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Research Article

Nonhomogeneous Boundary Value Problem for One-Dimensional Compressible Viscous Micropolar Fluid Model: Regularity of the Solution

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An initial-boundary value problem for 1D flow of a compressible viscous heat-conducting micropolar fluid is considered; the fluid is thermodynamically perfect and polytropic. Assuming that the initial data are Hölder continuous on $]0,1[$ and transforming the original problem into homogeneous one, we prove that the state function is Hölder continuous on $]0,1[\times]0,T[$, for each $T>0$. The proof is based on a global-in-time existence theorem obtained in the previous research paper and on a theory of parabolic equations.

1. Introduction

In this paper, we consider a nonstationary 1D flow of a compressible viscous and heat-conducting micropolar fluid, being in a thermodynamical sense perfect and polytropic. In [1–3], we considered the problem with homogeneous boundary conditions.

Here we study, as in [4, 5], the case of nonhomogeneous boundary conditions for velocity and microrotation which is called in gas dynamics "problem on piston" (see [6]). Assuming that the initial data are Hölder continuous on $]0,1[$ and transforming the original problem into homogeneous one, we prove that, for each $T>0$, the mass density, velocity, microrotation velocity, and temperature are Hölder continuous on $]0,1[\times]0,T[$. The proof is based on a global-in-time existence theorem [5] and on a theory of parabolic equations [7]. We use some ideas of Antontsev et al. [8] applied to the case of classical fluid with homogeneous boundary conditions, results from [3] as well and some inequalities for Hölder norms obtained by the Nirenberg-Gagliardo inequality.

2. Statement of the problem and its equivalent setting

Let ρ, v, ω, and θ denote, respectively, the mass density, velocity, microrotation velocity, and temperature of the fluid in the Lagrangean description. Then the problem which we consider has the formulation as follows [1]:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - K\frac{\partial}{\partial x}(\rho\theta), \tag{2.2}$$

$$\rho\frac{\partial \omega}{\partial t} = A\left[\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial \omega}{\partial x}\right) - \omega\right], \tag{2.3}$$

$$\rho\frac{\partial \theta}{\partial t} = -K\rho^2\theta\frac{\partial v}{\partial x} + \rho^2\left(\frac{\partial v}{\partial x}\right)^2 + \rho^2\left(\frac{\partial \omega}{\partial x}\right)^2 + \omega^2 + D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial \theta}{\partial x}\right) \tag{2.4}$$

in $]0,1[\times]0,T[$, $T > 0$, where K, A, and D are positive constants. Equations (2.1)–(2.4) are, respectively, local forms of the conservations laws for the mass, momentum, momentum moment, and energy. We take the following nonhomogeneous initial and boundary conditions:

$$\rho(x,0) = \rho_0(x), \tag{2.5}$$

$$v(x,0) = v_0(x), \tag{2.6}$$

$$\omega(x,0) = \omega_0(x), \tag{2.7}$$

$$\theta(x,0) = \theta_0(x), \tag{2.8}$$

$$v(0,t) = \mu_0(t), \qquad v(1,t) = \mu_1(t), \tag{2.9}$$

$$\omega(0,t) = \nu_0(t), \qquad \omega(1,t) = \nu_1(t), \tag{2.10}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0, \tag{2.11}$$

for $x \in \Omega =]0,1[$, $t \in]0,T[$. Here $\rho_0, v_0, \omega_0, \theta_0, \mu_0, \mu_1, \nu_0$, and ν_1 are given functions. We assume the compatibility conditions

$$v_0(0) = \mu_0(0), \qquad v_0(1) = \mu_1(0), \tag{2.12}$$

$$\omega_0(0) = \nu_0(0), \qquad \omega_0(1) = \nu_1(0), \tag{2.13}$$

$$\frac{\partial \theta_0}{\partial x}(0) = \frac{\partial \theta_0}{\partial x}(1) = 0, \tag{2.14}$$

and the inequalities

$$0 < m \le \rho_0(x) \le M, \quad m \le \theta_0(x) \le M \quad \text{for } x \in \Omega, \tag{2.15}$$

where $m, M \in R^+$. We assume also that there exists a constant $\delta > 0$ such that

$$l(t) = \int_0^1 \frac{1}{\rho_0(x)}dx + \int_0^t \left[\mu_1(\tau) - \mu_0(\tau)\right]d\tau \ge \delta, \quad t \in]0,T[. \tag{2.16}$$

In the previous work [5] we proved that for

$$\begin{aligned} &\mu_0, \mu_1, \nu_0, \nu_1 \in H^2(]0,T[), \\ &\rho_0, v_0, \omega_0, \theta_0 \in H^1(\Omega), \end{aligned} \tag{2.17}$$

the problems (2.1)–(2.4) have a unique generalized solution

$$(x,t) \longrightarrow (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T = \Omega \times]0,T[, \tag{2.18}$$

$$\rho \in L^\infty(0,T;H^1(\Omega)) \cap H^1(Q_T), \quad \inf_{Q_T} \rho > 0, \tag{2.19}$$

$$v, \omega, \theta \in L^\infty(0,T;H^1(\Omega)) \cap H^1(Q_T) \cap L^2(0,T;H^2(\Omega)), \tag{2.20}$$

that satisfies (2.1)–(2.4) a.e. in Q_T and conditions (2.5)–(2.11) in the sense of traces. Moreover,

$$\theta > 0 \text{ in } \overline{Q}_T. \tag{2.21}$$

From embedding and interpolation theorems (e.g., [9]) one can conclude that from (2.19) and (2.20) it follows:

$$\rho \in L^\infty(0,T;C(\overline{\Omega})) \cap C([0,T],L^2(\Omega)), \tag{2.22}$$

$$v, \omega, \theta \in L^2(0,T;C^1(\overline{\Omega})) \cap C([0,T],H^1(\Omega)), \tag{2.23}$$

$$v, \omega, \theta \in C(\overline{Q}_T). \tag{2.24}$$

Now, instead of the velocity v and microrotation ω we introduce new functions V and W in order to obtain a problem with the homogeneous boundary conditions.

