EXISTENCE OF INFINITELY MANY NODAL SOLUTIONS FOR A SUPERLINEAR NEUMANN BOUNDARY VALUE PROBLEM

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We study the existence of a class of nonlinear elliptic equation with Neumann boundary condition, and obtain infinitely many nodal solutions. The study of such a problem is based on the variational methods and critical point theory. We prove the conclusion by using the symmetric mountain-pass theorem under the Cerami condition.

1. Introduction

Consider the Neumann boundary value problem:

$$\begin{aligned} -\triangle u + \alpha u &= f(x,u), \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\alpha > 0$ is a constant. Denote by $\sigma(-\triangle) := \{\lambda_i \mid 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \dots\}$ the eigenvalues of the eigenvalue problem:

$$\begin{aligned} -\triangle u &= \lambda u, \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.2}$$

Let $F(x,s) = \int_0^s f(x,t)dt$, $G(x,s) = f(x,s)s - 2F(x,s)$. Assume

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R})$, $f(0) = 0$, and for some $2 < p < 2^* = 2N/(N-2)$ (for $N = 1,2$, take $2^* = \infty$), $c > 0$ such that

$$|f(x,u)| \leq c(1 + |u|^{p-1}), \quad (x,u) \in \Omega \times \mathbb{R}. \tag{1.3}$$

(f_2) There exists $L \geq 0$, such that $f(x,s) + Ls$ is increasing in s.

(f_3) $\lim_{|s|\to\infty}(f(x,s)s)/|s|^2 = +\infty$ uniformly for a.e. $x \in \Omega$.

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(f_4) There exist $\theta \geq 1$, $s \in [0,1]$ such that

$$\theta G(x,t) \geq G(x,st), \quad (x,u) \in \Omega \times \mathbb{R}. \tag{1.4}$$

(f_5) $f(x,-t) = -f(x,t)$, $(x,u) \in \Omega \times \mathbb{R}$.

Because of (f_3), (1.1) is called a superlinear problem. In [6, Theorem 9.38], the author obtained infinitely many solutions of (1.1) under (f_1)–(f_5) and

(AR) $\exists \mu > 2$, $R > 0$ such that

$$x \in \Omega, \quad |s| \geq R \Longrightarrow 0 < \mu F(x,s) \leq f(x,s)s. \tag{1.5}$$

Obviously, (f_3) can be deduced from (AR). Under (AR), the (PS) sequence of corresponding energy functional is bounded, which plays an important role for the application of variational methods. However, there are indeed many superlinear functions not satisfying (AR), for example, take $\theta = 1$, the function

$$f(x,t) = 2t\log(1+|t|) \tag{1.6}$$

while it is easy to see that the above function satisfies (f_1)–(f_5). Condition (f_4) is from [2] and (1.6) is from [4].

In view of the variational point, solutions of (1.1) are critical points of corresponding functional defined on the Hilbert space $E := W^{1,2}(\Omega)$. Let $X := \{u \in C^1(\Omega) \mid \partial u/\partial \nu = 0,\ x \in \partial\Omega\}$ a Banach space. We consider the functional

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \alpha u^2)\,dx - \int_\Omega F(x,u)\,dx, \tag{1.7}$$

where E is equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 + \alpha \int_\Omega u^2\right)^{1/2}. \tag{1.8}$$

By (f_1), J is of C^1 and

$$\langle J'(u), v\rangle = \int_\Omega (\nabla u \nabla v + \alpha u v)\,dx - \int_\Omega f(x,u)v\,dx, \quad u,v \in E. \tag{1.9}$$

Now, we can state our main result.

Theorem 1.1. *Under assumptions (f_1)–(f_5), (1.1) has infinitely many nodal solutions.*

Remark 1.2. [8, Theorem 3.2] obtained infinitely many solutions under (f_1)–(f_5) and

$(f_3)'$ $\lim_{|u|\to\infty} \inf (f(x,u)u)/|u|^\mu \geq c > 0$ uniformly for $x \in \Omega$, where $\mu > 2$.

$(f_4)'$ $f(x,u)/u$ is increasing in $|u|$.

