}$$

and for k *large,*

$$c_2 < \sup_{s,t\geq 0} I(su_1 - tw_k) < c_1 + S^\infty. \tag{4.7}$$

Proof. To prove (4.6), it suffices to show that there exist $s > 0$ and $t > 0$ such that

$$s(u_1 - tw_k)^+ \in \mathcal{N}^-, \qquad s(u_1 - tw_k)^- \in \mathcal{N}^-. \tag{4.8}$$

To this purpose, let

$$t_1 = \min_{\Omega} \frac{u_1}{w_k}, \qquad t_2 = \max_{\Omega} \frac{u_1}{w_k}. \tag{4.9}$$

For $t \in (t_1, t_2)$, denote by $s_+(t)$ and $s_-(t)$ the positive values given by Lemma 3.1 according to which we have

$$s_+(t)(u_1 - tw_k)^+ \in \mathcal{N}^-, \qquad -s_-(t)(u_1 - tw_k)^- \in \mathcal{N}^-. \tag{4.10}$$

Note that $s_+(t)$ and $s_-(t)$ are continuous with respect to t satifying

$$\begin{aligned} &\lim_{t\to t_1^+} s_+(t) = t^+\left((u_1 - t_1 w_k)^+\right) < +\infty, \quad \lim_{t\to t_2^-} s_+(t) = +\infty, \\ &\lim_{t\to t_1^+} s_-(t) = +\infty, \quad \lim_{t\to t_2^-} s_-(t) = t^+\left(-(u_1 - t_2 w_k)^-\right) < +\infty. \end{aligned} \tag{4.11}$$

Therefore, by the continuity of $s_\pm(t)$, we can find $t_0 \in (t_1, t_2)$ such that $s_+(t_0) = s_-(t_0) = s_0 > 0$. This gives (4.8) with $t = t_0$ and $s = s_0$.

To prove (4.7), we only need to estimate $I(su_1 - tw_k)$ for $s \geq 0$ and $t \geq 0$. First, it is obvious that the structure of I guarantees the existence of $r_0 > 0$ (independent of k large) such that $I(su_1 - tw_k) \leq c_1 < c_1 + S^\infty$, for all $s^2 + t^2 \geq r_0^2$. On the other hand, for $s^2 + t^2 \leq r_0^2$, since I is continuous in $H_0^1(\Omega)$, there exists $\bar{t} \in (0, r_0)$ small enough such that

$$I(su_1 - tw_k) < I(u_1) + I^\infty(w) = c_1 + S^\infty, \quad \forall\ s^2 + t^2 \leq r_0^2,\ t < \bar{t}. \tag{4.12}$$

At this point, we find large $k_0 \geq 1$, such that $I(su_1 - tw_k) < c_1 + S^\infty$ holds for all $s^2 + t^2 \leq r_0^2$ and $t \geq \bar{t}$:

$$\begin{aligned}
I(su_1 - tw_k) &= \frac{1}{2}\|su_1 - tw_k\|^2 - \frac{1}{p}\int a(x)|su_1 - tw_k|^p - \int f(su_1 - tw_k) \\
&= \left(\frac{1}{2}\|su_1\|^2 - \frac{1}{p}\int a(x)|su_1|^p - \int fsu_1\right) + \left(\frac{1}{2}\|tw_k\|^2 - \frac{1}{p}\int |tw_k|^p\right) \\
&\quad - st\int (\nabla u_1 \nabla w_k + u_1 w_k) - \frac{1}{p}\int \left(a(x)|su_1 - tw_k|^p - a(x)|su_1|^p - |tw_k|^p\right) \\
&\quad + \int ftw_k \\
&= I(su_1) + I^\infty(tw_k) - st\int u_1 w_k^{p-1} + \frac{1}{p}\int a(x)\left(|su_1|^p + |tw_k|^p - |su_1 - tw_k|^p\right) \\
&\quad - \frac{1}{p}\int (a(x) - a^\infty)|tw_k|^p + t\int ftw_k.
\end{aligned} \tag{4.13}$$

