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Research Article

Blow up of the Solutions of Nonlinear Wave Equation

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We construct for every fixed $n \geq 2$ the metric $g_s = h_1(r)dt^2 - h_2(r)dr^2 - k_1(\omega)d\omega_1^2 - \cdots - k_{n-1}(\omega)d\omega_{n-1}^2$, where $h_1(r)$, $h_2(r)$, $k_i(\omega)$, $1 \leq i \leq n-1$, are continuous functions, $r = |x|$, for which we consider the Cauchy problem $(u_{tt} - \Delta u)_{gs} = f(u) + g(|x|)$, where $x \in \mathbb{R}^n$, $n \geq 2$; $u(1,x) = u_\circ(x) \in L^2(\mathbb{R}^n)$, $u_t(1,x) = u_1(x) \in \dot{H}^{-1}(\mathbb{R}^n)$, where $f \in \mathscr{C}^1(\mathbb{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathscr{C}(\mathbb{R}^+)$, $g(r) \geq 0$, $r = |x|$, a and b are positive constants. When $g(r) \equiv 0$, we prove that the above Cauchy problem has a nontrivial solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r)$ for which $\lim_{t\to 0}\|u\|_{L^2([0,\infty))} = \infty$. When $g(r) \neq 0$, we prove that the above Cauchy problem has a nontrivial solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r)$ for which $\lim_{t\to 0}\|u\|_{L^2([0,\infty))} = \infty$.

1. Introduction

In this paper, we study the properties of the solutions of the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad x \in \mathbb{R}^n,\ n \geq 2, \tag{1}$$

$$u(1,x) = u_\circ(x) \in L^2(\mathbb{R}^n), \qquad u_t(1,x) = u_1(x) \in \dot{H}^{-1}(\mathbb{R}^n), \tag{2}$$

where g_s is the metric

$$g_s = h_1(r)dt^2 - h_2(r)dr^2 - k_1(\omega)d\omega_1^2 - \cdots - k_{n-1}(\omega)d\omega_{n-1}^2, \tag{1.1}$$

the functions $h_1(r)$, $h_2(r)$ satisfy the conditions

$$h_1(r), h_2(r) \in \mathscr{C}^1([0,\infty)), \quad h_1(r) > 0,\ h_2(r) \geq 0 \quad \forall r \in [0,\infty),$$

$$\int_0^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau < \infty, \qquad \int_0^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{h_1(\tau)h_2(\tau)} d\tau\, ds < \infty,$$

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} C_1 + C_2 \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr < \infty,$$

C_1, C_2 are arbitrary nonnegative constants,

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} C_1 + C_2 \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right) ds \right)^2 dr < \infty,$$

C_1, C_2 are arbitrary nonnegative constants,

$$\max_{r \in [0,\infty)} \sqrt{h_1(r)h_2(r)} < \infty, \tag{i1}$$

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau\, ds \right)^2 dr < \infty,$$

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty h_1(\tau)h_2(\tau) d\tau \right)^{1/2} ds \right)^2 dr < \infty,$$

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \right)^2 dr < \infty,$$

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr < \infty,$$

$k_i(\omega) \in \mathscr{C}^1([0,2\pi] \times \cdots \times [0,2\pi])$, $i = 1, \ldots, n-1$, $f \in \mathscr{C}^1(\mathscr{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, a and b are positive constants, $g \in \mathscr{C}(\mathbb{R}^1)$, $g(|x|) \geq 0$ for $|x| \in [0,\infty)$. (In Section 2 we will give example for such metric g_s.)

We search a solution $u = u(t,r)$ to the Cauchy problem (1), (2). Therefore, if the Cauchy problem (1), (2) has such solution, it will satisfy the Cauchy problem

$$\frac{1}{h_1(r)} u_{tt} - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r \right) = f(u) + g(r), \tag{1.2}$$

$$u(1,r) = u_\circ \in L^2([0,\infty)), \qquad u_t(1,r) = u_1 \in \dot{H}^{-1}([0,\infty)). \tag{1.3}$$

In this paper, we will prove that the Cauchy problem (1), (2) has nontrivial solution $u = u(t,r)$ for which

$$\lim_{t \to 0} \|u\|_{L^2([0,\infty))} = \infty. \tag{1.4}$$

Our main results are the following.

Theorem 1.1. *Suppose $n \geq 2$ is fixed, $h_1(r)$, $h_2(r)$ satisfy the conditions (i1), $g \equiv 0$, $f \in \mathscr{C}^1(\mathbb{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, a and b are positive constants. Then the homogeneous problem of Cauchy (1), (2) has nontrivial solution $u = u(t,r) \in \mathscr{C}((0,1]L^2([0,\infty)))$ for which*

$$\lim_{t\to 0} \|u\|_{L^2([0,\infty))} = \infty. \tag{1.5}$$

Theorem 1.2. *Suppose $n \geq 2$ is fixed, $h_1(r)$, $h_2(r)$ satisfy the conditions (i1). Suppose also that a and b are fixed positive constants, $a \leq b$, $f \in \mathscr{C}^1(\mathbb{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $b/2 \geq f(1) \geq a/2$, $g \neq 0$, $g \in \mathscr{C}([0,\infty))$, $g(r) \geq 0$ for every $r \geq 0$, $g(r) \leq b/2 - f(1)$ for every $r \in [0,\infty)$. Then the nonhomogeneous problem of Cauchy (1), (2) has nontrivial solution $u = u(t,r) \in \mathscr{C}((0,1]L^2([0,\infty)))$ for which*

$$\lim_{t\to 0} \|u\|_{L^2([0,\infty))} = \infty. \tag{1.6}$$

When g_s is the Minkowski metric and $u_0, u_1 \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ in [1] (see also [2, Section 6.3]), it is proved that there exists $T > 0$ and a unique local solution $u \in \mathscr{C}^2([0,T) \times \mathbb{R}^3)$ for the Cauchy problem

$$\begin{gathered}(u_{tt} - \Delta u)_{g_s} = f(u), \quad f \in \mathscr{C}^2(\mathbb{R}),\ t \in [0,T],\ x \in \mathbb{R}^3,\\ u\big|_{t=0} = u_0, \qquad u_t\big|_{t=0} = u_1,\end{gathered} \tag{1.7}$$

for which

$$\sup_{t<T,x\in\mathbb{R}^3} \big|u(t,x)\big| = \infty. \tag{1.8}$$

When g_s is the Minkowski metric, $1 \leq p < 5$ and initial data are in $\mathscr{C}_0^\infty(\mathbb{R}^3)$ in [1] (see also [2, Section 6.3]), it is proved that the initial value problem

$$\begin{gathered}(u_{tt} - \Delta u)_{g_s} = u|u|^{p-1}, \quad t \in [0,T],\ x \in \mathbb{R}^3,\\ u\big|_{t=0} = u_0, \qquad u_t\big|_{t=0} = u_1\end{gathered} \tag{1.9}$$

admits a global smooth solution.

When g_s is the Minkowski metric and initial data are in $\mathscr{C}_0^\infty(\mathbb{R}^3)$ in [3] (see also [2, Section 6.3]) it is proved that there exists a number $\epsilon_0 > 0$ such that for any data $(u_0,u_1) \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ with $E(u(0)) < \epsilon_0$, the initial value problem

$$\begin{gathered}(u_{tt} - \Delta u)_{g_s} = u^5, \quad t \in [0,T],\ x \in \mathbb{R}^3,\\ u\big|_{t=0} = u_0, \qquad u_t\big|_{t=0} = u_1\end{gathered} \tag{1.10}$$

admits a global smooth solution.

When g_s is the Reissner-Nordström metric in [4], it is proved that the Cauchy problem

$$\begin{gathered}(u_{tt} - \Delta u)_{g_s} + m^2 u = f(u), \quad t \in [0,1],\ x \in \mathbb{R}^3,\\ u(1,x) = u_0 \in \dot{B}^{\gamma}_{p,p}(\mathbb{R}^3), \qquad u_t(1,x) = u_1 \in \dot{B}^{\gamma-1}_{p,p}(\mathbb{R}^3),\end{gathered} \tag{1.11}$$

where $m \neq 0$ is constant and $f \in \mathscr{C}^2(\mathbb{R}^1)$, $a|u| \le |f^{(l)}(u)| \le b|u|$, $l = 0,1$, a and b are positive constants, has unique nontrivial solution $u(t,r) \in \mathscr{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$, $r = |x|$, $p > 1$, for which

$$\lim_{t\to 0} \|u\|_{\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)} = \infty. \tag{1.12}$$

When g_s is the Minkowski metric in [5], it is proved that the Cauchy problem

$$\begin{aligned} (u_{tt} - \Delta u)_{g_s} &= f(u), \quad t \in [0,1],\ x \in \mathbb{R}^3, \\ u(1,x) &= u_0, \qquad u_t(1,x) = u_1 \end{aligned} \tag{1.13}$$

has global solution. Here $f \in \mathscr{C}^2(\mathbb{R})$, $f(0) = f'(0) = f''(0) = 0$,

$$|f''(u) - f''(v)| \le B|u - v|^{q_1} \tag{1.14}$$

for $|u| \le 1$, $|v| \le 1$, $B > 0$, $\sqrt{2} - 1 < q_1 \le 1$, $u_0 \in \mathscr{C}^5_\circ(\mathbb{R}^3)$, $u_1 \in \mathscr{C}^4_\circ(\mathbb{R}^3)$, $u_0(x) = u_1(x) = 0$ for $|x - x_0| > \rho$, x_0 and ρ are suitable chosen.

When g_s is the Reissner-Nordström metric, $n = 3$, $p > 1$, $q \ge 1$, $\gamma \in (0,1)$ are fixed constants, $f \in \mathscr{C}^1(\mathbb{R}^1)$, $f(0) = 0$, $a|u| \le f'(u) \le b|u|$, $g \in \mathscr{C}(\mathbb{R}^+)$, $g(|x|) \ge 0$, $g(|x|) = 0$ for $|x| \ge r_1$, a and b are positive constants, $r_1 > 0$ is suitable chosen, in [6], it is proved that the initial value problem (1), (2) has nontrivial solution $u \in \mathscr{C}((0,1]\dot{B}^{\gamma}_{p,q}(\mathbb{R}^+))$ in the form

$$u(t,r) = \begin{cases} v(t)\omega(r), & \text{for } r \le r_1,\ t \in [0,1], \\ 0, & \text{for } r \ge r_1,\ t \in [0,1], \end{cases} \tag{1.15}$$

where $r = |x|$, for which $\lim_{t\to 0} \|u\|_{\dot{B}^{\gamma}_{p,q}(\mathbb{R}^+)} = \infty$.

The paper is organized as follows. In Section 2, we will prove some preliminary results. In Section 3, we will prove Theorem 1.1. In Section 4, we will prove Theorem 1.2. In the appendix we will prove some results which are used for the proof of Theorems 1.1 and 1.2.

2. Preliminary results

Proposition 2.1. *Let $h_1(r)$, $h_2(r)$ satisfy the conditions (i1), $f \in \mathscr{C}(-\infty,\infty)$, $g \equiv 0$. If for every fixed $t \in [0,1]$ the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies (1), then the function $u(t,r) = v(t)\omega(r)$ satisfies the integral equation*

$$u(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}f(u)\right) d\tau\, ds \tag{1*}$$

for every fixed $t \in [0,1]$.