It turns out that $(f_3)'$ and $(f_4)'$ are stronger than (f_3) and (f_4), respectively, furthermore, the function (1.6) does not satisfy $(f_3)'$, then Theorem 1.1 applied to Dirichlet boundary value problem improves [8, Theorem 3.2].

Remark 1.3. [1, Theorem 7.3] also got infinitely many nodal solutions for (1.1) under assumption that the functional is of C^2.

2. Preliminaries

Let E be a Hilbert space and $X \subset E$, a Banach space densely embedded in E. Assume that E has a closed convex cone P_E and that $P =: P_E \bigcap X$ has interior points in X, that is, $P = \overset{\circ}{P} \bigcup \partial P$, with $\overset{\circ}{P}$ the interior and ∂P the boundary of P in X. Let $J \in C^1(E,\mathbb{R})$, denote $K = \{u \in E : J'(u) = 0\}$, $J^c = \{u \in E : J(u) \le c\}$, $K_c = \{u \in K : J(u) = c\}$, $c \in \mathbb{R}$.

Definition 2.1. We say that J satisfies Cerami condition (C), if for all $c \in \mathbb{R}$

(i) Any bounded sequence $\{u_n\} \subset E$ satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ possesses a convergent subsequence.

(ii) There exist $\sigma, R, \beta > 0$ such that for any $u \in J^{-1}([c-\sigma, c+\sigma])$ with $\|u\| \ge R$, $\|J'(u)\|\|u\| \ge \beta$.

Definition 2.2 (see [3]). Let $M \subset X$ be an invariant set under σ. We say M is an admissible invariant set for J, if (a) M is the closure of an open set in X, that is, $M = \overset{\circ}{M} \bigcup \partial M$; (b) if $u_n = \sigma(t_n, v)$ for some $v \notin M$ and $u_n \to u$ in E as $t_n \to \infty$ for some $u \in K$, then $u_n \to u$ in X; (c) if $u_n \in K \bigcap M$ such that $u_n \to u$ in E, then $u_n \to u$ in X; (d) for any $u \in \partial M \backslash K$, $\sigma(t,u) \in \overset{\circ}{M}$ for $t > 0$.

In [5], we proved $J \in C^1(E,\mathbb{R})$ satisfier the deformation Lemma 2.3 under (PS) condition and assumption (Φ): $K(J) \subset X$, $J'(u) = u - A(u)$ for $u \in E$, $A : X \to X$ is continuous. It turns out that the same lemma still holds if J satisfies (C), that is.

Lemma 2.3. *Assume $J \in C^1(E,\mathbb{R})$ satisfies (Φ) and (C) condition. Let $M \subset X$ be an admissible invariant set to the pseudo-gradient flow σ of J. Define $K_c^1 = K_c \bigcap \overset{\circ}{M}$, $K_c^2 = K_c \bigcap (X \backslash M)$ for some c. Assume $K_c \bigcap \partial M = \emptyset$, there exits $\delta > 0$ such that $(K_c^1)_{4\delta} \bigcap (K_c^2)_{4\delta} = \emptyset$, where $(K_c^i)_{4\delta} = \{u \in E : d_E(u, K_c^i) < 4\delta\}$ for $i = 1,2$. Then there is $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$ and any compact subset $A \subset (J^{c+\varepsilon} \bigcap X) \bigcup M$, there is $\eta \in C([0,1] \times X, X)$ such that*

(i) *$\eta(t,u) = u$, if $t = 0$ or $u \notin J^{-1}([c-3\varepsilon_0, c+3\varepsilon_0]) \backslash (K_c^2)_\delta$;*

(ii) *$\eta(1, A \backslash (K_c^2)_{3\delta}) \subset J^{c-\varepsilon} \bigcup M$, and $\eta(1,A) \subset J^{c-\varepsilon} \bigcup M$ if $K_c^2 = \emptyset$;*

(iii) *$\eta(t,\cdot)$ is a homeomorphism of X for $t \in [0,1]$;*

(iv) *$J(\eta(\cdot,u))$ is nonincreasing for any $u \in X$;*

(v) *$\eta(t,M) \subset M$ for any $t \in [0,1]$;*

(vi) *$\eta(t,\cdot)$ is odd, if J is even and M is symmetric about the origin.*

Indeed, $\sigma > \varepsilon_0 > 0$ can be chosen small, where σ is from (ii) *of* (C), *such that $\|J'(u)\|^2/(1 + 2\|J'(u)\|) \ge 6\varepsilon_0/\delta$, $\forall u \in J^{-1}([c-3\varepsilon_0, c+3\varepsilon_0]) \backslash (K_c)_\delta$.*

In [3, 5], a version of symmetric mountain-pass theorem holds under (PS). (C) is weaker than (PS), but by above deformation Lemma 2.3, a version of "symmetric mountain-pass theorem" still follows.

