# MULTIDIMENSIONAL KOLMOGOROV-PETROVSKY TEST FOR THE BOUNDARY REGULARITY AND IRREGULARITY OF SOLUTIONS TO THE HEAT EQUATION

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*Dedicated to I. G. Petrovsky*

This paper establishes necessary and sufficient condition for the regularity of a characteristic top boundary point of an arbitrary open subset of $\mathbb{R}^{N+1}$ ($N \geq 2$) for the diffusion (or heat) equation. The result implies asymptotic probability law for the standard $N$-dimensional Brownian motion.

## 1. Introduction and main result

Consider the domain

$$\Omega_\delta = \{(x,t) \in \mathbb{R}^{N+1} : |x| < h(t),\ -\delta < t < 0\}, \tag{1.1}$$

where $\delta > 0$, $N \geq 2$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $t \in \mathbb{R}$, $h \in C[-\delta, 0]$, $h > 0$ for $t < 0$ and $h(t) \downarrow 0$ as $t \uparrow 0$.

For $u \in C^{2,1}_{x,t}(\Omega_\delta)$, we define the diffusion (or heat) operator

$$Du = u_t - \Delta u = u_t - \sum_{i=1}^{N} u_{x_i x_i}, \quad (x,t) \in \Omega_\delta. \tag{1.2}$$

A function $u \in C^{2,1}_{x,t}(\Omega_\delta)$ is called parabolic in $\Omega_\delta$ if $Du = 0$ for $(x,t) \in \Omega_\delta$. Let $f : \partial\Omega \to \mathbb{R}$ be a bounded function. First boundary value problem (FBVP) may be formulated as follows.

Find a function $u$ which is parabolic in $\Omega_\delta$ and satisfies the conditions

$$f_* \leq u_* \leq u^* \leq f^* \quad \text{for } z \in \partial\Omega_\delta, \tag{1.3}$$

where $f_*$, $u_*$ (or $f^*$, $u^*$) are lower (or upper) limit functions of $f$ and $u$, respectively.

Assume that $u$ is the generalized solution of the FBVP constructed by Perron's supersolutions or subsolutions method (see [1, 6]). It is well known that, in general, the generalized solution does not satisfy (1.3). We say that a point $(x_0, t_0) \in \partial\Omega_\delta$ is regular if, for any bounded function $f : \partial\Omega \to \mathbb{R}$, the generalized solution of the FBVP constructed by Perron's method satisfies (1.3) at the point $(x_0, t_0)$. If (1.3) is violated for some $f$, then $(x_0, t_0)$ is called irregular point.


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The principal result of this paper is the characterization of the regularity and irregularity of the origin ($\mathbb{O}$) in terms of the asymptotic behavior of $h$ as $t \uparrow 0$.

We write $h(t) = 2(t\log\rho(t))^{1/2}$, and assume that $\rho \in C[-\delta,0]$, $\rho(t) > 0$ for $-\delta \le t < 0$; $\rho(t) \downarrow 0$ as $t \uparrow 0$ and

$$\log\rho(t) = o\big(\log(|t|)\big) \quad \text{as } t \uparrow 0 \tag{1.4}$$

(see Remark 1.2 concerning this condition). The main result of this paper reads as follows.

Theorem 1.1. *The origin ($\mathbb{O}$) is regular or irregular according as*

$$\int^{0-} \frac{\rho(t)\,|\log\rho(t)|^{N/2}}{t}\,dt \tag{1.5}$$

*diverges or converges.*

For example, (1.5) diverges for each of the following functions

$$\begin{gathered}\rho(t) = |\log|t||^{-1}, \quad \rho(t) = \{|\log|t||\log^{(N+2)/2}|\log|t||\}^{-1},\\ \rho(t) = \left\{|\log|t||\log^{(N+2)/2}|\log|t||\prod_{k=3}^{n}\log_k|t|\right\}^{-1}, \quad n = 3,4,\dots,\end{gathered} \tag{1.6}$$

where we use the following notation:

$$\log_2|t| = \log|\log|t||, \qquad \log_n|t| = \log\log_{n-1}|t|, \quad n \ge 3. \tag{1.7}$$