Notice that using (2.9) from (2.1) we get

$$\int_0^1 \frac{dx}{\rho(x,t)} = l(t), \quad t \in]0,T[, \tag{2.25}$$

where the function l is defined by (2.16). We introduce the functions

$$v_1(x,t) = \frac{\mu(t)}{l(t)} \int_0^x \frac{d\xi}{\rho(\xi,t)} + \mu_0(t), \tag{2.26}$$

$$\omega_1(x,t) = \frac{\nu(t)}{l(t)} \int_0^x \frac{d\xi}{\rho(\xi,t)} + \nu_0(t) \text{ on } Q_T, \tag{2.27}$$

where $\mu(t) = \mu_1(t) - \mu_0(t)$ and $\nu(t) = \nu_1(t) - \nu_0(t)$. It is evident that

$$\begin{aligned} &v_1(0,t) = \mu_0(t), \qquad v_1(1,t) = \mu_1(t), \\ &\omega_1(0,t) = \nu_0(t), \quad \omega_1(1,t) = \nu_1(t), \quad t \in]0,T[. \end{aligned} \tag{2.28}$$

Inserting

$$V(x,t) = v(x,t) - v_1(x,t), \qquad W(x,t) = \omega(x,t) - \omega_1(x,t) \tag{2.29}$$

into (2.1)–(2.4) we get the following equivalent system:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial V}{\partial x} + \frac{\mu}{l}\rho = 0, \tag{2.30}$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial V}{\partial x}\right) - K\frac{\partial}{\partial x}(\rho\theta) - \frac{\partial v_1}{\partial t}, \tag{2.31}$$

$$\rho\frac{\partial W}{\partial t} = A\left[\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial W}{\partial x}\right) - \omega_1 - W\right] - \rho\frac{\partial \omega_1}{\partial t}, \tag{2.32}$$

$$\begin{aligned}\rho\frac{\partial \theta}{\partial t} = {}& -K\rho^2\theta\frac{\partial V}{\partial x} - K\rho\theta\frac{\mu}{l} + \rho^2\left(\frac{\partial V}{\partial x}\right)^2 + 2\rho\frac{\partial V}{\partial x}\frac{\mu}{l} + \left(\frac{\mu}{l}\right)^2 \\ & + \rho^2\left(\frac{\partial W}{\partial x}\right)^2 + 2\rho\frac{\partial W}{\partial x}\frac{\nu}{l} + \left(\frac{\nu}{l}\right)^2 + (W+\omega_1)^2 + D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial \theta}{\partial x}\right),\end{aligned} \tag{2.33}$$

with the homogeneous boundary conditions

$$V(0,t) = V(1,t) = 0, \qquad W(0,t) = W(1,t) = 0, \tag{2.34}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \tag{2.35}$$

for $t \in]0,T[$ and initial conditions

$$\rho(x,0) = \rho_0(x), \qquad V(x,0) = V_0(x), \tag{2.36}$$

$$W(x,0) = W_0(x), \qquad \theta(x,0) = \theta_0(x), \tag{2.37}$$

for $x \in \Omega$, where

$$\begin{aligned} V_0(x) &= v_0(x) - \frac{\mu(0)}{l(0)}\int_0^x \frac{1}{\rho_0(\xi)}d\xi - \mu_0(0), \\ W_0(x) &= \omega_0(x) - \frac{\nu(0)}{l(0)}\int_0^x \frac{1}{\rho_0(\xi)}d\xi - \nu_0(0) \end{aligned} \tag{2.38}$$

are known functions. In the article [5], we proved that the problems (2.30)–(2.37) have a unique generalized solution (ρ, V, W, θ) in the domain Q_T with property (2.21) as well. Moreover, we obtained that

$$v_1, \omega_1 \in L^\infty(0,T;H^2(\Omega)), \qquad \frac{\partial v_1}{\partial t}, \frac{\partial \omega_1}{\partial t} \in L^\infty(0,T;L^2(\Omega)). \tag{2.39}$$

In the following $C^{k+\alpha}(\overline{Q}_T)$ $(k \in N \cup \{0\}, 0 < \alpha < 1)$ is the Banach space of functions of the class $C^k(\overline{Q}_T)$, having kth derivatives Hölder continuous with the exponent α on $\overline{Q}_T$; the norm is defined by

$$|f|_{k+\alpha,Q_T} = \sum_{m+j=0}^{k} \left|D_x^m D_t^j f\right|_{0,Q_T} + \sum_{m+j=k} H^\alpha\left(D_x^m D_t^j f\right), \tag{2.40}$$

where

$$H^\alpha(f) = \sup_{y,z \in Q_T} \frac{\left|f(y) - f(z)\right|}{|y-z|^\alpha}, \tag{2.41}$$