Theorem 2.4. *Assume $J \in C^1(E,\mathbb{R})$ is even, $J(0) = 0$ satisfies (Φ) and $(C)_c$ condition for $c > 0$. Assume that P is an admissible invariant set for J, $K_c \bigcap \partial P = \emptyset$ for all $c > 0$, $E = \overline{\bigoplus_{j=1}^{\infty} E_j}$, where E_j are finite dimensional subspaces of X, and for each k, let $Y_k = \bigoplus_{j=1}^{k} E_j$ and $Z_k = \overline{\bigoplus_{j=k}^{\infty} E_j}$. Assume for each k there exist $\rho_k > \gamma_k > 0$, such that $\overline{\lim}_{k\to\infty} a_k < \infty$, where $a_k = \max_{Y_k \bigcap \partial B_{\rho_k}(0)} J(x)$, $b_k = \inf_{Z_k \bigcap \partial B_{\gamma_k}(0)} J(x) \to \infty$ as $k \to \infty$. Then J has a sequence of critical* *points* $u_n \in X\backslash(P\bigcup(-P))$ *such that* $J(u_n) \to \infty$ *as* $n \to \infty$, *provided* $Z_k \bigcap \partial B_{\gamma_k}(0) \bigcap P = \varnothing$ *for large* k.

3. Proof of Theorem 1.1

Proposition 3.1. *Under* (f_1)–(f_3) *and* (f_4), J *satisfies the* (C) *condition.*

Proof. For all $c \in \mathbb{R}$, since Sobolev embedding $H^1(\Omega) \to L^2(\Omega)$ is compact, the proof of (i) in (C) is trivial.

About (ii) of (C). If not, there exist $c \in \mathbb{R}$ and $\{u_n\} \subset H^1(\Omega)$ satisfying, as $n \to \infty$

$$J(u_n) \longrightarrow c, \quad ||u_n|| \longrightarrow \infty, \quad ||J'(u_n)||\,||u_n|| \longrightarrow 0, \tag{3.1}$$

then we have

$$\lim_{n\to\infty} \int_\Omega \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx = \lim_{n\to\infty} \left(J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle\right) = c. \tag{3.2}$$

Denote $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$, that is, $\{v_n\}$ is bounded in $H^1(\Omega)$, thus for some $v \in H^1(\Omega)$, we get

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{in } H^1(\Omega), \\ v_n &\longrightarrow v \quad \text{in } L^2(\Omega), \\ v_n &\longrightarrow v \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.3}$$

If $v = 0$, as the similar proof in [2], define a sequence $\{t_n\} \in \mathbb{R}$:

$$J(t_n u_n) = \max_{t\in[0,1]} J(tu_n). \tag{3.4}$$

If for some $n \in \mathbb{N}$, there is a number of t_n satisfying (3.4), we choose one of them. For all $m > 0$, let $\bar{v}_n = 2\sqrt{m}v_n$, it follows that

$$\lim_{n\to\infty} \int_\Omega F(x,\bar{v}_n)\,dx = \lim_{n\to\infty} \int_\Omega F(x, 2\sqrt{m}v_n)\,dx = 0. \tag{3.5}$$

Then for n large enough

$$J(t_n u_n) \ge J(\bar{v}_n) = 2m - \int_\Omega F(x,\bar{v}_n)\,dx \ge m, \tag{3.6}$$

that is, $\lim_{n\to\infty} J(t_n u_n) = +\infty$. Since $J(0) = 0$ and $J(u_n) \to c$, then $0 < t_n < 1$. Thus

$$\begin{aligned} &\int_\Omega \left(|\nabla(t_n u_n)|^2 + \alpha(t_n u_n)^2\right) - \int_\Omega f(x, t_n u_n)t_n u_n \\ &\quad = \langle J'(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\bigg|_{t=t_n} J(tu_n) = 0. \end{aligned} \tag{3.7}$$