From another side, (1.5) converges for each function

$$\begin{gathered}\rho(t) = |\log|t||^{-(1+\epsilon)}, \quad \rho(t) = \{|\log|t||\log^{(N+2)/2+\epsilon}|\log|t||\}^{-1},\\ \rho(t) = \{|\log|t||\log^{(N+2)/2}|\log|t||\log_3^{1+\epsilon}|t|\}^{-1},\\ \rho(t) = \{|\log|t||\log^{(N+2)/2}|\log|t||\log_3|t|\log_4^{1+\epsilon}|t|\}^{-1},\end{gathered} \tag{1.8}$$

and so forth, where $\epsilon > 0$ is sufficiently small number.

If we take $N = 1$, then Theorem 1.1 coincides with the result of Petrovsky's celebrated paper [6]. From the proof of Theorem 1.1, it follows that if (1.5) converges (in particular, for any example from (1.8)), then the function $u(x,t)$ which is parabolic in $\Omega_\delta$, vanishes on the lateral boundary of $\Omega_\delta$ and is positive on its bottom, cannot be continuous at the point $\mathbb{O}$, and its upper limit at $\mathbb{O}$ must be positive.

It should be mentioned that Wiener-type necessary and sufficient condition for boundary regularity is proved in [2]. However, it seems impossible to derive Theorem 1.1 from Wiener condition.

As in [6], a particular motivation for the consideration of the domain $\Omega_\delta$ is the problem about the local asymptotic behavior of the Brownian motion trajectories for the diffusion processes. We briefly describe the probabilistic counterpart of Theorem 1.1 in the context of the multidimensional Brownian motion. Consider the standard $N$-dimensional Brownian motion

$$\mathfrak{D} = \left[\xi(t) = \left(x_1(t), x_2(t), \ldots, x_N(t)\right) : t \geq 0, P_\bullet\right], \tag{1.9}$$

in which the coordinates of the sample path are independent standard 1-dimensional Brownian motions and $P_\bullet(B)$ is the probability of $B$ as a function of the starting point $\xi(0)$ of the $N$-dimensional Brownian path (see [3]). Consider the radial part $r(t) = (x_1^2(t) + x_2^2(t) + \cdots + x_N^2(t))^{1/2} : t \geq 0$ of the standard $N$-dimensional Brownian path. Blumenthal's 01 law implies that $P_0[r(t) < h(t),\ t \downarrow 0] = 0$ or 1; $h$ is said to belong to the upper class if this probability is 1 and to the lower class otherwise. The probabilistic analog of Theorem 1.1 states that if $h \in \uparrow$ and if $t^{-1/2}h \in \downarrow$ for small $t > 0$, then $h$ belongs to the upper class or to the lower class according as

$$\int_{0+} t^{-N/2-1} h^N(t) \exp\left(-\frac{h^2}{2t}\right) dt \tag{1.10}$$

converges or diverges. When $N = 1$, this is well-known Kolmogorov-Petrovsky test. Note that the integral (1.10) reduces to (1.5) (with coefficient $2^{N/2}$) if we replace $h^2(t)$ with $-2t\log\rho(-t)$. By adapting the examples (1.6) and (1.8), we easily derive that for any positive integer $n > 1$, the function

$$h(t) = \left(2t\left[\log_2\frac{1}{t} + \frac{N+2}{2}\log_3\frac{1}{t} + \log_4\frac{1}{t} + \cdots + \log_{n-1}\frac{1}{t} + (1+\epsilon)\log_n\frac{1}{t}\right]\right)^{1/2} \tag{1.11}$$

belongs to the upper or to the lower class according as $\epsilon > 0$ or $\epsilon \leq 0$.

