MULTIPLE POSITIVE SOLUTIONS OF SINGULAR p-LAPLACIAN PROBLEMS BY VARIATIONAL METHODS

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We obtain multiple positive solutions of singular p-Laplacian problems using variational methods. The techniques are applicable to other types of singular problems as well.

1. Introduction

We consider the singular quasilinear elliptic boundary value problem

$$\begin{aligned} -\Delta_p u &= a(x)u^{-\gamma} + \lambda f(x,u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^n, n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$, $a \geq 0$ is a nontrivial measurable function, $\gamma > 0$ is a constant, $\lambda > 0$ is a parameter, and f is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying

$$\sup_{(x,t)\in\Omega\times[0,T]} |f(x,t)| < \infty \quad \forall T > 0. \tag{1.2}$$

The semilinear case $p = 2$ with $\gamma < 1$ and $f = 0$ has been studied extensively in both bounded and unbounded domains (see [5, 6, 7, 10, 11, 12, 14, 20] and their references). In particular, Lair and Shaker [11] showed the existence of a unique (weak) solution when Ω is bounded and $a \in L^2(\Omega)$. Their result was extended to the sublinear case $f(t) = t^\beta$, $0 < \beta \leq 1$ by Shi and Yao [15] and Wiegner [18]. In the superlinear case $1 < \beta < 2^* - 1$ and for small λ, Coclite and Palmieri [4] obtained a solution when $a = 1$ and Sun et al. [16] obtained two solutions using the Ekeland's variational principle for more general a's. Zhang [19] extended their multiplicity result to more general superlinear terms $f(t) \geq 0$ using critical point theory on closed convex sets. The ODE case $n = 1$ was studied by Agarwal and O'Regan [1] using fixed point theory and by Agarwal et al. [2] using variational methods. The purpose of the present paper is to treat the general quasilinear case $p \in (1,\infty)$, $\gamma \in (0,\infty)$, and f is allowed to change sign. We use a simple cutoff argument and only the basic critical point theory. Our results seem to be new even for $p = 2$.

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First we assume

(H_1) $\exists\varphi \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > n$ such that $a\varphi^{-\gamma} \in L^q(\Omega)$.

This does not require $\gamma < 1$ as usually assumed in the literature. For example, when Ω is the unit ball, $a(x) = (1-|x|^2)^\sigma$, $\sigma \geq 0$, and $\gamma < \sigma + 1/n$, we can take $\varphi(x) = 1-|x|^2$ and $q < 1/(\gamma-\sigma)$ (resp., q with no additional restrictions) if $\gamma > \sigma$ (resp., $\gamma \leq \sigma$).

Theorem 1.1. *If (H_1) and (1.2) hold and $f \geq 0$, then $\exists\lambda_0 > 0$ such that problem (1.1) has a solution $\forall\lambda \in (0,\lambda_0)$.*

Corollary 1.2. *Problem (1.1) with $f = 0$ has a solution if (H_1) holds.*

Next we allow f to change sign, but strengthen (H_1) to

(H_2) $a \in L^\infty(\Omega)$ with $a_0 := \inf_\Omega a > 0$ and $\gamma < 1/n$.

This implies that $a\varphi^{-\gamma} \in L^q(\Omega)$ for any φ whose interior normal derivative $\partial\varphi/\partial\nu > 0$ on $\partial\Omega$ and $q < 1/\gamma$.

Theorem 1.3. *If (H_2) and (1.2) hold, then $\exists\lambda_0 > 0$ such that problem (1.1) has a solution $\forall\lambda \in (0,\lambda_0)$.*

Finally we assume that f is C^1 in t, satisfies

$$|f_t(x,t)| \leq C(t^{r-2}+1) \tag{1.3}$$

for some $2 \leq r < p^*$, and p-superlinear:

$$0 < \theta F(x,t) \leq tf(x,t), \quad t \text{ large} \tag{1.4}$$

for some $\theta > p$. Here $p^* = np/(n-p)$ (resp., ∞) if $p < n$ (resp., $p \geq n$) is the critical Sobolev exponent and C denotes a generic positive constant.

Theorem 1.4. *If $p \geq 2$, (H_1), (1.3), and (1.4) hold, and $f \geq 0$, then $\exists\lambda_0 > 0$ such that problem (1.1) has two solutions $\forall\lambda \in (0,\lambda_0)$.*

Theorem 1.5. *If $p \geq 2$ and (H_2), (1.3), and (1.4) hold, then $\exists\lambda_0 > 0$ such that problem (1.1) has two solutions $\forall\lambda \in (0,\lambda_0)$.*