By (4.13) and the following elementary inequality:

$$\left|\alpha + \beta\right|^p \geq |\alpha|^p + \left|\beta\right|^p - C_5\left(|\alpha|^{p-1}\left|\beta\right| + |\alpha|\left|\beta\right|^{p-1}\right), \quad \forall\ \alpha, \beta \in \mathbb{R}, p > 1, \tag{4.14}$$

where C_5 is some positive constant, we have

$$\begin{aligned}
\sup_{s^2+t^2 \leq r_0^2,\, s \geq 0,\, t \geq \bar{t}} I(su_1 - tw_k) &= \sup_{0 \leq s \leq r_0,\, \bar{t} \leq t \leq r_0} I(su_1 - tw_k) \\
&\leq \sup_{s \geq 0} I(su_1) + \sup_{t \geq 0} I^\infty(tw_k) + \frac{\|a\|_\infty}{p} C_5 r_0^{p-1} \int \left(u_1^{p-1} w_k + u_1 w_k^{p-1}\right) \\
&\quad - \frac{\bar{t}^p}{p}\int (a(x) - a^\infty) w_k^p + r_0 \int f w_k.
\end{aligned} \tag{4.15}$$

Without loss of generality, we may assume $R_0 = \overline{R}$, and $\epsilon \in (0, \overline{\delta}_0)$ where R_0, $\overline{R}$ and $\overline{\delta}_0$ are given in $(f3)$ and $(a3)$, respectively.

(i) First, by the Hölder inequality and (2.9),

$$\begin{aligned}\int_{\Omega_{R_0}} u_1^{p-1} w_k &\le \left(\int_{\Omega_{R_0}} u_1^p\right)^{(p-1)/p} \left(\int_{\Omega_{R_0}} w_k^p\right)^{1/p} \\ &\le C_6 \left(\int_\omega \int_{\{z:|z|\le R_0\}} \phi^p(y) e^{-p\sqrt{1+\mu_1-\epsilon}|z+ke_n|} dy\, dz\right)^{1/p} \\ &\le C_7 e^{-\sqrt{1+\mu_1-\epsilon}k}.\end{aligned} \tag{4.16}$$

From (2.9), (2.10), and applying Lemma 3.8, there exists a k_1 such that for $k \ge k_1$

$$\begin{aligned}\int_{\Omega\setminus R_0} u_1^{p-1} w_k &\le C_8 \int_{\{z:|z|\ge R_0\}} e^{-(p-1)\sqrt{1+\mu_1-\epsilon}|z|} e^{-\sqrt{1+\mu_1-\epsilon}|z+ke_n|} dz \\ &\le C_9 e^{-\sqrt{1+\mu_1-\epsilon}k}.\end{aligned} \tag{4.17}$$

Similarly, we also obtain

$$\begin{gathered}\int_{\Omega_{R_0}} w_k^{p-1} u_1 \le C_{10} e^{-(p-1)\sqrt{1+\mu_1-\epsilon}k}, \\ \int_{\Omega_{R_0}} |a(x) - a^\infty| w_k^p \le C_{11} e^{-p\sqrt{1+\mu_1-\epsilon}k}, \\ \int_{\Omega_{R_0}} |f(x)| w_k \le C_{12} e^{-\sqrt{1+\mu_1-\epsilon}k},\end{gathered} \tag{4.18}$$

and there exists a $k_2 \ge k_1$ such that for $k \ge k_2$

$$\int_{\Omega\setminus R_0} w_k^{p-1} u_1 \le C_{13} e^{-\sqrt{1+\mu_1-\epsilon}k}. \tag{4.19}$$

(ii) Since $a(x)$ satisfies assumption $(a3)$ and by Lemma 3.8, there exists a $k_3 \ge k_2$ such that for $k \ge k_3$,

$$\int_{\Omega\setminus\Omega_{R_0}} (a(x) - a^\infty) w_k^p \ge C_{14} e^{-\sqrt{1+\mu_1-\overline{\delta}_0}k}. \tag{4.20}$$