We see that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2} f(s, t_n u_n) t_n u_n - F(x, t_n u_n)\right) dx \\
&\qquad = \frac{1}{2}\int_\Omega \left(|\nabla(t_n u_n)|^2 + \alpha(t_n u_n)^2\right) - \int_\Omega F(x, t_n u_n) \\
&\qquad = J(t_n u_n) \longrightarrow +\infty, \quad n \longrightarrow \infty.
\end{aligned} \tag{3.8}$$

From above, we infer that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2} f(s, u_n) u_n - F(x, u_n)\right) dx \\
&\qquad = \frac{1}{2}\int_\Omega G(x, u_n)\, dx \geq \frac{1}{2\theta}\int_\Omega G(x, t_n u_n)\, dx \\
&\qquad = \frac{1}{\theta}\int_\Omega \left(\frac{1}{2} f(s, t_n u_n) t_n u_n - F(x, t_n u_n)\right) dx \longrightarrow +\infty, \quad n \longrightarrow \infty,
\end{aligned} \tag{3.9}$$

which contradicts (3.2).

If $v \not\equiv 0$, by (3.1)

$$\int_\Omega \left(|\nabla u_n|^2 + \alpha u_n^2\right) - \int_\Omega f(x, u_n) u_n = \langle J'(u_n), u_n \rangle = o(1), \tag{3.10}$$

that is,

$$1 - o(1) = \int_\Omega \frac{f(x, u_n) u_n}{\|u_n\|^2} dx = \left(\int_{v \neq 0} + \int_{v=0}\right) \frac{f(x, u_n) u_n}{|u_n|^2} |v_n|^2 dx. \tag{3.11}$$

For $x \in \Omega' := \{x \in \Omega : v(x) \neq 0\}$, we get $|u_n(x)| \to +\infty$. Then by (f_3)

$$\frac{f(x, u_n(x)) u_n(x)}{|u_n(x)|^2} |v_n(x)|^2 dx \longrightarrow +\infty, \quad n \longrightarrow \infty. \tag{3.12}$$

By using Fatou lemma, since $|\Omega'| > 0$ ($|\cdot|$ is the Lebesgue measure in $\mathbb{R}^N$),

$$\int_{v \neq 0} \frac{f(x, u_n) u_n}{|u_n|^2} |v_n|^2 dx \longrightarrow +\infty, \quad n \longrightarrow \infty. \tag{3.13}$$

On the other hand, by (f_3), there exists $\gamma > -\infty$, such that $f(x,s)s/|s|^2 \geq \gamma$ for $(x,s) \in \Omega \times \mathbb{R}$. Moreover,

$$\int_{v=0} |v_n|^2 dx \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.14}$$

Now, there exists $\Lambda > -\infty$ such that

$$\int_{v=0} \frac{f(x, u_n) u_n}{|u_n|^2} |v_n|^2 dx \geq \gamma \int_{v=0} |v_n|^2 dx \geq \Lambda > -\infty, \tag{3.15}$$

together with (3.11) and (3.13), it is a contradiction.

This proves that J satisfies (C). $\square$

Proposition 3.2. *Under $(f_4)'$, then for $|t| \geq |s|$ and $ts \geq 0$, $G(x,t) \geq G(x,s)$, that is, (f_4) holds for $\theta = 1$.*

Proof. for $0 \leq s \leq t$,

$$\begin{aligned} G(x,t) - G(x,s) &= 2\left[\frac{1}{2}\big(f(x,t)t - f(x,s)s\big) - \big(F(x,t) - F(x,s)\big)\right] \\ &= 2\left[\int_0^t \frac{f(x,t)}{t}\tau d\tau - \int_0^s \frac{f(x,s)}{s}\tau d\tau - \int_s^t \frac{f(x,\tau)}{\tau}\tau d\tau\right] \\ &= 2\left[\int_s^t \left(\frac{f(x,t)}{t} - \frac{f(x,\tau)}{\tau}\right)\tau d\tau + \int_0^s \left(\frac{f(x,t)}{t} - \frac{f(x,s)}{s}\right)\tau d\tau\right] \geq 0. \end{aligned} \tag{3.16}$$