and $|\cdot|_{0,Q_T}$ is the norm on $C(\overline{Q}_T)$; D_x^m and D_t^j are, respectively, the mth derivatives with respect to x and the jth derivatives with respect to t. $C^{k+\alpha,m+\beta}(\overline{Q}_T)$ $(k, m \in N \cup \{0\},\ 0 < \alpha,\ \beta < 1)$ is the Banach space of functions which have kth derivatives with respect to x and mth derivatives with respect to t Hölder continuous on $\overline{Q}_T$. The norm is defined by

$$|f|_{k+\alpha,m+\beta,Q_T} = \sum_{l=0}^{k} |D_x^l f|_{0,Q_T} + \sum_{j=1}^{m} \left|D_t^j f\right|_{0,Q_T} + H_x^\alpha\left(D_x^k f\right) + H_t^\beta\left(D_x^k f\right) + H_x^\alpha\left(D_t^m f\right) + H_t^\beta\left(D_t^m f\right), \tag{2.42}$$

where

$$\begin{aligned} H_x^\alpha(f) &= \sup_{(x_1,t),(x_2,t) \in Q_T} \frac{\left|f(x_1,t) - f(x_2,t)\right|}{\left|x_1 - x_2\right|^\alpha}, \\ H_t^\beta(f) &= \sup_{(x,t_1),(x,t_2) \in Q_T} \frac{\left|f(x,t_1) - f(x,t_2)\right|}{\left|t_1 - t_2\right|^\beta}. \end{aligned} \tag{2.43}$$

By $C \in \mathbf{R}^+$ we denote a generic constant, having possibly different values at different places. Also we use some inequalities for Hölder norms obtained by the following Nirenberg-Gagliardo interpolation inequality

$$|f|_{1/\mu} \le |f|_{1/\lambda}^{(\nu-\mu)/(\nu-\lambda)} |f|_{1/\nu}^{(\mu-\lambda)/(\nu-\lambda)}, \tag{2.44}$$

where $\mu, \nu, \lambda \in \mathbf{R}$ and $\lambda \le \mu \le \nu$. Here, for bounded domain $D \subset \mathbf{R}^n$ and $f : D \to \mathbf{R}$ the norm $|f|_q$ is defined by

$$|f|_q = \begin{cases} \|f\|_{L^q(D)}, & q > 0, \\ |f|_{k+\beta,D}, & q < 0, \end{cases} \tag{2.45}$$

where $k = [-n/q]$ and $\beta = -n/q - k$ (e.g., [8, page 27]). Some of our considerations are very similar or identical to that of [8] or [3]. In these cases we omit proofs or details of proofs, making reference to correspondent pages of the book [8] or article [3]; we use the notation $\|\cdot\| = \|\cdot\|_{L^2}$.

3. The main results

The aim of this paper is to prove the following regularity result.

Theorem 3.1. *Let the functions*

$$\mu_0, \mu_1, \nu_0, \nu_1 \in C^2([0,T]), \tag{3.1}$$

$$\rho_0 \in C^{1+\alpha}(\overline{\Omega}), \quad v_0, \omega_0, \theta_0 \in C^{2+\alpha}(\overline{\Omega}), \quad 0 < \alpha < 1 \tag{3.2}$$

satisfy the compatibility conditions

$$\frac{d}{dx}\left(\rho_0 \frac{dv_0}{dx}\right) - K\frac{d}{dx}(\rho_0\theta_0) = \begin{cases} \dfrac{d\mu_0(0)}{dt}, & \text{for } x = 0 \\ \dfrac{d\mu_1(0)}{dt}, & \text{for } x = 1, \end{cases} \tag{3.3}$$

$$A\left[\frac{d}{dx}\left(\rho_0 \frac{d\omega_0}{dx}\right) - \frac{\omega_0}{\rho_0}\right] = \begin{cases} \dfrac{d\nu_0(0)}{dt}, & \text{for } x = 0 \\ \dfrac{d\nu_1(0)}{dt}, & \text{for } x = 1, \end{cases} \tag{3.4}$$

and (2.12)–(2.16). *Then the generalized solution of the problems* (2.1)–(2.11) *has the properties*

$$\rho \in C^{1+\alpha}(\overline{Q}_T), \quad v, \omega, \theta \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T), \quad 0 < \alpha < 1. \tag{3.5}$$

Notice that because of (3.1) and (3.2) we have

$$V_0, W_0 \in C^{2+\alpha}(\overline{\Omega}), \qquad l \in C^3([0,T]), \tag{3.6}$$

and for $t = 0$ we can easily conclude that

$$v_1\big|_{t=0}, \omega_1\big|_{t=0}, \frac{\partial v_1}{\partial t}\bigg|_{t=0}, \frac{\partial \omega_1}{\partial t}\bigg|_{t=0} \in C^{2+\alpha}(\overline{\Omega}). \tag{3.7}$$

Now, (2.12), (2.13), (3.3), and (3.4) become the following compatibility conditions for the problems (2.30)–(2.37)

$$V_0(0) = V_0(1) = 0, \qquad W_0(0) = W_0(1) = 0, \tag{3.8}$$

$$\frac{d}{dx}\left(\rho_0\frac{dV_0}{dx}\right)-K\frac{d}{dx}(\rho_0\theta_0)=\begin{cases}\dfrac{d\mu_0(0)}{dt}, & \text{for } x=0\\[2mm] \dfrac{d\mu_1(0)}{dt}, & \text{for } x=1,\end{cases} \tag{3.9}$$

$$\frac{d}{dx}\left(\rho_0\frac{dW_0}{dx}\right)-\frac{W_0}{\rho_0}=\begin{cases}\dfrac{\nu_0(0)}{\rho_0}+A^{-1}\dfrac{d\nu_0(0)}{dt}, & \text{for } x=0\\[2mm] \dfrac{\nu_1(0)}{\rho_0}+A^{-1}\dfrac{d\nu_1(0)}{dt}, & \text{for } x=1.\end{cases} \tag{3.10}$$

In this paper, we will prove the following result first.