Obviously, one can replace the integral (1.10) with the simpler one for the function $h_1(t) = t^{-1/2}h(t)$:

$$\int_{0+} t^{-1} h_1^N(t) \exp\left(-\frac{h_1^2}{2}\right) dt. \tag{1.12}$$

It should be mentioned that the described probabilistic counterpart of Theorem 1.1 is well known (see survey article [5, page 181]) and there are various known proofs of the $N$-dimensional Kolmogorov-Petrovsky test in the probabilistic literature (see [3]). Recently in [4], a martingale proof of the $N$-dimensional Kolmogorov-Petrovsky test for Wiener processes is given.

*Remark 1.2.* It should be mentioned that we do not need the condition (1.4) for the proof of the irregularity assertion of Theorem 1.1 and it may be replaced with the weaker assumption that $t\log(\rho(t)) \to 0$ as $t \downarrow 0$. The latter is needed just to make $\mathbb{O}$ the top boundary point of $\Omega_\delta$. For the regularity assertion of Theorem 1.1, the assumption (1.4) makes almost no loss of generality. First of all, this condition is satisfied for all examples from (1.6) and (1.8). Secondly, note that the class of functions satisfying (1.4) contains the class of functions satisfying the following inequality:

$$\rho(t) \geq \rho_C^M = \left|\log(Ct)\right|^{-M} \tag{1.13}$$

for all small $|t|$ and for some $C < 0$, $M > 1$. Since the integral (1.5) is divergent, the function $\rho(t)$ may not satisfy (1.13) with reversed inequality and for all small $|t|$ because (1.5) is convergent for each function $\rho_C^M(t)$. Accordingly, the condition (1.13), together with divergence of (1.5), excludes only pathological functions with the property that in any small interval $-\epsilon < t < 0$ they intersect infinitely many times all the functions $\rho_C^M$, with $C < 0$, $M > 1$. We handle this kind of pathological functions in Section 3 within the proof of the irregularity assertion. Finally, we have to mention that the assumption (1.4) (or even (1.13)) makes no loss of generality in the probabilistic context. Indeed, since (1.10) is divergent, any function $h(t) = (-2t\log\rho_C^M(-t))^{1/2}$ with $C < 0$ and $M > 1$ belongs to the lower class. Hence, to get improved lower functions, it is enough to stay in the class of functions $h(t) = (-2t\log\rho(-t))^{1/2}$ with $\rho$ satisfying (1.13) (or (1.4)).

We present some preliminaries in Section 2. The proof of the cheap irregularity part of Theorem 1.1 is presented in Section 3, while a regularity assertion is proved in Section 4.

## 2. Preliminary results

Let $\Omega \subset \mathbb{R}^{N+1}$ ($N \geq 2$) denote any bounded open subset and $\partial\Omega$ its topological boundary. For a given point $z_0 = (x^0, t_0)$ and a positive number $\epsilon$, define the cylinder

$$Q(z_0,\epsilon) = \{z = (x,t) : |x - x_0| < \epsilon,\ t_0 - \epsilon < t < t_0\}. \tag{2.1}$$

For the definition of the parabolic boundary $\mathcal{P}\Omega$, lateral boundary $\mathcal{S}\Omega$, and basic facts about Perron's solution, super- and subsolutions of the FBVP, we refer to the paper in [1]. It is a standard fact in the classical potential theory that the boundary point $z_0 \in \mathcal{S}\Omega$ is regular if there exists a so-called "regularity barrier" $\overline{u}$ with the following properties:

(a) $\overline{u}$ is superparabolic in $U = Q(z_0,\epsilon) \cap \Omega$ for some $\epsilon > 0$;

(b) $\overline{u}$ is continuous and nonnegative in $\overline{U}$, vanishing only at $z_0$.