By $(f3)$, (2.9), and Lemma 3.8, there exists a $k_4 \ge k_3$ such that for $k \ge k_4$,

$$\begin{aligned}\int_{\Omega\setminus R_0} f w_k &\le C_{10} \int_{\{z:|z|\ge R_0\}} e^{-\sqrt{1+\mu_1+\epsilon_0}|z|} e^{-\sqrt{1+\mu_1-\epsilon}|z+ke_n|} dz \\ &\le C_{11} e^{-\sqrt{1+\mu_1-\epsilon}k}.\end{aligned} \tag{4.21}$$

(iii) Note that the constants C_i $(5 \le i \le 11)$ in (i), (ii) are independent of k. Thus, by (i), (ii), $2 < p < 2N/(N-2)$ and let $\epsilon = \overline{\delta}_0/2$, we can find a $k_0 \ge k_4$ such that for $k \ge k_0$,

$$\frac{\|a\|_\infty}{p} C_5 r_0^{p-1} \int \left(u_1^{p-1} w_k + u_1 w_k^{p-1} \right) - \frac{\bar{t}^p}{p} \int (a(x) - a^\infty) w_k^p + r_0 \int f w_k < 0. \tag{4.22}$$

Combining (4.15) and (4.22), we obtain that there exists a $k_0 \ge k_4$ such that for $k \ge k_0$,

$$\sup_{s^2+t^2 \le r_0^2,\, s \ge 0,\, t \ge \bar{t}} I(su_1 - tw_k) < \sup_{s \ge 0} I(su_1) + \sup_{t \ge 0} I^\infty(tw_k) = c_1 + S^\infty. \tag{4.23}$$

This completes the proof of Lemma 4.2. □

Proposition 4.3. *Assume* $(a1), (a2), (f1)$ *and* $(f2)$ *hold. If* $\beta_1 \ge c_1$ *and* $\beta_2 \ge c_1$, *then the minimization problem* $c_2 = \inf_{\mathcal{N}_*^-} I(u)$ *attains its infimum at* $u_2 \in \mathcal{N}_*^-$ *which defines a changing sign critical point of* I.

Proof. It is obvious that $\mathcal{N}_*^-$ is closed. Exactly as in the proof of [6, Proposition 3.2], by means of Ekeland's principle, we derive a $(PS)_{c_2}$-sequence $\{u_n\} \subset \mathcal{N}_*^-$ for I. In particular, we have $0 < b_1 \le \|u_n^\pm\| \le b_2$, for some constants b_1 and b_2. Thus, we can take a subsequence, also denoted by $\{u_n\}$, such that $u_n^\pm \rightharpoonup u^\pm$ weakly in $H_0^1(\Omega)$. We start by showing that $u^\pm \not\equiv 0$.

Indeed, if by contradiction we assume, for instant, that $u^+ \equiv 0$, then we can deduce that

$$\|u_n^+\|^2 - \int a(x)|u_n^+|^p = o(1). \tag{4.24}$$

On the other hand,

$$I(u_n^+) = \frac{1}{2}\|u_n^+\|^2 - \frac{1}{p}\int a(x)|u_n^+|^p - \int f u_n^+ = \frac{1}{2}\|u_n^+\|^2 - \frac{1}{p}\int a(x)|u_n^+|^p + o(1). \tag{4.25}$$

By (4.24) and $\|u_n^+\| \ge b_1 > 0$, we may assume that

$$\|u_n^+\|^2 \longrightarrow b, \quad \int a(x)|u_n^+|^p \longrightarrow b. \tag{4.26}$$

Using the argument in the proof of Proposition 3.5, by (2.7), (4.24), and (4.25), we can deduce that $b \ge S^{p/(p-2)}$ and

$$I(u_n^+) = \left(\frac{1}{2} - \frac{1}{p}\right) b + o(1) \ge \left(\frac{1}{2} - \frac{1}{p}\right) S^{p/(p-2)} + o(1) = S^\infty + o(1). \tag{4.27}$$

However, by Lemma 4.2, $I(u_n^+) = c_2 - I(-u_n^-) + o(1) \le c_2 - c_1 + o(1)$; that is, $\lim_{n\to\infty} I(u_n^+) = c_2 - c_1 < S^\infty$ which contradicts (4.27). A similar argument applies to u^-. Therefore, $u_2 = u^+ - u^- \not\equiv 0$ is a weak solution of problem (1.2) changing sign and $u_2 \in \mathcal{N}, I(u_2) \ge c_0$.