Proof. Suppose that $t \in [0,1]$ is fixed and the function $u(t,r) = v(t)\omega(r)$, $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies (1). Then for every fixed $t \in [0,1]$ and for $r \in [0,\infty)$ we have

$$\begin{aligned}
u_{tt}(t,r) &= \frac{v''(t)}{v(t)}u(t,r),\\
\frac{1}{h_1(r)}\frac{v''(t)}{v(t)}u(t,r) - \frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= f(u),\\
\frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= \frac{1}{h_1(r)}\frac{v''(t)}{v(t)}u(t,r) - f(u),\\
\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= \sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,r) - \sqrt{h_1(r)h_2(r)}f(u).
\end{aligned} \tag{2.1}$$

Now we integrate the last equality from r to ∞ here we suppose that $u_r(t,r) = v(t)\omega'(r)$, $u_r(t,\infty) = v(t)\omega'(\infty) = 0$, then we get

$$\begin{aligned}
-\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r) &= \int_r^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau,\\
-\sqrt{\frac{h_1(r)}{h_2(r)}}u_r(t,r) &= \int_r^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau,\\
-u_r(t,r) &= \sqrt{\frac{h_2(r)}{h_1(r)}}\int_r^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau.
\end{aligned} \tag{2.2}$$

Now we integrate the last equality from r to ∞; we use that $u(t,\infty) = v(t)\omega(\infty) = 0$, then we get

$$u(t,r) = \int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau\,ds, \tag{2.3}$$

that is, for every fixed $t \in [0,1]$ if the function $u(t,r) = v(t)\omega(r)$ satisfies (1), then the function $u(t,r) = v(t)\omega(r)$ satisfies the integral equation (1*). Here $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$. □

Proposition 2.2. *Let $h_1(r)$, $h_2(r)$ satisfy the conditions (i1), $f \in \mathscr{C}(-\infty,\infty)$, $g \equiv 0$. If for every fixed $t \in [0,1]$ the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies the integral equation (1*) then the function $u(t,r) = v(t)\omega(r)$ satisfies (1) for every fixed $t \in [0,1]$.*

Proof. Let $t \in [0,1]$ be fixed and let the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfy the integral equation (1*). From here and from $f \in \mathscr{C}(-\infty,\infty)$, for every fixed $t \in [0,1]$ we have

$u(t,r) \in \mathscr{C}^2([0,\infty))$ and

$$
\begin{aligned}
u_r(t,r) &= -\sqrt{\frac{h_2(r)}{h_1(r)}}\int_r^\infty\left(\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau,\\
\sqrt{\frac{h_1(r)}{h_2(r)}}u_r(t,r) &= -\int_r^\infty\left(\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau,\\
\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r) &= -\int_r^\infty\left(\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}f(u)\right)d\tau,\\
\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= \sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,r)-\sqrt{h_1(r)h_2(r)}f(u),\\
\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,r)-\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= \sqrt{h_1(r)h_2(r)}f(u),\\
\frac{1}{h_1(r)}\frac{v''(t)}{v(t)}u(t,r)-\frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= f(u).
\end{aligned}
\tag{2.4}
$$

Since for every fixed $t \in [0,1]$ we have

$$
\frac{v''(t)}{v(t)}u(t,r) = u_{tt}(t,r), \tag{2.5}
$$

we get

$$
\frac{1}{h_1(r)}u_{tt}(t,r)-\frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) = f(u), \tag{2.6}
$$

that is, for every fixed $t \in [0,1]$ if the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies (1*), then it satisfies (1) for every fixed $t \in [0,1]$. □

Proposition 2.3. *Let $h_1(r)$, $h_2(r)$ satisfy the conditions (i1), $f \in \mathscr{C}(-\infty,\infty)$, $g \in \mathscr{C}([0,\infty))$, $g(r) \geq 0$ for every $r \geq 0$. If for every fixed $t \in [0,1]$ the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies (1), then the function $u(t,r) = v(t)\omega(r)$ satisfies the integral equation*

$$
u(t,r) = \int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}\left(f(u)+g(r)\right)\right)d\tau\, ds \tag{1**}
$$

for every fixed $t \in [0,1]$.

Proof. Let $t \in [0,1]$ be fixed and let the function $u(t,r) = v(t)\omega(r)$, $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfy (1). Then for every fixed $t \in [0,1]$ and for $r \in [0,\infty)$ we have

$$\begin{aligned}
u_{tt}(t,r) &= \frac{v''(t)}{v(t)}u(t,r),\\
\frac{1}{h_1(r)}\frac{v''(t)}{v(t)}u(t,r) - \frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= (f(u)+g(r)),\\
\frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= \frac{1}{h_1(r)}\frac{v''(t)}{v(t)}u(t,r) - (f(u)+g(r)),\\
\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) &= \sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,r) - \sqrt{h_1(r)h_2(r)}(f(u)+g(r)).
\end{aligned} \tag{2.7}$$

Now we integrate the last equality from r to ∞; here we suppose that $u_r(t,r) = v(t)\omega'(r)$, $u_r(t,\infty) = v(t)\omega'(\infty) = 0$, then we get

$$\begin{aligned}
-\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r) &= \int_r^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau,\\
-\sqrt{\frac{h_1(r)}{h_2(r)}}u_r(t,r) &= \int_r^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau,\\
-u_r(t,r) &= \sqrt{\frac{h_2(r)}{h_1(r)}}\int_r^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau.
\end{aligned} \tag{2.8}$$

Now we integrate the last equality from r to ∞; we suppose that $u(t,\infty) = v(t)\omega(\infty) = 0$, then we get

$$u(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau\, ds, \tag{2.9}$$

that is, for every fixed $t \in [0,1]$ if the function $u(t,r) = v(t)\omega(r)$ satisfies (1), then the function $u(t,r) = v(t)\omega(r)$ satisfies the integral equation (1**). Here $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$. □

Proposition 2.4. *Let $h_1(r)$, $h_2(r)$ satisfy the conditions (i1), $f \in \mathscr{C}(-\infty,\infty)$, $g \in \mathscr{C}([0,\infty))$, $g(r) \geq 0$ for every $r \geq 0$. If for every fixed $t \in [0,1]$ the function $u(t,r) = v(t)\omega(r)$, where*

*$v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies the integral equation (1**), then the function $u(t,r) = v(t)\omega(r)$ satisfies (1) for every fixed $t \in [0,1]$.*

Proof. Let $t \in [0,1]$ be fixed and let the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfy the integral equation (1**). From here and from $f \in \mathscr{C}(-\infty,\infty)$, $g \in \mathscr{C}([0,\infty))$, for every fixed $t \in [0,1]$ we have $u(t,r) \in \mathscr{C}^2([0,\infty))$ and

$$u_r(t,r) = -\sqrt{\frac{h_2(r)}{h_1(r)}}\int_r^\infty \left(\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau,$$

$$\sqrt{\frac{h_1(r)}{h_2(r)}}u_r(t,r) = -\int_r^\infty \left(\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau,$$

$$\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r) = -\int_r^\infty \left(\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(r))\right)d\tau,$$

$$\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) = \sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,r) - \sqrt{h_1(r)h_2(r)}(f(u)+g(r)),$$

$$\sqrt{\frac{h_2(r)}{h_1(r)}}\frac{v''(t)}{v(t)}u(t,r) - \partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) = \sqrt{h_1(r)h_2(r)}(f(u)+g(r)),$$

$$\frac{1}{h_1(r)}\frac{v''(t)}{v(t)}u(t,r) - \frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) = f(u)+g(r). \tag{2.10}$$

Since for every fixed $t \in [0,1]$ we have

$$\frac{v''(t)}{v(t)}u(t,r) = u_{tt}(t,r), \tag{2.11}$$

we get

$$\frac{1}{h_1(r)}u_{tt}(t,r) - \frac{1}{\sqrt{h_1(r)h_2(r)}}\partial_r\left(\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}}u_r(t,r)\right) = f(u)+g(r), \tag{2.12}$$

that is, for every fixed $t \in [0,1]$ if the function $u(t,r) = v(t)\omega(r)$, where $v(t) \in \mathscr{C}^4([0,1])$, $v(t) \neq 0$ for every $t \in [0,1]$, $\omega(r) \in \mathscr{C}^2([0,\infty))$, $\omega(\infty) = \omega'(\infty) = 0$, satisfies (1**), then it satisfies (1) for every fixed $t \in [0,1]$. □

For fixed $n \geq 2$, $h_1(r)$, $h_2(r)$ which satisfy the conditions $(i1)$ and fixed positive constants a and b, we suppose that the positive constants c, d, A, B, A_1, A_2 satisfy the conditions

$$\begin{gathered}
c \leq d, \qquad A \geq B, \qquad A_1 \leq A_2, \qquad A_1 - \frac{b}{2B} > 0, \\
\sqrt{\frac{h_2(r)}{h_1(r)}}A_1 - \frac{b}{B}\sqrt{h_1(r)h_2(r)} \geq 0 \quad \text{for every } r \in [0,\infty), \\
\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \geq 1 \quad \text{for } r \in [c,d], \\
\int_1^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(A_1\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} - \frac{b}{B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \geq \frac{A_1}{10^{10}}
\end{gathered} \tag{H1}$$

$$\begin{gathered}
\max_{r\in[0,\infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right) d\tau\, ds \leq 1, \\
\max_{r\in[0,\infty)} \sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right) d\tau \leq 1,
\end{gathered} \tag{H2}$$

$$\begin{gathered}
\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr < \frac{1}{7}, \\
\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \frac{b}{B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds\right)^2 dr < \infty,
\end{gathered} \tag{H3}$$

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr < 1. \tag{H4}$$

Example 2.5. Let $0 < \epsilon \ll 1/3$ be enough small, $n \geq 2$ is fixed. We choose $c > 0$, $d > 0$, $c \leq d < \infty$ such that for every $r \in [c,d]$ we have

$$\frac{\pi}{4} \leq \operatorname{arctg}(d+1-r)^3, \quad \operatorname{arctg} d^3 < \frac{\pi}{3}. \tag{2.13}$$

Let also $b = 8\epsilon^3$, $a = 4\epsilon^3$, $A = 60$, $B = 40$, $A_1 = \epsilon^3$, $A_2 = 2\epsilon^3$. Let

$$h_1(r) = \left(\frac{B}{b}\left(-1 + \sqrt{1 + 2\frac{A_1 b}{B}}\right)\right)^2, \qquad h_2(r) = \frac{144(d+1-r)^4}{\left[(d+1-r)^6+1\right]^2}. \tag{2.14}$$

We note that the functions $h_1(r)$ and $h_2(r)$ satisfy all conditions of $(i1)$ and

$$\frac{A_1}{\sqrt{h_1(r)}} - \frac{b}{2B}\sqrt{h_1(r)} = 1,$$

$$\begin{aligned}
&\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \\
&\quad \geq \int_r^{d+1} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^{d+1} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \geq 1 \quad \text{for } r \in [c,d].
\end{aligned} \tag{2.15}$$

We note that $\sqrt{h_1(r)} \sim 1{,}6$.

For fixed $n \geq 2$, $h_1(r)$, $h_2(r)$, which satisfy the conditions $(i1)$, the constants a, b, c, d, A, B, A_1, A_2 are fixed which satisfy the conditions $(H1), \dots, (H4)$, then we suppose that the function $v(t)$ is fixed function and satisfies the conditions

$$v(t) \in \mathscr{C}^4([0,1]), \quad \frac{v''(t)}{v(t)} > 0, \quad v(t) > 0, \quad \forall t \in [0,1], \tag{H5}$$

$$A_1 \leq \frac{v''(t)}{v(t)} \leq A_2, \qquad v'''(1) = 0, \qquad v'(1) = 0, \tag{H6}$$

$$\lim_{t\to 0}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right) = +0. \tag{H7}$$

Example 2.6. Let a, b, c, d, A_1, A_2, B, A be the constants from the above example. Then $a/2 = A_2$ and

$$v(t) = C\left(e^{\sqrt{A_2}(t-1)} + e^{-\sqrt{A_2}(t-1)}\right), \tag{2.16}$$

where C is arbitrary positive constant, satisfiing the hypotheses $(H5)$, $(H6)$, $(H7)$.

Here and below we suppose that $v(t)$ is fixed function which satisfies the conditions $(H5), \dots, (H7)$.

When $g(r) \equiv 0$ we put

$$\begin{gathered}
u_\circ := v(1)\omega(r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(v(1)\omega(\tau))\right) d\tau\, ds, \\
u_1 \equiv 0.
\end{gathered} \tag{1'}$$

In Section 3, we will prove that $(1')$ has unique nontrivial solution $\omega(r) \in L^2([0,\infty))$.

When $g(r) \neq 0$ we put

$$u_\circ := v(1)\omega(r)$$
$$= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)}\big(f(v(1)\omega(\tau)) + g(\tau)\big)\right) d\tau\, ds,$$
$$u_1 \equiv 0. \tag{1''}$$

In Section 4, we will prove that $(1'')$ has unique nontrivial solution $\omega(r) \in L^2([0,1))$.

3. Proof of Theorem 1.1

3.1. Local existence of nontrivial solutions of homogeneous Cauchy problem (1), (2). In this section, we will prove that the homogeneous Cauchy problem (1), (2) has nontrivial solution in the form $u(t,r) = v(t)\omega(r)$.