Theorem 3.2. *Under the assumptions of Theorem 3.1 the problems* (2.30)–(2.37) *have a generalized solution* (ρ, V, W, θ) *in* Q_T *with the properties*

$$\rho\in C^{1+\alpha}(\overline{Q}_T),\quad V,W,\theta\in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T),\quad 0<\alpha<1. \tag{3.11}$$

Moreover,

$$v_1,\omega_1\in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T). \tag{3.12}$$

Theorem 3.1 is an immediate consequence of this result. In the proof of Theorem 3.2 we apply, as in [3], the method of the book [8], where Theorem 3.1 was proved for the classical fluid ($\omega = 0$) with homogeneous boundary conditions.

In that what follows, we assume that the conditions (2.12)–(2.16) and (3.1)–(3.4) are fulfilled.

4. Some properties of the solution (ρ, V, W, θ) and functions v_1 and ω_1

Lemma 4.1. *It holds*

$$\frac{\partial^2 v_1}{\partial t^2},\frac{\partial^2\omega_1}{\partial t^2}\in L^2(0,T;L^2(\Omega)). \tag{4.1}$$

Proof. Using (2.30) from (2.26) and (2.27) we get

$$\frac{\partial v_1}{\partial t}=\left[\left(\frac{\mu}{l}\right)'+\left(\frac{\mu}{l}\right)^2\right]\int_0^x\frac{1}{\rho}d\xi+\frac{\mu}{l}V+\mu_0', \tag{4.2}$$

$$\frac{\partial\omega_1}{\partial t}=\left[\left(\frac{\nu}{l}\right)'+\frac{\mu\nu}{l^2}\right]\int_0^x\frac{1}{\rho}d\xi+\frac{\nu}{l}V+\nu_0'. \tag{4.3}$$

After differentiating (4.2) with respect to t, squaring, integrating over Ω and taking into account (2.25), (2.30), (3.1), and (3.6) we get

$$\left\|\frac{\partial^2 v_1}{\partial t^2}(t)\right\|^2\leq C\left(1+\|V(t)\|^2+\left\|\frac{\partial V}{\partial t}(t)\right\|^2\right). \tag{4.4}$$

With the help of (2.20) for V we conclude that

$$\int_0^T \left\| \frac{\partial^2 v_1}{\partial t^2}(\tau) \right\|^2 d\tau \le C\left(1 + \int_0^T \|V(\tau)\|^2 d\tau + \int_0^T \left\| \frac{\partial V}{\partial t}(\tau) \right\|^2 d\tau \right) \le C. \tag{4.5}$$

From (4.3) follows the same estimation for $\partial^2 \omega_1 / \partial t^2$. □

Lemma 4.2. *The inclusions*

$$\frac{\partial \rho}{\partial t}, \frac{\partial V}{\partial t}, \frac{\partial W}{\partial t}, \frac{\partial \theta}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \tag{4.6}$$

hold true.

Proof. Using (2.22) for ρ and (2.20) for V from (2.30) we get immediately that $\partial \rho / \partial t \in L^\infty(0,T;L^2(\Omega))$. Differentiating (2.30) with respect to x, using the inequalities

$$\begin{gathered} |f|^2 \le C\|f\|\|f'\| \le C\|f'\|^2, \\ |f'|^2 \le C\|f'\|\|f''\| \le C\|f''\|^2 \end{gathered} \tag{4.7}$$

(valid for a function f or its derivative vanishing at $x = 0$ and $x = 1$), (2.22) and (3.1) we obtain

$$\begin{aligned} \left\| \frac{\partial^2 \rho}{\partial x \partial t}(t) \right\|^2 &\le C\left(\int_0^1 \left| \frac{\partial \rho}{\partial x}(t) \right|^2 \left| \frac{\partial V}{\partial x} \right|^2 dx + \int_0^1 \left| \frac{\partial^2 V}{\partial x^2} \right|^2 dx + \left| \frac{\mu}{l} \right|^2 \int_0^1 \left| \frac{\partial \rho}{\partial x} \right|^2 dx \right) \\ &\le C\left(\left\| \frac{\partial \rho}{\partial x}(t) \right\|^2 \left\| \frac{\partial^2 V}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 V}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial \rho}{\partial x}(t) \right\|^2 \right). \end{aligned} \tag{4.8}$$

Taking into account (2.20) and (2.19) we get

$$\int_0^T \left\| \frac{\partial^2 \rho}{\partial x \partial t}(\tau) \right\|^2 d\tau \le C. \tag{4.9}$$

After differentiating (2.31) with respect to the time variable, multiplying by $\partial V / \partial t$ and integrating by parts over Ω we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\| \frac{\partial V}{\partial t}(t) \right\|^2 + \int_0^1 \rho \left(\frac{\partial^2 V}{\partial x \partial t} \right)^2 dx \\ &= -\int_0^1 \frac{\partial \rho}{\partial t} \frac{\partial V}{\partial x} \frac{\partial^2 V}{\partial x \partial t} dx + K\int_0^1 \frac{\partial \rho}{\partial t} \theta \frac{\partial^2 V}{\partial x \partial t} dx + K\int_0^1 \rho \frac{\partial \theta}{\partial t} \frac{\partial^2 V}{\partial x \partial t} dx - \int_0^1 \frac{\partial^2 v_1}{\partial t^2} \frac{\partial V}{\partial t} dx. \end{aligned} \tag{4.10}$$