2. Preliminaries on the p-Laplacian

Consider the problem

$$\begin{aligned} -\Delta_p u = g(x) \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Proposition 2.1. *If $g \in L^q(\Omega)$ for some $q > n$, then (2.1) has a unique weak solution $u \in C_0^1(\overline{\Omega})$. If, in addition, $g \geq 0$ is nontrivial, then*

$$u > 0 \quad \text{in } \Omega, \qquad \partial u/\partial\nu > 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Proof. The existence of a unique solution $u \in W_0^{1,p}(\Omega)$ is well-known. The problem

$$\begin{aligned} -\Delta v &= g(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.3}$$

has a unique solution $v \in W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, $\alpha = 1 - n/q$. Then u satisfies

$$\begin{aligned} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u - G(x)\right) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.4}$$

where $G = \nabla v \in C^{\alpha}(\overline{\Omega})$, and u is bounded by Guedda and Véron [8] since $q > n/p$ if $p \leq n$, so $u \in C_0^1(\overline{\Omega})$ by Lieberman [13]. The rest now follows from Vázquez [17]. □

3. Proofs of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Since $a \in L^q(\Omega)$ by (H_1), the problem

$$\begin{aligned} -\Delta_p v &= a(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

has a unique positive solution $v \in C_0^1(\overline{\Omega})$ with $\partial v/\partial \nu > 0$ on $\partial\Omega$ by Proposition 2.1. Then $\inf_{\Omega}(v/\varphi) > 0$ and hence $av^{-\gamma} \in L^q(\Omega)$. Fix $0 < \varepsilon \leq 1$ so small that $\underline{u} := \varepsilon^{1/(p-1)}v \leq 1$. Then

$$-\Delta_p \underline{u} - a(x)\underline{u}^{-\gamma} - \lambda f(x,\underline{u}) \leq -(1-\varepsilon)a(x) \leq 0, \tag{3.2}$$

so $\underline{u}$ is a subsolution of (1.1).

Since $a\underline{u}^{-\gamma} \in L^q(\Omega)$, the problem

$$\begin{aligned} -\Delta_p u &= a(x)\underline{u}(x)^{-\gamma} + 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.3}$$

has a unique solution $\overline{u} \in C_0^1(\overline{\Omega})$ by Proposition 2.1, and $\overline{u} \geq \underline{u}$ since

$$-\Delta_p \overline{u} \geq a(x) \geq \varepsilon a(x) = -\Delta_p \underline{u}. \tag{3.4}$$

Then

$$-\Delta_p \overline{u} - a(x)\overline{u}^{-\gamma} - \lambda f(x,\overline{u}) \geq 1 - \lambda \sup_{x\in\Omega,\, t\leq \max_{\Omega}\overline{u}} f(x,t), \tag{3.5}$$

so $\exists \lambda_0 > 0$ such that $\overline{u}$ is a supersolution of (1.1) $\forall \lambda \in (0,\lambda_0)$ by (1.2).

Let

$$g_{\lambda,\overline{u}}(x,t) = \begin{cases} a(x)\overline{u}(x)^{-\gamma} + \lambda f(x,\overline{u}(x)), & t > \overline{u}(x) \\ a(x)t^{-\gamma} + \lambda f(x,t), & \underline{u}(x) \leq t \leq \overline{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x,\underline{u}(x)), & t < \underline{u}(x), \end{cases}$$
$$G_{\lambda,\overline{u}}(x,t) = \int_0^t g_{\lambda,\overline{u}}(x,s)\,ds,$$
$$\Phi_{\lambda,\overline{u}}(u) = \int_\Omega |\nabla u|^p - pG_{\lambda,\overline{u}}(x,u), \quad u \in W_0^{1,p}(\Omega). \tag{3.6}$$

Since

$$0 \leq g_{\lambda,\overline{u}}(x,t) \leq a(x)\underline{u}(x)^{-\gamma} + \lambda \sup_{x\in\Omega,\, t\leq \max_\Omega \overline{u}} f(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{3.7}$$

and $a\underline{u}^{-\gamma} \in L^q(\Omega)$, $\Phi_{\lambda,\overline{u}}$ is bounded from below and has a global minimizer u_0, which then is a solution of (1.1) in the order interval $[\underline{u},\overline{u}]$. □

Proof of Theorem 1.3. The problem

$$-\Delta_p v = a_0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{3.8}$$

has a unique positive solution $v \in C_0^1(\overline{\Omega})$ with $\partial v/\partial \nu > 0$ on $\partial\Omega$. Fix $0 < \varepsilon < 1$ so small that $\underline{u} := \varepsilon^{1/(p-1)}\, v \leq 1$. Then

$$-\Delta_p \underline{u} - a(x)\underline{u}^{-\gamma} - \lambda f(x,\underline{u}) \leq -(1-\varepsilon)a_0 + \lambda \sup_{x\in\Omega,\, t\leq \max_\Omega \underline{u}} |f(x,t)|, \tag{3.9}$$

so $\exists \lambda_0 > 0$ such that $\underline{u}$ is a subsolution of (1.1) $\forall \lambda \in (0,\lambda_0)$. The rest of the proof now proceeds as above. □

4. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Define a Carathéodory function on $\Omega \times \mathbb{R}$ by

$$g_\lambda(x,t) = \begin{cases} a(x)t^{-\gamma} + \lambda f(x,t), & t \geq \underline{u}(x) \\ a(x)\underline{u}(x)^{-\gamma} + \lambda f(x,\underline{u}(x)), & t < \underline{u}(x) \end{cases} \tag{4.1}$$

and consider the problem

$$-\Delta_p u = g_\lambda(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Every solution of (4.2) is $\geq \underline{u}$ and hence also a solution of (1.1). By (1.3),

$$0 \leq g_\lambda(x,t) \leq a(x)\underline{u}(x)^{-\gamma} + \lambda C\left(\left(t^+\right)^{r-1} + 1\right), \quad \forall (x,t) \in \Omega \times \mathbb{R} \tag{4.3}$$

so solutions of (4.2) are the critical points of the C^1 functional

$$\Phi_\lambda(u) = \int_\Omega |\nabla u|^p - pG_\lambda(x,u), \quad u \in W_0^{1,p}(\Omega), \tag{4.4}$$

where $G_\lambda(x,t) = \int_0^t g_\lambda(x,s)ds$.

Since u_0 solves

$$\begin{gathered} -\Delta_p u = g_{\lambda,\overline{u}}(x,u_0(x)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.5}$$

and $g_{\lambda,\overline{u}}(\cdot,u_0(\cdot)) \in L^q(\Omega)$ by (3.7), $u_0 \in C_0^1(\overline{\Omega})$ by Proposition 2.1. Note that, with a possibly smaller λ_0, $2\overline{u}$ is also a supersolution of (1.1) $\forall \lambda \in (0,\lambda_0)$. We assume that u_0 is the global minimizer of the corresponding functional $\Phi_{\lambda,2\overline{u}}$ also, for otherwise we are done. Since

$$u_0 \le \overline{u} < 2\overline{u} \quad \text{in } \Omega, \qquad \partial u_0/\partial\nu \le \partial\overline{u}/\partial\nu < \partial(2\overline{u})/\partial\nu \quad \text{on } \partial\Omega, \tag{4.6}$$

$\Phi_{\lambda,2\overline{u}} = \Phi_\lambda$ in a $C_0^1(\overline{\Omega})$-neighborhood of u_0, so u_0 is a local minimizer of $\Phi_\lambda|_{C_0^1(\overline{\Omega})}$, and hence also of Φ_λ by Brezis and Nirenberg [3] for $p = 2$ and by Guo and Zhang [9] for $p > 2$. The mountain pass lemma now gives a second critical point as (1.4) implies that Φ_λ satisfies the (PS) condition and $\Phi_\lambda(t\underline{u}) \to -\infty$ as $t \to \infty$. □

Proof of Theorem 1.5 is similar and therefore omitted.

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References

[1] R. P. Agarwal and D. O'Regan, *Singular boundary value problems for superlinear second order ordinary and delay differential equations*, J. Differential Equations **130** (1996), no. 2, 333–355.

[2] R. P. Agarwal, K. Perera, and D. O'Regan, *Multiple positive solutions of singular problems by variational methods*, to appear in Proc. Amer. Math. Soc.

[3] H. Brezis and L. Nirenberg, *H^1 versus C^1 local minimizers*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 5, 465–472.

[4] M. M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations **14** (1989), no. 10, 1315–1327.

[5] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), no. 2, 193–222.

[6] J. I. Diaz, J.-M. Morel, and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations **12** (1987), no. 12, 1333–1344.

[7] A. L. Edelson, *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl. **139** (1989), no. 2, 523–532.

[8] M. Guedda and L. Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879–902.

[9] Z. Guo and Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), no. 1, 32–50.

[10] T. Kusano and C. A. Swanson, *Entire positive solutions of singular semilinear elliptic equations*, Japan. J. Math. (N.S.) **11** (1985), no. 1, 145–155.

[11] A. V. Lair and A. W. Shaker, *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. **211** (1997), no. 2, 371–385.

[12] A. C. Lazer and P. J. McKenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 721–730.

[13] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[14] A. W. Shaker, *On singular semilinear elliptic equations*, J. Math. Anal. Appl. **173** (1993), no. 1, 222–228.

[15] J. Shi and M. Yao, *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), no. 6, 1389–1401.

[16] Y. Sun, S. Wu, and Y. Long, *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, J. Differential Equations **176** (2001), no. 2, 511–531.

[17] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.

[18] M. Wiegner, *A degenerate diffusion equation with a nonlinear source term*, Nonlinear Anal. **28** (1997), no. 12, 1977–1995.

[19] Z. Zhang, *Critical points and positive solutions of singular elliptic boundary value problems*, J. Math. Anal. Appl. **302** (2005), no. 2, 476–483.

[20] M. A. del Pino, *A global estimate for the gradient in a singular elliptic boundary value problem*, Proc. Roy. Soc. Edinburgh Sect. A **122** (1992), no. 3-4, 341–352.

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