For fixed function $v(t)$, which satisfies the conditions $(H5)$, $(H6)$, and $(H7)$ we consider the integral equation

$$u(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u)\right) d\tau\, ds. \tag{1*}$$

Theorem 3.1. *Let $n \geq 2$ be fixed, let $h_1(r)$, $h_2(r)$ fixed, which satisfy the conditions $(i1)$, let the positive constants a, b be fixed, $a \leq b$, let the positive constants c, d. A, B, A_1, A_2 be fixed which satisfy the conditions $(H1), \dots, (H4)$ and $f \in \mathscr{C}^1((-\infty,\infty))$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$. Let also $v(t)$ be fixed function which satisfies the conditions $(H5), \dots, (H7)$. Then $(1^\star)$ has unique nontrivial solution $u(t,r) = v(t)\omega(r)$ for which $u(t,r) \in \mathscr{C}([0,1] \times [0,\infty))$, $u(t,r) \leq 1/B$ for every $t \in [0,1]$ and for every $r \in [0,\infty)$, $u(t,r) \geq 1/A$ for every $t \in [0,1]$ and for every $r \in [c,d]$, $u(t,r) \geq 0$ for every $t \in [0,1]$ and for every $r \in [0,\infty)$, $u(t,\infty) = u_r(t,\infty) = 0$ for every $t \in [0,1]$, $u(t,r) \in C((0,1]L^2([0,\infty)))$.*

Proof. Let M be the set

$$M = \Big\{u(t,r) : u(t,r) \in \mathscr{C}([0,1]\times[0,\infty)),\ u(t,\infty) = u_r(t,\infty) = 0\ \forall t \in [0,1],$$
$$u(t,r) \geq \frac{1}{A} \text{ for } t \in [0,1],\ r \in [c,d],\ u(t,r) \leq \frac{1}{B}\ \forall t \in [0,1],\ \forall r \in [0,\infty),$$
$$u(t,r) \geq 0\ \forall t \in [0,1],\ \forall r \in [0,\infty),\ u(t,r) \in L^2([0,\infty)) \text{ for every } t \in (0,1]\Big\}. \tag{3.1}$$

Let $t \in [0,1]$ be fixed. We define the operator L as follows:

$$L(u)(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u)\right) d\tau\, ds \tag{3.2}$$

for $u \in M$. First we will see that $L : M \to M$. Let $u \in M$. Then the following holds.

(1) Since $v(t) \in \mathscr{C}^4([0,1])$, $u(t,r) \in \mathscr{C}([0,1] \times [0,\infty))$, and $f \in \mathscr{C}(-\infty,\infty)$ and from $(i1)$ we have that $L(u) \in \mathscr{C}([0,1] \times [0,\infty))$. Also we have

$$\begin{gathered} L(u)\big|_{r=\infty} = 0, \\ \frac{\partial}{\partial r} L(u) = -\sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau) h_2(\tau)} f(u) \right) d\tau, \\ \frac{\partial}{\partial r} L(u)\big|_{r=\infty} = 0. \end{gathered} \tag{3.3}$$

(2) Now we will prove that for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have that $L(u) \geq 0$. Really,

$$\begin{aligned} L(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau) h_2(\tau)} f(u) \right) d\tau \, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \min_{t \in [0,1]} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau) h_2(\tau)} f(u) \right) d\tau \, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \sqrt{h_1(\tau) h_2(\tau)} f(u) \right) d\tau \, ds. \end{aligned} \tag{3.4}$$

Now we suppose that for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have $u(t,r) \geq 0$; from here $f'(u) \leq bu$, since $f(0) = 0$ and $u(t,r) \leq 1/B$ for every $t \in [0,1]$ and for every $r \in [0,\infty)$ we get

$$\begin{aligned} f(u) &\leq \frac{b}{2} u^2 \leq \frac{b}{2B} u(t,r) \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \frac{b}{2B} \sqrt{h_1(\tau) h_2(\tau)} u \right) d\tau \, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau) h_2(\tau)} \right) u \, d\tau \, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau \in [0,\infty)} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau) h_2(\tau)} \right) u \, d\tau \, ds. \end{aligned} \tag{3.5}$$

From $(H1)$ we have that

$$\min_{\tau \in [0,\infty)} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau) h_2(\tau)} \right) \geq 0 \tag{3.6}$$

and since $u(t,r) \geq 0$ for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we get

$$\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau\in[0,\infty)} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right) u\, d\tau\, ds \geq 0, \tag{3.7}$$

that is, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$L(u) \geq 0. \tag{3.8}$$

(3) Now we will see that for every fixed $t \in [0,1]$ and for every $r \in [c,d]$ we have $L(u) \geq 1/A$. Really, for every $r \in [c,d]$ we have

$$\begin{aligned} L'(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)} - \sqrt{h_1(\tau)h_2(\tau)}f'(u)\right) d\tau\, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \geq 0. \end{aligned} \tag{3.9}$$

(See $(H1)$.) From here, since for every $r \in [c,d]$ we have $u(t,r) \geq 1/A$, we get

$$\begin{aligned} L(u) \geq L\left(\frac{1}{A}\right) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}\frac{1}{A} - \sqrt{h_1(\tau)h_2(\tau)}f\left(\frac{1}{A}\right)\right) d\tau\, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{A_1}{A} - \frac{b}{2A^2}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \\ &= \frac{1}{A}\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{2A}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \\ &\geq \frac{1}{A}\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds \geq \frac{1}{A}. \end{aligned} \tag{3.10}$$

(Here we use $(H1)$.) Consequently, for every fixed $t \in [0,1]$ and for every $r \in [c,d]$ we have that

$$L(u) \geq \frac{1}{A}. \tag{3.11}$$

(4) Now we will prove that for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have that

$$L(u) \leq \frac{1}{B}. \tag{3.12}$$

Really, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$\begin{aligned} L(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau\, ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u + \sqrt{h_1(\tau)h_2(\tau)} \,|\, f(u) \,|\, \right) d\tau\, ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} |u| + \sqrt{h_1(\tau)h_2(\tau)} \,|\, f(u) \,|\, \right) d\tau\, ds. \end{aligned} \tag{3.13}$$

Now we suppose that

$$\begin{aligned} &|\, f(u) \,| \leq \frac{b}{2} u^2 \leq \frac{b}{2B} u, \qquad \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \leq A_2, \\ &\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 u + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} u \right) d\tau\, ds \\ &\qquad = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) u\, d\tau\, ds; \end{aligned} \tag{3.14}$$

here we use

$$\begin{aligned} u &\leq \frac{1}{B} \leq \frac{1}{B} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau\, ds \\ &\leq \frac{1}{B} \max_{r\in[0,\infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau\, ds \leq \frac{1}{B}. \end{aligned} \tag{3.15}$$

In the last inequality, we use $(H2)$. Consequently, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$L(u) \leq \frac{1}{B}. \tag{3.16}$$

(5) Now we will prove that for every fixed $t \in [0,1]$ we have that $L(u) \in L^2([0\infty))$. Really, for every fixed $t \in [0,1]$ after we use the inequality (3.13) we get

$$\begin{aligned} &\int_0^\infty |\, L(u) \,|^2 dr \\ &\quad \leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) |u|\, d\tau\, ds \right)^2 dr. \end{aligned} \tag{3.17}$$

Now we use the Hölder inequality

$$\begin{aligned}
&\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2}\right.\\
&\qquad \left.\times\left(\int_s^\infty |u|^2 d\tau\right)^{1/2} ds\right)^2 dr\\
&\quad\leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2}\right.\\
&\qquad\qquad \left.\times\left(\int_0^\infty |u|^2 d\tau\right)^{1/2} ds\right)^2 dr\\
&\quad= \|u\|^2_{L^2([0,\infty))}\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr\\
&\quad\leq \|u\|^2_{L^2([0,\infty))}\frac{1}{7}.
\end{aligned} \tag{3.18}$$

(In the last inequality, we use the condition $(H3)$.) Consequently, for every fixed $t\in[0,1]$ we have $L(u)\in L^2([0,\infty))$.

From (1), (2), (3), (4), and (5) we get that for every fixed $t\in[0,1]$ we have

$$L: M\longrightarrow M. \tag{3.19}$$

Now we will prove that the operator $L: M\to M$ is contractive operator. Let u_1 and u_2 be two elements of the set M. Then, for every fixed $t\in[0,1]$ we have

$$\begin{aligned}
&|L(u_1)-L(u_2)|\\
&\quad=\left|\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}(u_1-u_2)-\sqrt{h_1(\tau)h_2(\tau)}(f(u_1)-f(u_2))\right)d\tau\, ds\right|\\
&\quad\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}|u_1-u_2|+\sqrt{h_1(\tau)h_2(\tau)}|f(u_1)-f(u_2)|\right)d\tau\, ds,
\end{aligned} \tag{3.20}$$

then from the middle-point theorem we have $|f(u_1) - f(u_2)| = |f'(\xi)||u_1 - u_2|$, $|\xi| \le \max\{|u_1|, |u_2|\}$, $|f'(\xi)| \le b|\xi| \le b/B$,

$$\begin{aligned}
&\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} |u_1 - u_2| \right) d\tau\, ds \\
&\quad = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2|\, d\tau\, ds \\
&\quad \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2|\, d\tau\, ds \\
&\quad \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2|\, d\tau\, ds,
\end{aligned} \tag{3.21}$$

that is, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$|L(u_1) - L(u_2)| \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2|\, d\tau\, ds. \tag{3.22}$$

From here

$$\begin{aligned}
&\|L(u_1) - L(u_2)\|^2_{L^2([0,\infty))} \\
&\quad \le \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2|\, d\tau\, ds \right)^2 dr.
\end{aligned} \tag{3.23}$$

Now we will use the Hölder inequality

$$\begin{aligned}
&\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} \right. \\
&\qquad \left. \times \left(\int_s^\infty |u_1 - u_2|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
&\quad \le \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} \right. \\
&\qquad\qquad \left. \times \left(\int_0^\infty |u_1 - u_2|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
&\quad = \|u_1 - u_2\|^2_{L^2([0,\infty))} \\
&\quad \times \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr,
\end{aligned} \tag{3.24}$$

that is, for every fixed $t \in [0,1]$ we have

$$||L(u_1) - L(u_2)||^2_{L^2([0,\infty))} \leq ||u_1 - u_2||^2_{L^2([0,\infty))} \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr, \tag{3.25}$$

from $(H4)$ we have

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr < 1. \tag{3.26}$$

From here and from (3.25) we get

$$||L(u_1) - L(u_2)||^2_{L^2([0,\infty))} < ||u_1 - u_2||_{L^2([0,\infty))}. \tag{3.27}$$

Consequently, the operator $L : M \to M$ is contractive operator. We note that the set M is closed subset of the space $\mathscr{C}((0,1]L^2([0,\infty)))$ (for the proof see Lemma A.1 in the appendix of this paper). Therefore, $(1^\star)$ has unique nontrivial solution in the set M. □

Let $\tilde{u}$ be the solution from Theorem 3.1, that is, $\tilde{u}$ is a solution to the integral equation $(1^\star)$. From Proposition 2.2, we have that $\tilde{u}$ satisfies (1). Consequently, $\tilde{u}$ is solution to the Cauchy problem (1), (2) with initial data

$$u_\circ = v(1)\omega(r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(v(1)\omega(\tau)) \right) d\tau\, ds,$$
$$u_1 \equiv 0. \tag{3.28}$$

We have $\tilde{u} \in \mathscr{C}((0,1]L^2([0,\infty)))$, $u_\circ \in L^2([0,\infty))$, $u_1 \in \dot{H}^{-1}([0,\infty))$.