Applying (4.6) and (2.22) for ρ, (2.24) for θ, (4.7) for $\partial V/\partial x$ and the Young inequality with a parameter $\varepsilon > 0$ we obtain

$$\left|\int_0^1 \frac{\partial \rho}{\partial t}\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x \partial t}dx\right| \leq \varepsilon \int_0^1 \rho\left(\frac{\partial^2 V}{\partial x \partial t}\right)^2 dx + C\left\|\frac{\partial^2 V}{\partial x^2}(t)\right\|^2, \tag{4.11}$$

$$\left|K\int_0^1 \frac{\partial \rho}{\partial t}\theta\frac{\partial^2 V}{\partial x \partial t}dx\right| \leq C + \varepsilon \int_0^1 \rho\left(\frac{\partial^2 V}{\partial x \partial t}\right)^2 dx, \tag{4.12}$$

$$\left|K\int_0^1 \rho\frac{\partial \theta}{\partial t}\frac{\partial^2 V}{\partial x \partial t}dx\right| \leq \varepsilon \int_0^1 \rho\left(\frac{\partial^2 V}{\partial x \partial t}\right)^2 dx + C\left\|\frac{\partial \theta}{\partial t}(t)\right\|^2, \tag{4.13}$$

$$\left|\int_0^1 \frac{\partial^2 v_1}{\partial t^2}\frac{\partial V}{\partial t}dx\right| \leq C\left(\left\|\frac{\partial^2 v_1}{\partial t^2}(t)\right\|^2 + \left\|\frac{\partial V}{\partial t}(t)\right\|^2\right). \tag{4.14}$$

For sufficiently small $\varepsilon > 0$ from (4.10) and (4.11)–(4.14) it follows that for $t \in]0, T[$ we have

$$\begin{aligned}\left\|\frac{\partial V}{\partial t}(t)\right\|^2 + \int_0^t \left\|\frac{\partial^2 V}{\partial x \partial t}(\tau)\right\|^2 d\tau \leq C\Bigg(1 + \left\|\frac{\partial V}{\partial t}(0)\right\|^2 + \int_0^t \left\|\frac{\partial \theta}{\partial t}(\tau)\right\|^2 d\tau + \int_0^t \left\|\frac{\partial^2 V}{\partial x^2}(\tau)\right\|^2 d\tau \\ + \int_0^t \left\|\frac{\partial^2 v_1}{\partial t^2}(\tau)\right\|^2 d\tau + \int_0^t \left\|\frac{\partial V}{\partial t}(\tau)\right\|^2 d\tau\Bigg).\end{aligned} \tag{4.15}$$

Taking into account (3.2), (3.6), and (3.7) from (2.31) we can easily conclude that

$$\left\|\frac{\partial V}{\partial t}(0)\right\| = \left\|\rho_0' V_0' + \rho_0 V_0'' - K\rho_0'\theta_0 - K\rho_0\theta_0' - \frac{\partial v_1}{\partial t}(0)\right\| \leq C, \tag{4.16}$$

and using (2.20) and (4.1) we get that inclusion (4.6) is satisfied for the function V. In the similar way from (2.32) and (2.33) we obtain (4.6) for W and θ. □

Now, taking into account (4.6) and (2.20), we can introduce the following inequalities for

$$\eta \in \{V, W, \theta\} \tag{4.17}$$

derived in [3] by the Nirenberg-Gagliardo inequality (2.44).

Lemma 4.3 (see [3, Lemmas 2.2–2.4]). *For $0 < \alpha < 1$ and $\varepsilon > 0$, the function η satisfies the inequalities*

$$\left|\frac{\partial \eta}{\partial x}\right|_{0,Q_T} \leq C|\eta|^{a}_{2+\alpha,1+\alpha/2,Q_T}, \tag{4.18}$$

$$\left|\frac{\partial \eta}{\partial x}\right|_{\alpha,\alpha/2,Q_T} \leq C\left(\varepsilon|\eta|_{2+\alpha,1+\alpha/2,Q_T} + \sup_{Q_T}\left|\frac{\partial \eta}{\partial t}\right| + 1\right), \tag{4.19}$$

$$\left|\frac{\partial \eta}{\partial x}\right|_{\alpha,Q_T} \leq C|\eta|^{d}_{2+\alpha,1+\alpha/2,Q_T}\left(|\eta|_{2+\alpha,1+\alpha/2,Q_T} + 1\right)^{1-d}, \tag{4.20}$$

where $a = 1/(3+2\alpha)$ and $d = \alpha/(2-\alpha)$. For $0 < \alpha \leq 1/2$ it holds

$$\left|\frac{\partial \eta}{\partial x}\right|_{\alpha,\alpha/2,Q_T} \leq C|\eta|^{b}_{2+\alpha,1+\alpha/2,Q_T}, \tag{4.21}$$

where $b = (1+2\alpha)/(3+2\alpha)$.

5. The proofs of Theorems 3.1 and 3.2

The conclusions of Theorems 3.1 and 3.2 are immediate consequences of the following lemmas.