Set $u_n^+ = u^+ + v_n^+$ and $u_n^- = u^- + v_n^-$ with $v_n^\pm \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. Note that

$$\|v_n^\pm\|^2 - \int a(x)\left|v_n^\pm\right|^p = o(1). \tag{4.28}$$

In view of Proposition 3.9 and Lemma 4.2, we also have

$$\begin{aligned} \lim_{n\to\infty}\left(I(v_n^+) + I\left(v_n^-\right)\right) &= \lim_{n\to\infty} I(v_n) = \lim_{n\to\infty} I(u_n) - I(u_2) \\ &\le c_2 - c_0 < c_1 + S^\infty - c_0 < 2S^\infty. \end{aligned} \tag{4.29}$$

Therefore, we must have

$$\min\left\{\lim_{n\to\infty} I(v_n^+), \lim_{n\to\infty} I\left(-v_n^-\right)\right\} < S^\infty. \tag{4.30}$$

Without loss of generality, we suppose

$$\lim_{n\to\infty} I(v_n^+) < S^\infty. \tag{4.31}$$

By (4.24), we have

$$I(v_n^+) = \frac{1}{2}\|v_n^+\|^2 - \frac{1}{p}\int a(x)|v_n^+|^p + o(1). \tag{4.32}$$

We claim that $\lim_{n\to\infty}\|v_n^+\|^2 = 0$. Indeed, we assume $\{v_n^+\}$ is bounded below, as above, (4.28) and (4.32) imply $I(v_n^+) \ge S^\infty + o(1)$, contradicting (4.31). In the same way, if $\lim_{n\to\infty} I(-v_n^-) < S^\infty$, we can also prove $\lim_{n\to\infty}\|v_n^-\|^2 = 0$. Hence we have $\lim_{n\to\infty}\|v_n^+\|^2 = 0$ or $\lim_{n\to\infty}\|v_n^-\|^2 = 0$; that is, $u_2 = u^+ - u^- \in \mathcal{N}_1^-$ or $u_2 = u^+ - u^- \in \mathcal{N}_2^-$. By assumptions $\beta_1 \ge c_1$ and $\beta_2 \ge c_2$, we conclude that $I(u_2) \ge c_1$.

If we write $u_n = u_2 + w_n$ with $w_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, we have

$$\begin{aligned} \|w_n\|^2 - \int a(x)|w_n|^p &= o(1), \\ \lim_{n\to\infty} I(u_n) - I(u_2) &= \lim_{n\to\infty}\left(\frac{1}{2}\|w_n\|^2 - \frac{1}{p}\int a(x)|w_n|^p\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{p}\right)\|w_n\|^2. \end{aligned} \tag{4.33}$$

Furthermore, by Lemma 4.2, we have

$$\lim_{n\to\infty} I(u_n) - I(u_2) = c_2 - I(u_2) \le c_2 - c_1 < S^\infty. \tag{4.34}$$

We claim that $\lim_{n\to\infty}\|w_n\|^2 = 0$. Indeed, we assume $\{w_n\}$ is bounded below, as above, (4.33) imply $I(w_n) \geq S^\infty + o(1)$, contradicting (4.34). Consequently, $u_n \to u_2$ strongly in $H_0^1(\Omega)$ and $I(u_2) = c_2$. □

The Proof of Theorems 1.2–1.4

The conclusion of Theorem 1.2 follows immediately from Theorem 1.2 and Propositions 4.1 and 4.3. With the same argument, we also have that Theorems 1.3 and 1.4 hold for $\Omega = \mathbb{R}^N$.

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