3.2. Blow up of the solutions of homogeneous Cauchy problem (1), (2). Let $v(t)$ be the same function as in Theorem 3.1.

THEOREM 3.2. *Let $n \geq 2$ be fixed, let $h_1(r)$, $h_2(r)$ fixed, which satisfy the conditions $(i1)$, be the positive constants a, b be fixed, let the positive constants c, d, A, B, A_1, A_2 be fixed which satisfy the conditions $(H1), \ldots, (H4)$ and $f \in \mathscr{C}^1((-\infty,\infty))$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$. Then for the solution $\tilde{u}$ of the Cauchy problem (1), (2) one has*

$$\lim_{t \to 0} ||\tilde{u}||_{L^2([0,\infty))} = \infty. \tag{3.29}$$

Proof. For every fixed $t \in (0,1]$ and for every $r \in [0,\infty)$ we have

$$\tilde{u}(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau\, ds. \tag{3.30}$$

Since for every fixed $t \in (0,1]$ and for every $r \in [c,d]$ we have that $\widetilde{u} \geq 1/A$ follows that there exists subinterval Δ in $[0,\infty)$ such that

$$\widetilde{u} \geq \frac{1}{A} \quad \text{for } r \in \Delta. \tag{3.31}$$

Let us fix the subinterval Δ. From here

$$\begin{aligned}
&\|\widetilde{u}\|^2_{L^2([0,\infty))} \\
&\quad= \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u}) \right) d\tau\, ds \right)^2 dr \\
&\quad= \int_{[0,\infty)\setminus\Delta} \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u}) \right) d\tau\, ds \right)^2 dr \\
&\qquad+ \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u}) \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{3.32}$$

Let

$$\begin{aligned}
I_1 :&= \int_{[0,\infty)\setminus\Delta} \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u}) \right) d\tau\, ds \right)^2 dr, \\
I_2 :&= \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u}) \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{3.33}$$

Then

$$\|\widetilde{u}\|^2_{L^2([0,\infty))} = I_1 + I_2. \tag{3.34}$$

For I_1 we have the following estimate:

$$\begin{aligned}
I_1 &\leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) + \sqrt{h_1(\tau)h_2(\tau)} \,|\, f(\widetilde{u}) \,|\, \right) d\tau\, ds \right)^2 dr \\
&\leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) + \sqrt{h_1(\tau)h_2(\tau)} \,|\, f(\widetilde{u}) \,|\, \right) d\tau\, ds \right)^2 dr \\
&\leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \widetilde{u}(t,\tau) + \sqrt{h_1(\tau)h_2(\tau)} \,|\, f(\widetilde{u}) \,|\, \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{3.35}$$

Now we suppose that $f'(\widetilde{u}) \leq b\widetilde{u}$, $f(0) = 0$, $\widetilde{u} \leq 1/B$ for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$. Therefore $|f(\widetilde{u})| \leq (b/2)\widetilde{u}^2 \leq (b/2B)\widetilde{u}$ for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$.

$$\begin{aligned}
&\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \widetilde{u}(t,\tau) + \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\widetilde{u}\right) d\tau\, ds\right)^2 dr \\
&\quad = \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right) \widetilde{u}\, d\tau\, ds\right)^2 dr.
\end{aligned} \tag{3.36}$$

Now we apply the Hölder inequality

$$\begin{aligned}
&\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)^2 d\tau\right)^{1/2} \left(\int_s^\infty |\widetilde{u}|^2 d\tau\right)^{1/2} ds\right)^2 dr \\
&\quad \leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)^2 d\tau\right)^{1/2} \left(\int_0^\infty |\widetilde{u}|^2 d\tau\right)^{1/2} ds\right)^2 dr \\
&\quad = \|\widetilde{u}\|^2_{L^2[0,\infty)} \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr.
\end{aligned} \tag{3.37}$$

Let

$$Q := \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr. \tag{3.38}$$

Then

$$I_1 \leq Q\|\widetilde{u}\|^2_{L^2([0,\infty))}. \tag{3.39}$$

Now we consider I_2. For it we have

$$\begin{aligned}
I_2 &= \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u})\right) d\tau\, ds\right)^2 dr \\
&\leq \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u})\right) d\tau\, ds\right)^2 dr \\
&= \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u})\right) d\tau\, ds\right. \\
&\qquad \left. + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0,\infty)\setminus\Delta} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \widetilde{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\widetilde{u})\right) d\tau\, ds\right)^2 dr.
\end{aligned} \tag{3.40}$$

Let

$$I_{21}=\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_\Delta\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}f(\widetilde{u})\right)d\tau\,ds,$$
$$I_{22}=\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{[0,\infty)\setminus\Delta}\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}f(\widetilde{u})\right)d\tau\,ds. \tag{3.41}$$

Consequently,

$$I_2=\int_\Delta\left(I_{21}+I_{22}\right)^2dr\le 2\int_\Delta I_{21}^2dr+2\int_\Delta I_{22}^2dr. \tag{3.42}$$

Also we have

$$\begin{aligned}
&\int_\Delta I_{21}^2dr\\
&\quad=\int_\Delta\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_\Delta\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau)-\sqrt{h_1(\tau)h_2(\tau)}f(\widetilde{u})\right)d\tau\,ds\right)^2dr\\
&\quad=\int_\Delta\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_\Delta\sqrt{h_1(\tau)h_2(\tau)}\left(\frac{1}{h_1(\tau)}\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau)-f(\widetilde{u})\right)d\tau\,ds\right)^2dr\\
&\quad\le\int_\Delta\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_\Delta\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\left(\max_{\tau\in[0,\infty)}\frac{1}{h_1(\tau)}\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau)-f(\widetilde{u})\right)d\tau\,ds\right)^2dr.
\end{aligned} \tag{3.43}$$

Case 1. Let

$$\max_{r\in[0,\infty)}\frac{1}{h_1(r)}\le\frac{1}{A}. \tag{3.44}$$

Then, after we suppose that for every fixed $t\in[0,1]$ and for every $r\in\Delta$ we have $f(\widetilde{u})\ge(a/2)\widetilde{u}^2\ge(a/2A)\widetilde{u}$, we get

$$\begin{aligned}
&\int_\Delta\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_\Delta\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\left(\max_{\tau\in[0,\infty)}\frac{1}{h_1(\tau)}\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau)-\frac{a}{2A}\widetilde{u}\right)d\tau\,ds\right)^2dr\\
&\quad\le\int_\Delta\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_\Delta\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)}-\frac{a}{2}\right)\widetilde{u}\,d\tau\,ds\right)^2dr.
\end{aligned} \tag{3.45}$$

Now we suppose that for every fixed $t \in [0,1]$ and for every $r \in \Delta$ we have $\widetilde{u} \geq 1/A$, $A\widetilde{u} \geq 1$,

$$\begin{aligned}
&\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta A\widetilde{u}\frac{1}{A} d\tau\, ds\right)^2 dr \\
&\quad \leq \left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta A^2\widetilde{u}^2\frac{1}{A} d\tau\, ds\right)^2 dr \\
&\quad \leq A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \widetilde{u}^2 d\tau\, ds\right)^2 dr \\
&\quad = \|\widetilde{u}\|^4_{L^2([0,\infty))} A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds\right)^2 dr,
\end{aligned} \tag{3.46}$$

that is,

$$\int_0^\infty I_{21}^2 dr \leq \|\widetilde{u}\|^4_{L^2([0,\infty))} A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds\right)^2 dr. \tag{3.47}$$

Let

$$F = A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds\right)^2 dr. \tag{3.48}$$

Then

$$\int_\Delta I_{21}^2 dr \leq F\|\widetilde{u}\|^4_{L^2[0,\infty)}. \tag{3.49}$$

Case 2. Let

$$\max_{r\in[0,\infty)} \frac{1}{h_1(r)} \geq \frac{1}{A}. \tag{3.50}$$

Let

$$\max_{r\in[0,\infty)} \frac{1}{h_1(r)} = G. \tag{3.51}$$

Then

$$\begin{aligned}
\int_\Delta I_{21}^2 dr \leq{}& 2\int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\left(\left(G - \frac{1}{A}\right)\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau) - f(\widetilde{u})\right) d\tau\, ds\right)^2 dr \\
&+ 2\int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)}\widetilde{u}(t,\tau) - \frac{a}{2}\widetilde{u}\right) d\tau\, ds\right)^2 dr.
\end{aligned} \tag{3.52}$$

As mentioned above

$$2Q\|\widetilde{u}\|^2_{L^2([0,\infty))} + 2F\|\widetilde{u}\|^4_{L^2([0,\infty))}, \tag{3.53}$$

that is,

$$\int_\Delta I_{21}^2 dr \leq 2Q\|\widetilde{u}\|^2_{L^2([0,\infty))} + 2F\|\widetilde{u}\|^4_{L^2([0,\infty))}. \tag{3.54}$$

From (3.49) and (3.54) we have

$$\int_\Delta I_{21}^2 dr \leq 2Q\|\widetilde{u}\|^2_{L^2([0,\infty))} + 2F\|\widetilde{u}\|^4_{L^2([0,\infty))}. \tag{3.55}$$

As in the estimate for I_1 we have

$$\int_0^\infty I_{22}^2 dr \leq Q\|\widetilde{u}\|^2_{L^2[0,\infty)}. \tag{3.56}$$

From (3.42), (3.55), and (3.56) we get

$$I_2 \leq 4F\|\widetilde{u}\|^4_{L^2([0,\infty))} + 6Q\|\widetilde{u}\|^2_{L^2[0,\infty)}. \tag{3.57}$$

From the last inequality and from (3.34), (3.39) we have

$$\begin{gathered}\|\widetilde{u}\|^2_{L^2([0,\infty))} \leq 4F\|\widetilde{u}\|^4_{L^2([0,\infty))} + 7Q\|\widetilde{u}\|^2_{L^2([0,\infty))}, \\ (1-7Q)\|\widetilde{u}\|^2_{L^2([0,\infty))} \leq 4F\|\widetilde{u}\|^4_{L^2([0,\infty))}.\end{gathered} \tag{3.58}$$

From $(H3)$ we have

$$Q < \frac{1}{7}, \tag{3.59}$$

from here

$$\|\widetilde{u}\|^2_{L^2([0,\infty))} \geq \frac{1-7Q}{4F}. \tag{3.60}$$

From $(H7)$ we have

$$\lim_{t\to 0} F = +0. \tag{3.61}$$

Therefore

$$\lim_{t\to 0} \|\widetilde{u}\|_{L^2([0,\infty))} = \infty. \tag{3.62}$$

□

4. Proof of Theorem 1.2

4.1. Local existence of nontrivial solutions of nonhomogeneous Cauchy problem (1), (2). In this section we will prove that the nonhomogeneous Cauchy problem (1), (2) has nontrivial solution in the form $u(t,r) = v(t)\omega(r)$.

Let us consider the integral equation

$$u(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u)+g(\tau)) \right) d\tau\, ds. \tag{$1^{\star\star}$}$$

Theorem 4.1. *Let $n \geq 2$ be fixed, let $h_1(r)$, $h_2(r)$ be fixed, which satisfy the conditions $(i1)$, the positive constants a, b, $a \leq b$ are fixed, and let the positive constants c, d, A, B, A_1, A_2 be fixed which satisfy the conditions $(H1), \ldots, (H4)$ and $f \in \mathscr{C}^1((-\infty,\infty))$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $b/2 \geq f(1) \geq a/2$, $g \in \mathscr{C}([0,\infty))$, $g(r) \geq 0$ for every $r \in [0,\infty)$, $g(r) \leq b/2 - f(1)$ for every $r \in [0,\infty)$. Let also $v(t)$ be fixed function which satisfies the conditions $(H5), \ldots, (H7)$. Then $(1^{\star\star})$ has unique nontrivial solution $u(t,r) = v(t)\omega(r)$ for which $u(t,r) \in \mathscr{C}([0,1] \times [0,\infty))$, $u(t,r) \leq 1/B$ for every $t \in [0,1]$ and for every $r \in [0,\infty)$, for every $t \in [0,1]$ and for every $r \in [c,d]$, $u(t,r) \geq 1/A$, $u(t,r) \geq 0$ for every $t \in [0,1]$ and for every $r \in [0,\infty)$, $u(t,\infty) = u_r(t,\infty) = 0$ for every $t \in [0,1]$, $u(t,r) \in C((0,1]L^2([0,\infty)))$.*

Proof. Let M be the set of the proof of Theorem 3.1. Let $t \in [0,1]$ be fixed. We define the operator R as follows:

$$R(u)(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u)+g(\tau)) \right) d\tau\, ds \tag{4.1}$$

for $u \in M$. First we will see that $R : M \to M$. Let $u \in M$. Then the following holds.