Lemma 5.1. *It holds*

$$\rho, V, W, \theta \in C^{1/2,1/2}(\overline{Q}_T). \tag{5.1}$$

Moreover,

$$v_1, \omega_1 \in C^{1+1/2,1+1/2}(\overline{Q}_T). \tag{5.2}$$

Proof. Taking into account (4.6) we get inclusion (5.1) for the functions V, W, and θ in the same way as for ρ in (see [8, pages 54-55]). Using (3.1), (3.6), and (5.1) from (2.26), (2.27), (4.2), and (4.3) we get (5.2) immediately. □

Lemma 5.2. *For $0 < \alpha < 1$ and $\gamma = \min\{1/2, \alpha\}$ it holds*

$$\frac{\partial \rho}{\partial x} \in C^{\gamma,\gamma}(\dot{Q}_T). \tag{5.3}$$

Proof. With the help of (5.1), (5.2), and (3.2) we obtain (5.3) in the similar way as in (see [8, pages 57-58]). □

Lemma 5.3. *For* $0 < \alpha < 1$, $\gamma = \min\{1/2, \alpha\}$, $a = 1/(3+2\alpha)$, *and* $b = (1+2\gamma)/(3+2\gamma)$ *the inequalities*

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \leq C\left(1 + |\theta|_{1+\gamma,\gamma/2,Q_T}\right), \tag{5.4}$$

$$|W|_{2+\gamma,1+\gamma/2,Q_T} \leq C, \tag{5.5}$$

$$|\theta|_{2+\gamma,1+\gamma/2,Q_T} \leq C\Big(|V|^{a}_{2+\gamma,1+\gamma/2,Q_T} + |V|^{b}_{2+\gamma,1+\gamma/2,Q_T} + |V|^{a+b}_{2+\gamma,1+\gamma/2,Q_T} + |V|^{2a}_{2+\gamma,1+\gamma/2,Q_T} + 1\Big) \tag{5.6}$$

hold true.

Proof. We write (2.31), (2.32), and (2.33) in the form

$$\begin{aligned}
&\frac{\partial V}{\partial t} - \rho\frac{\partial^2 V}{\partial x^2} - \frac{\partial \rho}{\partial x}\frac{\partial V}{\partial x} = -K\frac{\partial \rho}{\partial x}\theta - K\rho\frac{\partial \theta}{\partial x} - \frac{\partial v_1}{\partial t},\\
&\frac{\partial W}{\partial t} - A\rho\frac{\partial^2 W}{\partial x^2} - A\frac{\partial \rho}{\partial x}\frac{\partial W}{\partial x} + A\frac{W}{\rho} = -A\frac{\omega_1}{\rho} - \frac{\partial \omega_1}{\partial t},\\
&\frac{\partial \theta}{\partial t} - D\rho\frac{\partial^2 \theta}{\partial x^2} - D\frac{\partial \rho}{\partial x}\frac{\partial \theta}{\partial x} = -K\rho\theta\frac{\partial V}{\partial x} - K\theta\frac{\mu}{l} + \rho\left(\frac{\partial V}{\partial x}\right)^2 + 2\frac{\partial V}{\partial x}\frac{\mu}{l} + \frac{1}{\rho}\left(\frac{\mu}{l}\right)^2\\
&\qquad + \rho\left(\frac{\partial W}{\partial x}\right)^2 + 2\frac{\partial W}{\partial x}\frac{\nu}{l} + \frac{1}{\rho}\left(\frac{\nu}{l}\right)^2 + \frac{1}{\rho}(W+\omega_1)^2,
\end{aligned} \tag{5.7}$$

and we consider them as parabolic equations for V, W, and θ, respectively, with Hölder continuous coefficients with exponent $\gamma = \min\{1/2, \alpha\}$. Taking into account the compatibility conditions (3.8)–(3.10), (2.14), and $|fg|_{\alpha,\alpha/2} \leq |f|_0|g|_{\alpha,\alpha/2} + |f|_{\alpha,\alpha/2}|g|_0$, from a parabolic theory (see [7, Theorems 5.2 and 5.3]) we conclude that the solutions V, W, and θ satisfy the following inequalities

$$\begin{aligned}
|V|_{2+\gamma,1+\gamma/2,Q_T} \leq C\bigg(&\left|\frac{\partial \rho}{\partial x}\right|_{\gamma,\gamma/2,Q_T}|\theta|_{0,Q_T} + \left|\frac{\partial \rho}{\partial x}\right|_{0,Q_T}|\theta|_{\gamma,\gamma/2,Q_T} + |\rho|_{\gamma,\gamma/2,Q_T}\left|\frac{\partial \theta}{\partial x}\right|_{0,Q_T}\\
&+ |\rho|_{0,Q_T}\left|\frac{\partial \theta}{\partial x}\right|_{\gamma,\gamma/2,Q_T} + \left|\frac{\partial v_1}{\partial t}\right|_{\gamma,\gamma/2,Q_T} + \left|V_0\right|_{2+\gamma,\Omega}\bigg),
\end{aligned} \tag{5.8}$$

$$|W|_{2+\gamma,1+\gamma/2,Q_T} \leq \left(\left|\frac{1}{\rho}\right|_{0,Q_T}\left|\omega_1\right|_{\gamma,\gamma/2,Q_T} + \left|\frac{1}{\rho}\right|_{\gamma,\gamma/2,Q_T}\left|\omega_1\right|_{0,Q_T} + \left|\frac{\partial \omega_1}{\partial t}\right|_{\gamma,\gamma/2,Q_T} + \left|W_0\right|_{2+\gamma,\Omega}\right), \tag{5.9}$$