In like manner, for $t \leq s \leq 0$, $G(x,t) - G(x,s) \geq 0$. □

On $E := H^1(\Omega)$, let us define $P_E = \{u \in E : u(x) \geq 0, \text{ a.e. in } \Omega\}$, which is a closed convex cone. Let $X = C^1_\nu(\Omega)$, which is a Banach space and embedded densely in E. Set $P = P_E \bigcap X$, then P is a closed convex cone in X. Furthermore, $P = \overset{\circ}{P} \bigcup \partial P$ under the topology of X, that is, there exist interior points in X. We may define a partial order relation: $u, v \in X$, $u > v \Leftrightarrow u - v \in P \setminus \{0\}$, $u \gg v \Leftrightarrow u - v \in \overset{\circ}{P}$.

As the proof of those propositions in [5, Section 5], it turns out that condition $\mathbf{\Phi}$ is satisfied and P is an admissible invariant set for J under (f_1), (f_2), and (C) condition.

Proof of Theorem 1.1. Let $E_i = \ker(-\triangle - \lambda_i)$, $Y_k = \bigoplus_{i=1}^k E_i$ and $Z_k = \bigoplus_{i=k}^\infty E_i$. It shows that J is continuously differentiable by (f_1) and satisfies the $(C)_c$ condition for every $c \in R$ by Proposition 3.1.

(1) As the proof of [7, Theorem 3.7(3)], there exists $\gamma_k > 0$ such that for $u \in Z_k$, $\|u\| = \gamma_k$, we have

$$b_k := \inf_{Z_k \bigcap \partial B_{\gamma_k}(0)} J(u) \longrightarrow \infty, \quad k \longrightarrow \infty. \tag{3.17}$$

(2) Since $\dim Y_k < +\infty$ and all norms are equivalent on the finite dimensional space, there exists $C_k > 0$, for all $u \in Y_k$, we get

$$\frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \alpha u^2\right) = \frac{1}{2}\|u\|^2 \leq C_k |u|_2^2 \equiv C_k \int_\Omega |u|^2 dx. \tag{3.18}$$

Next, by (f_3), there exists $R_k > 0$ such that $F(x,s) \geq 2C_k|s|^2$ for $|s| \geq R_k$. Take $M_k := \max\{0, \inf_{|s| \leq R_k} F(x,s)\}$, then for all $(x,s) \in \Omega \times \mathbb{R}$, we obtain

$$F(x,s) \geq 2C_k|s|^2 - M_k. \tag{3.19}$$

It follows from (3.18) and (3.19) that, for all $u \in Y_k$

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \alpha u^2\right) - \int_\Omega F(x,u) \\ &\leq -C_k|u|_2^2 + M_k|\Omega| \leq -\frac{1}{2}\|u\|^2 + M_k|\Omega|, \end{aligned} \tag{3.20}$$

which implies that for ρ_k large enough ($\rho_k > \gamma_k$),

$$a_k := \max_{Y_k \bigcap \partial B_{\rho_k}(0)} J(u) \leq 0. \tag{3.21}$$

Moreover, for $k \geq 2$, $Z_k \bigcap P = \{0\}$. This can be seen by noting that for all $u \in P \setminus \{0\}$, $\int_\Omega u\phi_1(x)dx > 0$, while for $u \in Z_k$, $\int_\Omega u\phi_1(x)dx = 0$, where ϕ_1 is the first eigenfunction corresponding to λ_1, which implies $Z_k \bigcap \partial B_{\gamma_k}(0) \bigcap P = \emptyset$.

By Theorem 2.4, J has a sequence of critical points $u_n \in X \setminus (P \bigcup (-P))$ such that $J(u_n) \to \infty$ as $n \to \infty$, that is, (1.1) has infinitely many nodal solutions. □

Example 3.3. By Theorem 1.1, the following equation with $\alpha > 0$

$$\begin{gathered} -\triangle u + \alpha u = 2u \log(1 + |u|), \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega \end{gathered} \tag{3.22}$$

has infinitely many nodal solutions, while the result cannot be obtained by either [6, Theorem 9.12] or [8, Theorem 3.2].

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