(1) Since $v(t) \in \mathscr{C}^4([0,1])$, $u(t,r) \in \mathscr{C}([0,1] \times [0,\infty))$ and $f \in \mathscr{C}(-\infty,\infty)$ and from $(i1)$ we have that $R(u) \in \mathscr{C}([0,1] \times [0,\infty))$. Also we have

$$\begin{gathered} R(u)|_{r=\infty} = 0, \\ \frac{\partial}{\partial r} R(u) = -\sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u)+g(\tau)) \right) d\tau, \\ \frac{\partial}{\partial r} R(u)|_{r=\infty} = 0. \end{gathered} \tag{4.2}$$

(2) Now we will prove that for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have that $R(u) \geq 0$. Really,

$$R(u) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(\tau)) \right) d\tau\, ds$$
$$\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(\tau)) \right) d\tau\, ds$$
$$\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \sqrt{h_1(\tau)h_2(\tau)}(f(u)+g(\tau)) \right) d\tau\, ds. \tag{4.3}$$

Now we suppose that for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have $u(t,r) \geq 0$, from here $f'(u) \leq bu$ after integration from 1 to u then we get that

$$f(u) - f(1) \leq \frac{b}{2}u^2 - \frac{b}{2},$$
$$f(u) \leq \frac{b}{2}u^2 - \frac{b}{2} + f(1), \tag{4.4}$$
$$f(u)+g(r) \leq \frac{b}{2}u^2 - \frac{b}{2} + f(1) + g(r).$$

From the conditions of Theorem 4.1 we have that

$$g(r) \leq \frac{b}{2} - f(1). \tag{4.5}$$

Therefore

$$f(u)+g(r) \leq \frac{b}{2}u^2. \tag{4.6}$$

From here and from $u(t,r) \leq 1/B$ for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we get

$$f(u)+g(r) \leq \frac{b}{2B}u. \tag{4.7}$$

Consequently, from (4.3) we get

$$R(u) \geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)} u \right) d\tau\, ds$$
$$\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)} \right) u\, d\tau\, ds \tag{4.8}$$
$$\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau\in[0,\infty)} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)} \right) u\, d\tau\, ds.$$

From $(H1)$ we have that

$$\min_{\tau\in[0,\infty)}\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)\geq 0 \tag{4.9}$$

and since $u(t,r)\geq 0$ for every fixed $t\in[0,1]$ and for every $r\in[0,\infty)$ from (4.8) we get

$$\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\min_{\tau\in[0,\infty)}\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)u\,d\tau\,ds\geq 0, \tag{4.10}$$

that is, for every fixed $t\in[0,1]$ and for every $r\in[0,\infty)$ we have

$$R(u)\geq 0. \tag{4.11}$$

(3) Now we will see that for every fixed $t\in[0,1]$ and for every $r\in[c,d]$ we have $R(u)\geq 1/A$. Really, after we use (4.7),

$$R(u)\geq\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)u\,d\tau\,ds. \tag{4.12}$$

Let

$$R_1(u)=\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)u\,d\tau\,ds. \tag{4.13}$$

From here

$$R_1'(u)=\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)d\tau\,ds\geq 0. \tag{4.14}$$

Since for every $r\in[c,d]$ we have $u(t,r)\geq 1/A$, we get

$$\begin{aligned}R_1(u)\geq R_1\left(\frac{1}{A}\right)&=\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)\frac{1}{A}d\tau\,ds\\&=\frac{1}{A}\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_1-\frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)}\right)d\tau\,ds\geq\frac{1}{A}\end{aligned} \tag{4.15}$$

(in the last inequality we use $(H1)$). Consequently, for every fixed $t \in [0,1]$ and for every $r \in [c,d]$ we have that

$$R(u) \geq \frac{1}{A}. \tag{4.16}$$

(4) Now we will prove that for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have that

$$R(u) \leq \frac{1}{B}. \tag{4.17}$$

Really, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$\begin{aligned} R(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)}\,(f(u)+g(\tau)) \right) d\tau\, ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u + \sqrt{h_1(\tau)h_2(\tau)}\,(f(u)+g(\tau)) \right) d\tau\, ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} |u| + \sqrt{h_1(\tau)h_2(\tau)}\,(f(u)+g(\tau)) \right) d\tau\, ds. \end{aligned} \tag{4.18}$$

Now we use (4.7)

$$\begin{aligned} &\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 u + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} u \right) d\tau\, ds \\ &\quad = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) u\, d\tau\, ds. \end{aligned} \tag{4.19}$$

here we use

$$\begin{aligned} u &\leq \frac{1}{B} \leq \frac{1}{B} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau\, ds \\ &\leq \frac{1}{B} \max_{r\in[0,\infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau\, ds \leq \frac{1}{B}. \end{aligned} \tag{4.20}$$

In the last inequality, we use ($H2$). Consequently, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$R(u) \leq \frac{1}{B}. \tag{4.21}$$

(5) Now we will prove that for every fixed $t \in [0,1]$ we have that $R(u) \in L^2([0\infty))$. Really, for every fixed $t \in [0,1]$ after we use the inequality (4.18) we get

$$\begin{aligned}&\int_0^\infty |R(u)|^2 dr\\&\quad\leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)|u|\,d\tau\,ds\right)^2 dr.\end{aligned} \tag{4.22}$$

Now we use the Hölder inequality

$$\begin{aligned}&\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2}\right.\\&\qquad\left.\times\left(\int_s^\infty |u|^2 d\tau\right)^{1/2} ds\right)^2 dr\\&\quad\leq \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2}\right.\\&\qquad\qquad\left.\times\left(\int_0^\infty |u|^2 d\tau\right)^{1/2} ds\right)^2 dr\\&\quad= \|u\|^2_{L^2([0,\infty))}\\&\qquad\times\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{2B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr\\&\quad\leq \|u\|^2_{L^2([0,\infty))}\frac{1}{7}.\end{aligned} \tag{4.23}$$

(In the last inequality, we use the condition ($H3$).) Consequently, for every fixed $t \in [0,1]$ we have $R(u) \in L^2([0,\infty))$.

From (1), (2), (3), (4), and (5) we get that for every fixed $t \in [0,1]$ we have

$$R : M \longrightarrow M. \tag{4.24}$$

Now we will prove that the operator $R : M \to M$ is contractive operator. Let u_1 and u_2 be two elements of the set M. Then for every fixed $t \in [0,1]$ we have

$$\begin{aligned} &|R(u_1) - R(u_2)| \\ &\quad = \left| \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} (u_1 - u_2) - \sqrt{h_1(\tau) h_2(\tau)} (f(u_1) - f(u_2)) \right) d\tau \, ds \right| \\ &\quad \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau) h_2(\tau)} |f(u_1) - f(u_2)| \right) d\tau \, ds, \end{aligned} \tag{4.25}$$

then from the middle-point theorem we have $|f(u_1) - f(u_2)| = |f'(\xi)||u_1 - u_2|$, $|\xi| \le \max\{|u_1|, |u_2|\}$, $|f'(\xi)| \le b|\xi| \le b/B$,

$$\begin{aligned} &\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} |u_1 - u_2| \right) d\tau \, ds \\ &\quad = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| \, d\tau \, ds \\ &\quad \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| \, d\tau \, ds \\ &\quad \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| \, d\tau \, ds, \end{aligned} \tag{4.26}$$

that is, for every fixed $t \in [0,1]$ and for every $r \in [0,\infty)$ we have

$$|R(u_1) - R(u_2)| \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| \, d\tau \, ds. \tag{4.27}$$

From here

$$
\begin{aligned}
&||R(u_1)-R(u_2)||^2_{L^2([0,\infty))} \\
&\quad \le \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)|u_1-u_2|\,d\tau\, ds\right)^2 dr.
\end{aligned}
\tag{4.28}
$$

Now we will use the Hölder inequality

$$
\begin{aligned}
&\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2}\right. \\
&\qquad \left.\times\left(\int_s^\infty |u_1-u_2|^2 d\tau\right)^{1/2} ds\right)^2 dr \\
&\quad \le \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2}\right. \\
&\qquad\qquad \left.\times\left(\int_0^\infty |u_1-u_2|^2 d\tau\right)^{1/2} ds\right)^2 dr \\
&\quad = ||u_1-u_2||^2_{L^2([0,\infty))} \\
&\qquad \times\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr,
\end{aligned}
\tag{4.29}
$$

that is, for every fixed $t \in [0,1]$ we have

$$
\begin{aligned}
&||R(u_1)-R(u_2)||^2_{L^2([0,\infty))} \\
&\quad \le ||u_1-u_2||^2_{L^2([0,\infty))} \\
&\qquad \times\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr,
\end{aligned}
\tag{4.30}
$$

from $(H4)$ we have

$$
\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr < 1. \tag{4.31}
$$

From here and from (4.30) we get

$$||R(u_1) - R(u_2)||^2_{L^2([0,\infty))} < ||u_1 - u_2||_{L^2([0,\infty))}. \tag{4.32}$$

Consequently, the operator $R : M \to M$ is contractive operator. We note that the set M is closed subset of the space $\mathcal{C}((0,1]L^2([0,\infty)))$ (for the proof see Lemma A.1 in the appendix of this paper). Therefore $(1^{\star\star})$ has unique nontrivial solution in the set M. □

Let $\overline{u}$ be the solution from Theorem 4.1, that is, $\overline{u}$ is a solution to the integral equation $(1^{\star\star})$. From Proposition 2.4, we have that $\widetilde{u}$ satisfies (1). Consequently ,$\overline{u}$ is solution to the nonhomogeneous Cauchy problem (1), (2) with initial data

$$\begin{aligned} u_\circ &= v(1)\omega(r) \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)}\left(f\left(v(1)\omega(\tau)\right) + g(\tau)\right) \right) d\tau\, ds, \\ u_1 &\equiv 0. \end{aligned} \tag{4.33}$$

We have $\overline{u} \in \mathcal{C}((0,1]L^2([0,\infty)))$, $u_\circ \in L^2([0,\infty))$, $u_1 \in \dot{H}^{-1}([0,\infty))$.

4.2. Blow up of the solutions of nonhomogeneous Cauchy problem (1), (2). Let $v(t)$ be the same function as in Theorem 4.1.

Theorem 4.2. *Let $n \geq 2$ be fixed, let $h_1(r)$, $h_2(r)$ be fixed, which satisfy the conditions ($i1$), let the positive constants a, b, $a \leq b$ be fixed, and let the positive constants c, d, A, B, A_1, A_2 be fixed which satisfy the conditions ($H1$), …, ($H4$) and $f \in \mathcal{C}^1((-\infty,\infty))$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $b/2 \geq f(1) \geq a/2$, $g(r) \in \mathcal{C}([0,\infty))$, $g(r) \geq 0$ for every $r \geq 0$, $g(r) \leq b/2 - f(1)$ for every $r \geq 0$. Then for the solution $\overline{u}$ of the Cauchy problem (1), (2) one has*

$$\lim_{t\to 0} \|\overline{u}\|_{L^2([0,\infty))} = \infty. \tag{4.34}$$

Proof. For every fixed $t \in (0,1]$ and for every $r \in [0,\infty)$ we have

$$\overline{u}(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}\left(f(\overline{u}) + g(\tau)\right) \right) d\tau\, ds. \tag{4.35}$$

Since for every fixed $t \in (0,1]$ and for every $r \in [c,d]$ we have that $\overline{u} \geq 1/A$ follows that there exists subinterval Δ in $[0,\infty)$ such that

$$\overline{u} \geq \frac{1}{A} \quad \text{for } r \in \Delta. \tag{4.36}$$

Let us fix the subinterval Δ. From here

$$\begin{aligned}
\|\overline{u}\|^2_{L^2([0,\infty))}
&= \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
&\qquad \left.\left. - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr \\
&= \int_{[0,\infty)\setminus\Delta} \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
&\qquad \left.\left. - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr \\
&\quad + \int_{\Delta} \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
&\qquad \left.\left. - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{4.37}$$

Let

$$\begin{aligned}
I_1 :&= \int_{[0,\infty)\setminus\Delta} \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
&\qquad \left.\left. - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr, \\
I_2 :&= \int_{\Delta} \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
&\qquad \left.\left. - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{4.38}$$

Then

$$\|\overline{u}\|^2_{L^2([0,\infty))} = I_1 + I_2. \tag{4.39}$$

For I_1 we have the following estimate:

$$\begin{aligned}
I_1 \le & \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
& \qquad \left.\left. + \sqrt{h_1(\tau)h_2(\tau)} \left(f(\overline{u}) + g(\tau)\right) \right) d\tau\, ds \right)^2 dr \\
\le & \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right.\right. \\
& \qquad \left.\left. + \sqrt{h_1(\tau)h_2(\tau)} \left(f(\overline{u}) + g(\tau)\right) \right) d\tau\, ds \right)^2 dr \\
\le & \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \overline{u}(t,\tau) \right.\right. \\
& \qquad \left.\left. + \sqrt{h_1(\tau)h_2(\tau)} \left(f(\overline{u}) + g(\tau)\right) \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{4.40}$$

Now we use (4.7)

$$\begin{aligned}
& \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \overline{u}(t,\tau) + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \overline{u} \right) d\tau\, ds \right)^2 dr \\
& = \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) \overline{u}\, d\tau\, ds \right)^2 dr.
\end{aligned} \tag{4.41}$$

Now we apply the Hölder inequality

$$\begin{aligned}
& \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} \left(\int_s^\infty |\overline{u}|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
& \le \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} \left(\int_0^\infty |\overline{u}|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
& = \|\overline{u}\|^2_{L^2[0,\infty)} \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr.
\end{aligned} \tag{4.42}$$

Let

$$Q := \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau) h_2(\tau)} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr. \tag{4.43}$$