$$
\begin{aligned}
|\theta|_{2+\gamma,1+\gamma/2,Q_T} \le C\Bigg(& |\rho\theta|_{0,Q_T}\left|\frac{\partial V}{\partial x}\right|_{\gamma,\gamma/2,Q_T} + |\rho\theta|_{\gamma,\gamma/2,Q_T}\left|\frac{\partial V}{\partial x}\right|_{0,Q_T} + |\theta|_{\gamma,\gamma/2,Q_T} \\
& + \left|\rho\frac{\partial V}{\partial x}\right|_{0,Q_T}\left|\frac{\partial V}{\partial x}\right|_{\gamma,\gamma/2,Q_T} + \left|\rho\frac{\partial V}{\partial x}\right|_{\gamma,\gamma/2,Q_T}\left|\frac{\partial V}{\partial x}\right|_{0,Q_T} + \left|\frac{\partial V}{\partial x}\right|_{\gamma,\gamma/2,Q_T} \\
& + \left|\frac{1}{\rho}\right|_{\gamma,\gamma/2,Q_T} + \left|\rho\frac{\partial W}{\partial x}\right|_{0,Q_T}\left|\frac{\partial W}{\partial x}\right|_{\gamma,\gamma/2,Q_T} + \left|\frac{\partial W}{\partial x}\right|_{\gamma,\gamma/2,Q_T} \\
& + \left|\rho\frac{\partial W}{\partial x}\right|_{\gamma,\gamma/2,Q_T}\left|\frac{\partial W}{\partial x}\right|_{0,Q_T} + \left|\frac{W}{\rho}\right|_{0,Q_T}|W|_{\gamma,\gamma/2,Q_T} \\
& + \left|\frac{W}{\rho}\right|_{\gamma,\gamma/2,Q_T}|W|_{0,Q_T} + \left|\frac{\omega_1^2}{\rho}\right|_{\gamma,\gamma/2,Q_T} + |\theta_0|_{2+\gamma,\Omega}\Bigg).
\end{aligned}
\tag{5.10}
$$

Using the inequalities

$$
|f|_{0,Q_T} \le |f|_{\gamma,\gamma/2,Q_T} \le |f|_{\gamma,\gamma,Q_T}, \tag{5.11}
$$

and (3.6), (3.2), and (5.1)–(5.3), from (5.8)-(5.9) we get easily (5.4) and (5.5). With the help of (4.18), (4.21) for $\eta = W$ and (5.5) from (5.10) it follows

$$
|\theta|_{2+\gamma,1+\gamma/2,Q_T} \le C\left(\left|\frac{\partial V}{\partial x}\right|_{\gamma,\gamma/2,Q_T} + \left|\frac{\partial V}{\partial x}\right|_{0,Q_T} + \left|\frac{\partial V}{\partial x}\right|_{\gamma,\gamma/2,Q_T}\left|\frac{\partial V}{\partial x}\right|_{0,Q_T} + \left|\frac{\partial V}{\partial x}\right|^2_{0,Q_T} + 1\right). \tag{5.12}
$$

Using (4.18) and (4.21) for $\eta = V$ we get (5.6) immediately. □

Lemma 5.4. *For γ from Lemma 5.2 the estimations*

$$
|V|_{2+\gamma,1+\gamma/2,Q_T} \le C, \tag{5.13}
$$

$$
|\theta|_{2+\gamma,1+\gamma/2,Q_T} \le C, \tag{5.14}
$$

$$
|\rho|_{1+\gamma,Q_T} \le C, \tag{5.15}
$$

$$
|v_1|_{2+\gamma,1+\gamma/2,Q_T} \le C, \qquad |\omega_1|_{2+\gamma,1+\gamma/2,Q_T} \le C \tag{5.16}
$$

hold true.

Proof. For $\eta = \theta$ and $0 < \alpha < 1$ from (4.19) we can conclude that

$$
|\theta|_{1+\alpha,\alpha/2,Q_T} \le C\left(|\theta|_{2+\alpha,1+\alpha/2,Q_T} + 1\right), \tag{5.17}
$$

and from (5.4) it follows

$$
|V|_{2+\gamma,1+\gamma/2,Q_T} \le C\left(|\theta|_{2+\gamma,1+\gamma/2,Q_T} + 1\right). \tag{5.18}
$$

Inserting (5.6) on the right-hand side of (5.18) we obtain

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \le C\left(|V|^{a}_{2+\gamma,1+\gamma/2,Q_T} + |V|^{b}_{2+\gamma,1+\gamma/2,Q_T} + |V|^{a+b}_{2+\gamma,1+\gamma/2,Q_T} + |V|^{2a}_{2+\gamma,1+\gamma/2,Q_T} + 1\right), \tag{5.19}$$

where $a, b, a+b, 2a \in]0,1[$. Applying the Young inequality with a parameter $\varepsilon > 0$ we obtain

$$|V|_{2+\gamma,1+\gamma/2,Q_T} \le C\left(\varepsilon |V|_{2+\gamma,1+\gamma/2,Q_T} + 1\right), \tag{5.20}$$

and hence (5.13). Using this result from (5.6) follows (5.14). From (2.30) we get

$$\left|\frac{\partial \rho}{\partial t}\right|_{\gamma,Q_T} \le \left|\rho^2\right|_{0,Q_T}\left|\frac{\partial V}{\partial x}\right|_{\gamma,Q_T} + \left|\rho^2\right|_{\gamma,Q_T}\left|\frac{\partial V}{\partial x}\right|_{0,Q_T} + \left|\frac{\mu}{l}\rho\right|_{\gamma,Q_T}. \tag{5.21}$$