Then

$$I_1 \le Q \|\overline{u}\|^2_{L^2([0,\infty))}. \tag{4.44}$$

Now we consider I_2. For it we have

$$\begin{aligned}
I_2 &= \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr \\
&\le \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr \\
&= \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right. \\
&\qquad + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0,\infty)\setminus\Delta} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) \right. \\
&\qquad\qquad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds \right)^2 dr.
\end{aligned} \tag{4.45}$$

Let

$$\begin{aligned}
I_{21} &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds, \\
I_{22} &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0,\infty)\setminus\Delta} \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \overline{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} \big(f(\overline{u}) + g(\tau) \big) \right) d\tau\, ds.
\end{aligned} \tag{4.46}$$

Consequently,

$$I_2 \le \int_\Delta \left(I_{21} + I_{22} \right)^2 dr \le 2 \int_\Delta I_{21}^2 dr + 2 \int_\Delta I_{22}^2 dr. \tag{4.47}$$

Also we have

$$\begin{aligned}
\int_{\Delta} I_{21}^2 dr &= \int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}\overline{u}(t,\tau)\right.\right.\\
&\qquad\left.\left. - \sqrt{h_1(\tau)h_2(\tau)}\left(f(\overline{u})+g(\tau)\right)\right)d\tau\, ds\right)^2 dr\\
&= \int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\sqrt{h_1(\tau)h_2(\tau)}\left(\frac{1}{h_1(\tau)}\frac{v''(t)}{v(t)}\overline{u}(t,\tau)\right.\right.\\
&\qquad\left.\left. - \left(f(\overline{u})+g(\tau)\right)\right)d\tau\, ds\right)^2 dr\\
&\le \int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\right.\\
&\qquad\left.\times\left(\max_{\tau\in[0,\infty)}\frac{1}{h_1(\tau)}\frac{v''(t)}{v(t)}\overline{u}(t,\tau) - \left(f(\overline{u})+g(\tau)\right)\right)d\tau\, ds\right)^2 dr.
\end{aligned} \tag{4.48}$$

Case 1. Let

$$\max_{r\in[0,\infty)}\frac{1}{h_1(r)} \le \frac{1}{A}. \tag{4.49}$$

Then after we use that for every fixed $t\in[0,1]$ and for every $r\in\Delta$ we have $f'(\overline{u})\ge a\overline{u}$ and integrate the last inequality from 1 to $\overline{u}$, we get

$$\begin{aligned}
f(\overline{u}) &\ge \frac{a}{2}\overline{u}^2 + f(1) - \frac{a}{2},\\
f(\overline{u})+g(r) &\ge \frac{a}{2}\overline{u}^2 - \frac{a}{2} + f(1) + g(r)\\
&\ge \frac{a}{2}\overline{u}^2 - \frac{a}{2} + f(1) - f(1) + \frac{a}{2} \ge \frac{a}{2}\overline{u}^2 \ge \frac{a}{2A}\overline{u},
\end{aligned} \tag{4.50}$$

for every $r\in\Delta$,

$$\begin{aligned}
&\int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\left(\max_{\tau\in[0,\infty)}\frac{1}{h_1(\tau)}\frac{v''(t)}{v(t)}\overline{u}(t,\tau) - \frac{a}{2A}\overline{u}\right)d\tau\, ds\right)^2 dr\\
&\quad\le \int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\overline{u}\, d\tau\, ds\right)^2 dr.
\end{aligned} \tag{4.51}$$

Now we use that for every fixed $t \in [0,1]$ and for every $r \in \Delta$ we have $\overline{u} \geq 1/A$, $A\overline{u} \geq 1$,

$$\begin{aligned}
&\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta A\overline{u}\frac{1}{A} d\tau\, ds\right)^2 dr \\
&\quad\leq \left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_\Delta \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta A^2\overline{u}^2\frac{1}{A} d\tau\, ds\right)^2 dr \\
&\quad\leq A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \overline{u}^2 d\tau\, ds\right)^2 dr \\
&\quad\leq A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \overline{u}^2 d\tau\, ds\right)^2 dr \\
&\quad= \|\overline{u}\|^4_{L^2([0,\infty))} A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds\right)^2 dr,
\end{aligned} \tag{4.52}$$

that is,

$$\int_\Delta I_{21}^2 dr \leq \|\overline{u}\|^4_{L^2([0,\infty))} A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds\right)^2 dr. \tag{4.53}$$

Let

$$F = A^2\left(\max_{\tau\in[0,\infty)} \sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)} - \frac{a}{2}\right)\right)^2 \int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds\right)^2 dr. \tag{4.54}$$

Then

$$\int_\Delta I_{21}^2 dr \leq F\|\overline{u}\|^4_{L^2[0,\infty)}. \tag{4.55}$$

Case 2. Let

$$\max_{r\in[0,\infty)} \frac{1}{h_1(r)} \geq \frac{1}{A}. \tag{4.56}$$

Let

$$\max_{r\in[0,\infty)} \frac{1}{h_1(r)} = G. \tag{4.57}$$

Then

$$\int_{\Delta} I_{21}^2 dr \leq 2\int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\right.$$
$$\left.\times\left(\left(G-\frac{1}{A}\right)\frac{v''(t)}{v(t)}\overline{u}(t,\tau)-\left(f\left(\overline{u}+g(\tau)\right)\right)\right)d\tau\, ds\right)^2 dr$$
$$+2\int_{\Delta}\left(\int_r^{\infty}\sqrt{\frac{h_2(s)}{h_1(s)}}\int_{\Delta}\max_{\tau\in[0,\infty)}\sqrt{h_1(\tau)h_2(\tau)}\frac{1}{A}\left(\frac{v''(t)}{v(t)}\overline{u}(t,\tau)-\frac{a}{2}\overline{u}\right)d\tau\, ds\right)^2 dr. \tag{4.58}$$

As mentioned above, we get

$$2Q\|\overline{u}\|_{L^2([0,\infty))}^2+2F\|\overline{u}\|_{L^2([0,\infty))}^4, \tag{4.59}$$

that is,

$$\int_{\Delta} I_{21}^2 dr \leq 2Q\|\overline{u}\|_{L^2([0,\infty))}^2+2F\|\overline{u}\|_{L^2([0,\infty))}^4. \tag{4.60}$$

From (4.55) and (4.60) we have

$$\int_{\Delta} I_{21}^2 dr \leq 2Q\|\overline{u}\|_{L^2([0,\infty))}^2+2F\|\overline{u}\|_{L^2([0,\infty))}^4. \tag{4.61}$$

As in the estimate for I_1 we have

$$\int_{\Delta} I_{22}^2 dr \leq \int_0^{\infty} I_{22}^2 dr \leq Q\|\overline{u}\|_{L^2([0,\infty))}^2. \tag{4.62}$$

From (4.47), (4.61), and (4.62) we get

$$I_2 \leq 4F\|\overline{u}\|_{L^2([0,\infty))}^4+6Q\|\overline{u}\|_{L^2([0,\infty))}^2. \tag{4.63}$$

From the last inequality and from (4.39), (4.44) we have

$$\begin{gathered}\|\overline{u}\|_{L^2([0,\infty))}^2 \leq 4F\|\overline{u}\|_{L^2([0,\infty))}^4+7Q\|\overline{u}\|_{L^2([0,\infty))}^2,\\ (1-7Q)\|\overline{u}\|_{L^2([0,\infty))}^2 \leq 4F\|\overline{u}\|_{L^2([0,\infty))}^4.\end{gathered} \tag{4.64}$$

From $(H3)$ we have

$$Q<\frac{1}{7}, \tag{4.65}$$

from here

$$\|\overline{u}\|_{L^2([0,\infty))}^2 \geq \frac{1-7Q}{4F}. \tag{4.66}$$

From $(H7)$ we have

$$\lim_{t\to 0} F = +0. \tag{4.67}$$

Therefore

$$\lim_{t\to 0} \|\overline{u}\|_{L^2([0,\infty))} = \infty. \tag{4.68}$$

□

Appendix

LEMMA A.1. *The set M is closed subset of the space $\mathscr{C}((0,1]L^2([0,\infty)))$.*

Proof. Let $t \in (0,1]$ be fixed. Let $\{u_n\}$ be a sequence of elements of the set M for which

$$\lim_{n\to\infty} \|u_n - \widetilde{u}\|_{L^2([0,\infty))} = 0, \tag{A.1}$$

where $\widetilde{u} \in L^2([0,\infty))$.

First, we note that for $u \in M$ we have that $L(u)$ is continuous function because $f(u) \in \mathscr{C}^1(-\infty,\infty)$. Also for $u \in M$ we have that $L'(u)$ is continuous function and

$$L'(u) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - \sqrt{h_1(\tau)h_2(\tau)} f'(u)\right) d\tau\, ds. \tag{A.2}$$

Now, we will prove that $|L'(u)| > 0$ for $u \in M$. Really,

$$\begin{aligned} L'(u) &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - \sqrt{h_1(\tau)h_2(\tau)} f'(u)\right) d\tau\, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - b\sqrt{h_1(\tau)h_2(\tau)} |u|\right) d\tau\, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} - b\sqrt{h_1(\tau)h_2(\tau)} |u|\right) d\tau\, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(A_1\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} - \frac{b}{B}\sqrt{h_1(\tau)h_2(\tau)}\right) d\tau\, ds. \end{aligned} \tag{A.3}$$

From $(H1)$ we have

$$A_1\sqrt{\frac{h_2(r)}{h_1(r)}} - \frac{b}{B}\sqrt{h_1(r)h_2(r)} \geq 0 \tag{A.4}$$

for every $r \geq 0$. Therefore

$$L'(u) \geq 0 \tag{A.5}$$

for every fixed $t \in [0,1]$ and for every $r \geq 0$. Also for every $r \in [0,1]$ we have that $L'(u)(r)$ is decrease function of r and for every $r \in [0,1]$ we have

$$L'(u)(r) \geq \int_1^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(A_1 \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} - \frac{b}{B}\sqrt{h_1(\tau)h_2(\tau)} \right) d\tau\, ds \geq \frac{A_1}{10^{10}}. \tag{A.6}$$

(In the last inequality, we use $(H1)$.) From here it follows that for every $u \in M$ there exists

$$N := \min_{r\in[0,1]} \left| L'(u)(r) \right| > 0. \tag{A.7}$$

Let

$$M_1 = \max_{r\in[0,\infty)} \left| \frac{\partial}{\partial r} L'(u)(r) \right|. \tag{A.8}$$

Now, we will prove that for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that from $|x - y| < \delta$ we have

$$\left| u_m(x) - u_m(y) \right| < \epsilon \quad \forall m \in \mathcal{N}. \tag{A.9}$$

We suppose that there exists $\widetilde{\epsilon} > 0$ such that for every $\delta > 0$ there exist natural m and $x, y \in [0,\infty)$, $|x - y| < \delta$ for which $|u_m(x) - u_m(y)| \geq \widetilde{\epsilon}$. We choose $\widetilde{\widetilde{\epsilon}} > 0$ such that $\widetilde{\widetilde{\epsilon}} < N\widetilde{\epsilon}$. We note that $L(u_m)(x)$ is uniformly continuous function for $x \in [0,\infty)$ (for $u \in M$ $L(u)(r)$ is uniformly continuous function for $r \in [0,\infty)$ because $L(u)(r) \in \mathcal{C}([0,\infty))$ and as in the proof of Theorem 3.1 (after we use the second condition of $(H2)$) we have that $|(\partial/\partial r)L(u)(r)| \leq 1/B$). Then there exists $\delta_1 = \delta_1(\widetilde{\widetilde{\epsilon}}) > 0$ such that for every natural n we have

$$\left| L(u)(x) - L(u)(y) \right| < \widetilde{\widetilde{\epsilon}} \quad \forall x, y \in [0,\infty) : |x - y| < \delta_1. \tag{A.10}$$

Consequently, we can choose

$$0 < \delta < \min\left\{ 1, \delta_1, \frac{(N\widetilde{\epsilon} - \widetilde{\widetilde{\epsilon}})B}{M_1} \right\}, \tag{A.11}$$

such that there exist natural m and $x_1, x_2 \in [0,\infty)$ for which

$$|x_1 - x_2| < \delta, \qquad |u_m(x_1 - x_2) - u_m(0)| \geq \tilde{\epsilon}. \tag{A.12}$$