Using (4.20) and (5.13) for V, the inequality $|\rho|_{\gamma,Q_T} \le |\rho|_{\gamma,\gamma,Q_T}$, (5.1) and (3.1) we obtain

$$\left|\frac{\partial \rho}{\partial t}\right|_{\gamma,Q_T} \le C, \tag{5.22}$$

and with the help of (5.3) we get (5.15). Notice that from (5.15) follows

$$\int_0^x \rho^{-1} d\xi \in C^{\gamma,\gamma/2}(\overline{Q}_T), \tag{5.23}$$

and using (5.13) and (3.1) from (4.2) and (2.26) we obtain

$$\begin{aligned} \left|\frac{\partial v_1}{\partial t}\right|_{\gamma,\gamma/2,Q_T} &\le \left|\left(\frac{\mu}{l}\right)' + \left(\frac{\mu}{l}\right)^2 \int_0^x \frac{1}{\rho} d\xi\right|_{\gamma,\gamma/2,Q_T} + \left|\frac{\mu}{l}V\right|_{\gamma,\gamma/2,Q_T} + \left|\mu_0'\right|_{\gamma/2,]0,T[} \le C, \\ \left|\frac{\partial^2 v_1}{\partial x^2}\right|_{\gamma,\gamma/2,Q_T} &\le \left|\frac{\mu}{l\rho}\right|_{\gamma,\gamma/2,Q_T}\left|\frac{\partial \rho}{\partial x}\right|_{0,Q_T} + \left|\frac{\mu}{l\rho}\right|_{0,Q_T}\left|\frac{\partial \rho}{\partial x}\right|_{\gamma,\gamma/2,Q_T} + \left|\frac{\mu}{l}\right|_{\gamma,]0,T[} \le C. \end{aligned} \tag{5.24}$$

Taking into account (5.2) it is evident that the inequality

$$\left|v_1\right|_{2+\gamma,1+\gamma/2,Q_T} \le C \tag{5.25}$$

is satisfied. From (2.27) and (4.3) follows the same estimation for the function ω_1. □

Now, from the above estimations we derive the conclusion that if $\alpha \le 1/2$ then $\alpha = \gamma$ and Lemmas 5.1–5.4 are the proofs of Theorems 3.2 and 3.1. If $\alpha > 1/2$ we have

$$V, W, \theta, v_1, \omega_1 \in C^{2+1/2,1+1/4}(\overline{Q}_T), \qquad \rho \in C^{1+1/2}(\overline{Q}_T). \tag{5.26}$$

Lemma 5.5. *For* $1/2 < \alpha < 1$ *we have*

$$\rho, V, W, \theta \in C^{\alpha,\alpha}(\overline{Q}_T), \tag{5.27}$$

$$v_1, \omega_1 \in C^{1+\alpha,1+\alpha}(\overline{Q}_T), \tag{5.28}$$

$$\frac{\partial \rho}{\partial x} \in C^{\alpha,\alpha}(\dot{Q}_T). \tag{5.29}$$

Proof. Inclusions (5.27) follows directly from (5.26). Using this result from (2.26), (2.27), (4.2), and (4.3) we get (5.28). Estimation (5.29) is proved in (see [8, pages 57-58]). □

Lemma 5.6. *For* $1/2 < \alpha < 1$ *the estimations*

$$|V|_{2+\alpha,1+\alpha/2,Q_T} \le C, \tag{5.30}$$

$$|W|_{2+\alpha,1+\alpha/2,Q_T} \le C, \tag{5.31}$$

$$|\theta|_{2+\alpha,1+\alpha/2,Q_T} \le C, \tag{5.32}$$

$$|\rho|_{1+\alpha,Q_T} \le C. \tag{5.33}$$

$$|v_1|_{2+\alpha,1+\alpha/2,Q_T} \le C, \qquad |\omega_1|_{2+\alpha,1+\alpha/2,Q_T} \le C \tag{5.34}$$

are true.

Proof. We consider (2.31)–(2.33) again as parabolic equations for V, W, and θ, respectively, with Hölder continuous coefficients with exponent α. In the same way as before from (5.8) and (5.9) we get

$$|V|_{2+\alpha,1+\alpha/2,Q_T} \le C\left(1 + |\theta|_{2+\alpha,1+\alpha/2,Q_T}\right), \tag{5.35}$$

$$|W|_{2+\alpha,1+\alpha/2,Q_T} \le C, \tag{5.36}$$

and with the help of (5.26) from (5.10) we obtain

$$|\theta|_{2+\alpha,1+\alpha/2,Q_T} \le C\left(1 + \left|\frac{\partial V}{\partial x}\right|_{\alpha,\alpha/2,Q_T}\right). \tag{5.37}$$

Inserting (5.37) in (5.35), using (4.19) and (5.26) for the function V, we obtain (5.30). With the help of (4.19) and (5.30) from (5.37) it follows (5.32). In the same way as before we get

$$\left|\frac{\partial \rho}{\partial t}\right|_{\alpha,Q_T} \le \left|\rho^2\right|_{\alpha,Q_T}\left|\frac{\partial V}{\partial x}\right|_{0,Q_T} + \left|\rho^2\right|_{0,Q_T}\left|\frac{\partial V}{\partial x}\right|_{\alpha,Q_T} + \left|\frac{\mu}{l}\rho\right|_{\alpha,Q_T}. \tag{5.38}$$

Because of (5.27), (5.30), and (4.20) for $\eta = V$ we obtain

$$\left|\frac{\partial \rho}{\partial t}\right|_{\alpha,Q_T} \le C, \tag{5.39}$$

and using (5.29) we have (5.33). Taking into account (5.33) in the same way as in Lemma 5.4 we get (5.34). □

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