In particular,

$$|L(u_m)(x_1 - x_2) - L(u_m)(0)| < \tilde{\tilde{\epsilon}}. \tag{A.13}$$

Let us suppose for convenience that $x_1 - x_2 > 0$. Then, $x_1 - x_2 < 1$ and for every $u \in M$ we have $L'(u)(x_1 - x_2) \geq N$. Then, from the middle point theorem we have

$$\begin{aligned}
&L(0) = 0, \qquad L(u_m)(x_1 - x_2) = L'(\xi)(x_1 - x_2)u_m(x_1 - x_2),\\
&L(u_m)(0) = L'(\xi)(0)u_m(0),\\
&|L(u_m)(x_1 - x_2) - L(u_m)(0)|\\
&\quad = |L'(\xi)(x_1 - x_2)u_m(x_1 - x_2) - L'(\xi)(0)u_m(0)|\\
&\quad = |L'(\xi)(x_1 - x_2)u_m(x_1 - x_2) - L'(\xi)(x_1 - x_2)u_m(0)\\
&\qquad + L'(\xi)(x_1 - x_2)u_m(0) - L'(\xi)(0)u_m(0)|\\
&\quad \geq |L'(\xi)(x_1 - x_2)u_m(x_1 - x_2) - L'(\xi)(x_1 - x_2)u_m(0)|\\
&\qquad - |L'(\xi)(x_1 - x_2)u_m(0) - L'(\xi)(0)u_m(0)|\\
&\quad = |L'(\xi)(x_1 - x_2)|\,|u_m(x_1 - x_2) - u_m(0)| - |L'(\xi)(x_1 - x_2) - L'(\xi)(0)|\,|u_m(0)|\\
&\quad = |L'(\xi)(x_1 - x_2)|\,|u_m(x_1 - x_2) - u_m(0)| - \left|\frac{\partial}{\partial r}(L'(\xi))\right| |x_1 - x_2|\,|u_m(0)|\\
&\quad \geq N\tilde{\epsilon} - M_1\delta\frac{1}{B} \geq \tilde{\tilde{\epsilon}},
\end{aligned} \tag{A.14}$$

which is contradiction with (A.13). Therefore,

$$\begin{aligned}
&\forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon) > 0: \text{ from } |x - y| < \delta\\
&\quad \Longrightarrow |u_m(x) - u_m(y)| < \epsilon \quad \forall m.
\end{aligned} \tag{A.15}$$

On the other hand, from the definition of the set M we have

$$|u_m| \le \frac{1}{B} \quad \forall m \in \mathcal{N}. \tag{A.16}$$

From (A.15) and (A.16) it follows that the set $\{u_n\}$ is compact subset of the space $\mathcal{C}([0,\infty))$. Therefore, there exists a subsequence $\{u_{n_k}\}$ and function $u \in \mathcal{C}([0,\infty))$ for which

$$|u_{n_k}(x) - u(x)| < \epsilon \tag{A.17}$$

for every $x \in [0,\infty)$. From here and from

$$||u_{n_k} - \widetilde{u}||_{L^2([0,\infty))} = 0 \tag{A.18}$$

we have that for every $\epsilon > 0$ there exists $F = F(\epsilon) > 0$ such that for every $n_k > F$ we have

$$\max_{r\in[0,\infty)} |u_{n_k} - u| < \delta, \qquad ||u_{n_k} - \widetilde{u}||_{L^2([0,\infty))} < \epsilon. \tag{A.19}$$

Then, for every $n_k > F$ we have

$$|u - \widetilde{u}| = |u - u_{n_k} + u_{n_k} - \widetilde{u}| \le |u - u_{n_k}| + |u_{n_k} - \widetilde{u}| < \epsilon + |u_{n_k} - \widetilde{u}|. \tag{A.20}$$

Also

$$\begin{aligned}
&|L(u) - L(\widetilde{u})| \\
&\quad = \left| \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} (u - \widetilde{u}) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) - f(\widetilde{u})) \right) d\tau \, ds \right| \\
&\quad \le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u - \widetilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} |f(u) - f(\widetilde{u})| \right) d\tau \, ds.
\end{aligned} \tag{A.21}$$

Now we apply the middle point theorem

$$\begin{aligned}
&\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\frac{v''(t)}{v(t)}|u-\widetilde{u}| + \sqrt{h_1(\tau)h_2(\tau)}\,|\,f'(\xi)\,|\,|u-\widetilde{u}|\right)d\tau\, ds \\
&\quad \leq |\xi| \leq \max\{|u|,|\widetilde{u}|\} \\
&\quad \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u-\widetilde{u}| + \sqrt{h_1(\tau)h_2(\tau)}\,|\,f'(\xi)\,|\,|u-\widetilde{u}|\right)d\tau\, ds \\
&\quad \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2|u-\widetilde{u}| + \sqrt{h_1(\tau)h_2(\tau)}\,|\,f'(\xi)\,|\,|u-\widetilde{u}|\right)d\tau\, ds \\
&\quad \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2|u-\widetilde{u}| + \sqrt{h_1(\tau)h_2(\tau)}b|\xi||u-\widetilde{u}|\right)d\tau\, ds,
\end{aligned} \tag{A.22}$$

(1) $|\xi| \leq |u|$. Then, $|\xi| \leq 1/B$. From here

$$\begin{aligned}
|\,L(u)-L(\widetilde{u})\,| &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2|u-\widetilde{u}| + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}|u-\widetilde{u}|\right)d\tau\, ds \\
&= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)|u-\widetilde{u}|d\tau\, ds \\
&\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)(\epsilon + |\,u_{n_k}-\widetilde{u}\,|)d\tau\, ds \\
&= \epsilon\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds \\
&\quad + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)|\,u_{n_k}-\widetilde{u}\,|\,d\tau\, ds, \\
|\,L(u)-L(\widetilde{u})\,|^2 &\leq \left(\epsilon\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right. \\
&\quad \left. + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)|\,u_{n_k}-\widetilde{u}\,|\,d\tau\, ds\right)^2 \\
&\leq 2\epsilon^2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right)^2 \\
&\quad + 2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2 + \sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)|\,u_{n_k}-\widetilde{u}\,|\,d\tau\, ds\right)^2.
\end{aligned} \tag{A.23}$$

Now we apply the Hölder inequality

$$
\begin{aligned}
&2\epsilon^2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right)^2\\
&\qquad +2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2}\right.\\
&\qquad\qquad \left.\times\left(\int_s^\infty |u_{n_k}-\tilde{u}|^2 d\tau\right)^{1/2} ds\right)^2\\
&\le 2\epsilon^2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right)^2\\
&\quad +2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2}\right.\\
&\qquad \left.\times\left(\int_0^\infty |u_{n_k}-\tilde{u}|^2 d\tau\right)^{1/2} ds\right)^2\\
&= 2\epsilon^2\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right)^2\\
&\quad +2\|u_{n_k}-\tilde{u}\|^2_{L^2([0,\infty))}\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2\\
&\le 2\epsilon^2\left[\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right)^2\right.\\
&\qquad \left.+\left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2\right].
\end{aligned}
\tag{A.24}
$$

From here

$$
\begin{aligned}
&\int_0^\infty |L(u)-L(\tilde{u})|^2 dr\\
&\quad\le 2\epsilon^2\left[\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)d\tau\, ds\right)^2 dr\right.\\
&\qquad\quad \left.+\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}A_2+\sqrt{h_1(\tau)h_2(\tau)}\frac{b}{B}\right)^2 d\tau\right)^{1/2} ds\right)^2 dr\right].
\end{aligned}
\tag{A.25}
$$

From $(i1)$ we have that there exist constants C_1 and C_2 such that

$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau\, ds \right)^2 dr \le C_1,$$
$$\int_0^\infty \left(\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr \le C_2. \tag{A.26}$$

Consequently,

$$\int_0^\infty |L(u) - L(\widetilde{u})|^2 dr \le 2\epsilon^2 (C_1 + C_2). \tag{A.27}$$

(2) $|\xi| \le |\widetilde{u}|$. Then,

$$\begin{aligned}
|L(u) - L(\widetilde{u})| &\le \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + b\sqrt{h_1(\tau)h_2(\tau)} |\widetilde{u}| \right) (\epsilon + |u_{n_k} - \widetilde{u}|)\, d\tau\, ds \\
&= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left(\sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \epsilon A_2 + \epsilon b \sqrt{h_1(\tau)h_2(\tau)} |\widetilde{u}| + A_2 \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} |u_{n_k} - \widetilde{u}| \right. \\
&\qquad \left. + b\sqrt{h_1(\tau)h_2(\tau)} |\widetilde{u}|\, |u_{n_k} - \widetilde{u}| \right) d\tau\, ds \\
&= \epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau\, ds \\
&\quad + \epsilon b \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{h_1(\tau)h_2(\tau)} |\widetilde{u}|\, d\tau\, ds \\
&\quad + A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} |u_{n_k} - \widetilde{u}|\, d\tau\, ds \\
&\quad + b \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{h_1(\tau)h_2(\tau)} |\widetilde{u}|\, |u_{n_k} - \widetilde{u}|\, d\tau\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau\, ds \\
&\quad + \epsilon b \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty h_1(\tau) h_2(\tau) d\tau \right)^{1/2} \left(\int_s^\infty |\widetilde{u}|^2 d\tau \right)^{1/2} ds \\
&\quad + A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} \left(\int_s^\infty |u_{n_k} - \widetilde{u}|^2 d\tau \right)^{1/2} ds \\
&\quad + b \max_{r \in [0,\infty)} \sqrt{h_1(\tau) h_2(\tau)} \\
&\quad \times \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty |\widetilde{u}|^2 d\tau \right)^{1/2} \left(\int_s^\infty |u_{n_k} - \widetilde{u}|^2 d\tau \right)^{1/2} ds,
\end{aligned}
\tag{A.28}
$$

from $(i1)$ we have that there exists constant $C_3 > 0$ such that

$$
\begin{aligned}
&\max_{r \in [0,\infty)} \sqrt{h_1(r) h_2(r)} \leq C_3, \\
&\epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau\, ds \\
&\quad + \epsilon b \|\widetilde{u}\|_{L^2[0,\infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty h_1(\tau) h_2(\tau) d\tau \right)^{1/2} ds \\
&\quad + A_2 ||u_{n_k} - \widetilde{u}||_{L^2([0,\infty))} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \\
&\quad + b C_3 \|\widetilde{u}\|_{L^2([0,\infty))} ||u_{n_k} - \widetilde{u}||_{L^2([0,\infty))} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \\
&\leq \epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau\, ds \\
&\quad + \epsilon b \|\widetilde{u}\|_{L^2[0,\infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty h_1(\tau) h_2(\tau) d\tau \right)^{1/2} ds \\
&\quad + A_2 \epsilon \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left(\int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds + \epsilon b C_3 \|\widetilde{u}\|_{L^2([0,\infty))} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds,
\end{aligned}
\tag{A.29}
$$

from here

$$\begin{aligned}\left|L(u)-L(\tilde{u})\right|^2 \le 2\epsilon^2\Bigg[&A_2^2\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}d\tau\,ds\right)^2\\&+b^2\|\tilde{u}\|^2_{L^2[0,\infty)}\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty h_1(\tau)h_2(\tau)d\tau\right)\frac{1}{2}ds\right)^2\\&+A_2^2\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty\frac{h_2(\tau)}{h_1(\tau)}d\tau\right)^{1/2}ds\right)^2\\&+b^2C_3^2\|\tilde{u}\|^2_{L^2([0,\infty))}\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}ds\right)^2\Bigg],\end{aligned}\tag{A.30}$$

from here

$$\begin{aligned}&\int_0^\infty\left|L(u)-L(\tilde{u})\right|^2dr\\&\quad\le 2\epsilon^2\Bigg[A_2^2\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}d\tau\,ds\right)^2dr\\&\qquad+b^2\|\tilde{u}\|^2_{L^2[0,\infty)}\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty h_1(\tau)h_2(\tau)d\tau\right)\frac{1}{2}ds\right)^2dr\\&\qquad+A_2^2\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty\frac{h_2(\tau)}{h_1(\tau)}d\tau\right)^{1/2}ds\right)^2dr\\&\qquad+b^2C_3^2\|\tilde{u}\|^2_{L^2([0,\infty))}\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}ds\right)^2dr\Bigg].\end{aligned}\tag{A.31}$$

From $(i1)$ we have that there exist constants C_4, C_5, C_6, and C_7 such that

$$\begin{gathered}\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\int_s^\infty\sqrt{\frac{h_2(\tau)}{h_1(\tau)}}d\tau\,ds\right)^2dr\le C_4,\\\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty h_1(\tau)h_2(\tau)d\tau\right)^{1/2}ds\right)^2dr\le C_5,\\\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}\left(\int_s^\infty\frac{h_2(\tau)}{h_1(\tau)}d\tau\right)^{1/2}ds\right)^2dr\le C_6,\\\int_0^\infty\left(\int_r^\infty\sqrt{\frac{h_2(s)}{h_1(s)}}ds\right)^2dr\le C_7.\end{gathered}\tag{A.32}$$

Consequently,

$$\int_0^\infty |L(u) - L(\widetilde{u})|^2 dr \le 2\epsilon^2\left(A_2^2 C_4 + b^2\|\widetilde{u}\|^2_{L^2([0,\infty))} C_5 + C_6 A_2^2 + \|\widetilde{u}\|^2_{L^2([0,\infty))} b^2 C_3^2 C_7\right). \tag{A.33}$$

Since $\widetilde{u} \in L^2([0,\infty))$ and from the inequalities (A.27) and (A.33) it follows that there exists constant C_8 such that

$$\int_0^\infty |L(u) - L(\widetilde{u})|^2 dr \le C_8\epsilon^2. \tag{A.34}$$

From here it follows that $L(u) = L(\widetilde{u})$ a.e. in $[0,\infty)$. Now we suppose that it is not true $u = \widetilde{u}$ a.e. in $[0,\infty)$. Then there exist $\epsilon_1 > 0$ and subinterval $\Delta \subset [0,\infty)$ such that

$$|u - \widetilde{u}| > \epsilon_1 \quad \text{for } r \in \Delta. \tag{A.35}$$

From the middle point theorem we have

$$|L(u) - L(\widetilde{u})| = |L'(\xi)|\,|u - \widetilde{u}|, \quad |\xi| \le \{|u|, |\widetilde{u}|\}. \tag{A.36}$$

Also there exists constant $M_1 > 0$ such that

$$\min_{r\in\Delta} |L'(\xi)(r)| \ge M_1. \tag{A.37}$$

Then for

$$\epsilon < M_1^2 \mu(\Delta)\epsilon_1^2 \tag{A.38}$$

we have

$$\epsilon > \int_0^\infty |L(u) - L(\widetilde{u})|^2 dr \ge \int_\Delta |L(u) - L(\widetilde{u})|^2 dr \ge \int_\Delta |L'(\xi)|^2 |u - \widetilde{u}|^2 dr \ge M_1^2 \mu(\Delta)\epsilon_1^2 \tag{A.39}$$

which is contradiction with (A.38). From here it follows that $u = \widetilde{u}$ a.e. in $[0,\infty)$, $|u_n - u|^2 = |u_n - \widetilde{u}|^2$ a.e. in $[0,\infty)$,

$$||u_n - u||_{L^2([0,\infty))} = ||u_n - \widetilde{u}||_{L^2([0,\infty))}. \tag{A.40}$$

Consequently, for every sequence $\{u_n\}$ from elements of the set M which is convergent in $L^2([0,\infty))$ there exists function $u \in \mathscr{C}([0,\infty))$, $u \in L^2([0,\infty))$ for which

$$\lim_{n\to\infty} ||u_n - u||_{L^2([0,\infty))} = 0. \tag{A.41}$$

Below we suppose that the sequence $\{u_n\}$ is a sequence from elements of the set M which is convergent in $L^2([0,\infty))$. Then, there exists function $u \in \mathscr{C}([0,\infty))$, $u \in L^2([0,\infty))$ for which $\lim_{n\to\infty} \|u_n - u\|_{L^2([0,\infty))} = 0$.

Now we suppose that $u(t,\infty) \neq 0$. Since $u(t,r) \in \mathscr{C}([0,\infty))$, $u_n(t,r) \in \mathscr{C}([0,\infty))$, $u_n(t,\infty) = 0$ for every natural n we have that there exists $\epsilon_2 > 0$ and constant $Q > 0$ such that for every $r > Q$

$$|u_n| < \frac{\epsilon_2}{2}, \quad |u| > \epsilon_2. \tag{A.42}$$

Then, for every natural n and for every $r > Q$ we have

$$|u_n - u| \geq |u| - |u_n| > \frac{\epsilon_2}{2}. \tag{A.43}$$

Also there exist constants $M > 0$ and $M_1 > 0$ such that

$$\min_{r\in[Q,Q+1]} |L'(\xi)(r)| \geq M, \quad |L'(\xi)(r)| \leq \frac{1}{M_1} \quad \text{for } r \in [0,\infty). \tag{A.44}$$

Let $\epsilon_3 > 0$ be such that

$$\epsilon_3 < \frac{M\epsilon_2}{2(1/M_1)}. \tag{A.45}$$

Therefore, there exists natural n and for every $r > Q$ we have

$$|u_n(r) - u(r)| > \frac{\epsilon_2}{2}, \qquad ||u_n - u||_{L^2([0,\infty))} < \epsilon_3. \tag{A.46}$$

From here

$$\begin{aligned} M^2\frac{\epsilon_2^2}{4} &\leq M^2\int_Q^{Q+1} |u - u_n|^2 dr \leq \int_Q^{Q+1} |L(u) - L(u_n)|^2 dr \\ &= \int_Q^{Q+1} |L'(\xi)|^2 |u - u_n|^2 dr \leq \frac{1}{M_1^2}\int_0^{\infty} |u - u_n|^2 dr \\ &\leq \frac{1}{M_1^2}||u_n - u||^2_{L^2([0,\infty))} \leq \frac{1}{M_1^2}\epsilon_3^2 \end{aligned} \tag{A.47}$$

or

$$M^2\frac{\epsilon_2^2}{4} \leq \frac{1}{M_1^2}\epsilon_3^2 \tag{A.48}$$

or

$$\epsilon_3 \geq \frac{M\epsilon_2}{2(1/M_1)}, \tag{A.49}$$

which is contradiction with (A.45). Therefore, $u(t,\infty) = 0$ and for every enough large $Q > 0$ we have $u(t,r) = 0$ for every $r > Q$. From here $u_r(t,\infty) = 0$.

Now we suppose that the inequality

$$u(t,r) \leq \frac{1}{B} \tag{A.50}$$

is not true for every $r \in [0,\infty)$. Since $u \in \mathscr{C}([0,\infty))$, we can take $\epsilon_4 > 0$ and $\Delta_1 \subset [0,\infty)$ such that

$$u \geq \frac{1}{B} + \epsilon_4 \quad \text{for } r \in \Delta_1. \tag{A.51}$$

Then for every natural n and for every $r \in \Delta_1$ we have

$$|u_n - u| \geq |u| - |u_n| \geq \frac{1}{B} + \epsilon_4 - \frac{1}{B} = \epsilon_4. \tag{A.52}$$

Let $\epsilon_5 > 0$ be such that

$$\epsilon_5 \leq \epsilon_4 (\mu(\Delta_1))^{1/2}. \tag{A.53}$$

Then, there exists constant $F > 0$ such that for every $n > F$ we have

$$||u_n - u||_{L^2([0,\infty))} < \epsilon_5. \tag{A.54}$$

Therefore, for every $n > F$ and for every $r \in \Delta_1$ we have

$$|u_n(r) - u(r)| \geq \epsilon_4, \qquad ||u_n - u||_{L^2([0,\infty))} < \epsilon_5. \tag{A.55}$$

Also

$$\begin{aligned} \epsilon_4 \mu(\Delta_1) &\leq \int_{\Delta_1} |u_n - u|\, dx \leq (\mu(\Delta_1))^{1/2} \left(\int_{\Delta_1} |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_1))^{1/2} \left(\int_0^\infty |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_1))^{1/2} ||u_n - u||_{L^2([0,\infty))} < \epsilon_5 (\mu(\Delta_1))^{1/2} \end{aligned} \tag{A.56}$$

or

$$\epsilon_4 \mu(\Delta_1) < \epsilon_5 (\mu(\Delta_1))^{1/2}. \tag{A.57}$$

From here

$$\epsilon_4 (\mu(\Delta_1))^{1/2} < \epsilon_5, \tag{A.58}$$

which is contradiction with (A.53). Consequently, for every $r \in [0,\infty)$ we have

$$u(t,r) \leq \frac{1}{B}. \tag{A.59}$$

Now we suppose that the inequality

$$u(t,r) \geq \frac{1}{A} \tag{A.60}$$

is not true for every $r \in [c,d]$. Since $u \in \mathscr{C}([0,\infty))$, we can take $\epsilon_6 > 0$ and $\Delta_2 \subset [c,d]$ such that

$$|u| \le \frac{1}{A} - \epsilon_6 \quad \text{for } r \in \Delta_2. \tag{A.61}$$

Then, for every natural n and for every $r \in \Delta_2$ we have

$$|u_n - u| \ge |u_n| - |u| \ge \frac{1}{A} + \epsilon_6 - \frac{1}{A} = \epsilon_6. \tag{A.62}$$

Let $\epsilon_7 > 0$ be such that

$$\epsilon_7 \le \epsilon_6 (\mu(\Delta_2))^{1/2}, \tag{A.63}$$

then there exists natural n such that

$$\begin{gathered} ||u_n - u||_{L^2([0,\infty))} < \epsilon_7, \\ |u_n(r) - u(r)| \ge \epsilon_6, \quad r \in \Delta_2. \end{gathered} \tag{A.64}$$

Also

$$\begin{aligned} \epsilon_6 \mu(\Delta_2) &\le \int_{\Delta_2} |u_n - u| \, dx \le (\mu(\Delta_2))^{1/2} \left(\int_{\Delta_2} |u_n - u|^2 dx \right)^{1/2} \\ &\le (\mu(\Delta_2))^{1/2} \left(\int_0^\infty |u_n - u|^2 dx \right)^{1/2} \\ &\le (\mu(\Delta_2))^{1/2} ||u_n - u||_{L^2([0,\infty))} < \epsilon_7 (\mu(\Delta_2))^{1/2} \end{aligned} \tag{A.65}$$

or

$$\epsilon_6 \mu(\Delta_2) < \epsilon_7 (\mu(\Delta_2))^{1/2}. \tag{A.66}$$

From here

$$\epsilon_6 (\mu(\Delta_2))^{1/2} < \epsilon_7, \tag{A.67}$$

which is contradiction with (A.63). Consequently, for every $r \in [c,d]$ we have

$$u(t,r) \ge \frac{1}{A}. \tag{A.68}$$

Now we suppose that the inequality

$$u(t,r) \ge 0 \tag{A.69}$$

is not true for every $r \in [0,\infty)$. Since $u \in \mathscr{C}([0,\infty))$, we can take $\epsilon_8 > 0$ and $\Delta_3 \subset [0,\infty)$ such that

$$u < 0 \quad \text{for } r \in \Delta_3, \quad |u_n - u| \ge \epsilon_8. \tag{A.70}$$

Let $\epsilon_9 > 0$ be such that

$$\epsilon_9 \leq \epsilon_8 \left(\mu(\Delta_3)\right)^{1/2}, \tag{A.71}$$

then there exists natural n such that

$$\begin{gathered} ||u_n - u||_{L^2([0,\infty))} < \epsilon_9, \\ |u_n(r) - u(r)| \geq \epsilon_8, \quad r \in \Delta_3. \end{gathered} \tag{A.72}$$

Also

$$\begin{aligned} \epsilon_8 \mu(\Delta_3) &\leq \int_{\Delta_3} |u_n - u|\, dx \leq \left(\mu(\Delta_3)\right)^{1/2} \left(\int_{\Delta_3} |u_n - u|^2 dx \right)^{1/2} \\ &\leq \left(\mu(\Delta_3)\right)^{1/2} \left(\int_0^\infty |u_n - u|^2 dx \right)^{1/2} \\ &\leq \left(\mu(\Delta_3)\right)^{1/2} ||u_n - u||_{L^2([0,\infty))} < \epsilon_9 \left(\mu(\Delta_3)\right)^{1/2} \end{aligned} \tag{A.73}$$

or

$$\epsilon_8 \mu(\Delta_3) < \epsilon_9 \left(\mu(\Delta_3)\right)^{1/2}. \tag{A.74}$$

From here

$$\epsilon_8 \left(\mu(\Delta_3)\right)^{1/2} < \epsilon_9, \tag{A.75}$$

which is contradiction with (A.71). Consequently, for every $r \in [0,\infty)$ we have

$$u(t,r) \geq 0. \tag{A.76}$$

Consequently, for every sequence $\{u_n\} \subset M$ which is convergent in $L^2([0,\infty))$ there exists $u \in M$ such that

$$\lim_{n\to\infty} ||u_n - u||_{L^2([0,\infty))} = 0. \tag{A.77}$$

□

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