It is also a well-known fact in the classical potential theory that in order to prove the irregularity of the boundary point $z_0 \in \mathcal{S}\Omega$, it is essential to construct a so-called irregularity barrier $\underline{u}$ with the following properties:

(a) $\underline{u}$ is subparabolic in $U = Q(z_0,\epsilon) \cap \Omega$ for some $\epsilon > 0$;

(b) $\underline{u}$ is continuous on the boundary of $U$, possibly except at $z_0$, where it has a removable singularity;

(c) $\underline{u}$ is continuous in $\overline{U} \setminus \{z_0\}$ and

$$\limsup_{z \to z_0,\, z \in U} \underline{u} > \limsup_{z \to z_0,\, z \in \partial U} \underline{u}. \tag{2.2}$$

Obviously, we have

$$\mathcal{P}\Omega_\delta = \partial\Omega_\delta, \qquad \mathcal{S}\Omega_\delta = \{z : |x| = h(t),\ -\delta < t \leq 0\}. \tag{2.3}$$

Assume that all the boundary points $z \in \mathcal{S}\Omega \setminus \{\mathbb{O}\}$ are regular points. For example, this is the case if $\rho(t)$ is differentiable for $t < 0$. Then concerning the regularity or irregularity of $\mathbb{O}$, we have the following.

Lemma 2.1. *The origin ($\mathbb{O}$) is regular for $\Omega_\delta$ if and only if there exists a regularity barrier $\overline{u}$ for $\mathbb{O}$ regarded as a boundary point of $\Omega_\delta$ for sufficiently small $\delta$.*

The proof is similar to the proof of Lemma 2.1 of [1].

Lemma 2.2. *The origin ($\mathbb{O}$) is irregular for $\Omega_\delta$ if and only if there exists an irregularity barrier $\underline{u}$ for $\mathbb{O}$ regarded as a boundary point of $\Omega_\delta$ for sufficiently small $\delta$.*

*Proof.* The proof of the "if" part is standard (see [6]). Take a boundary function $f = \underline{u}$ at the points near $\mathbb{O}$ (at $\mathbb{O}$ define it by continuity) and $f = c$ at the rest of the boundary with $c > \sup|\underline{u}|$. Let $u = H_f^{\Omega_\delta}$ be Perron's solution. Applying the maximum principle to $u - \underline{u}$ in domains $\Omega_\delta \cap \{t < \epsilon < 0\}$ and passing to limit as $\epsilon \uparrow 0$, we derive that $u \geq \underline{u}$ in $\Omega_\delta$. In view of property (c) of the irregularity barrier, we have discontinuity of $u$ at $\mathbb{O}$. To prove the "only if" part, take $f = -t$ and let $\underline{u} = H_f^{\Omega_\delta}$ be Perron's solution. Since all the boundary points $z_0 \in \mathcal{P}\Omega_\delta$, $z_0 \neq \mathbb{O}$ are regular points, $u$ is continuous in $\overline{\Omega_\delta}\backslash\mathbb{O}$ and in view of the maximum principle, it is positive in $\Omega_\delta$. Therefore, $u$ must be discontinuous at $\mathbb{O}$. Otherwise, it is a regularity barrier and we have a contradiction with Lemma 2.1. The lemma is proved. □

The next lemma immediately follows from Lemmas 2.1 and 2.2.

Lemma 2.3. *Let $\Omega$ be a given open set in $\mathbb{R}^{N+1}$ and $\mathbb{O} \in \mathcal{P}\Omega$, $\Omega^- \neq \varnothing$, where $\Omega^- = \{z \in \Omega : t < 0\}$. If $\Omega^- \subset \Omega_\delta$, then from the regularity of $\mathbb{O}$ for $\Omega_\delta$, it follows that $\mathbb{O}$ is regular for $\Omega$. Otherwise speaking, from the irregularity of $\mathbb{O}$ for $\Omega$ or $\Omega^-$, it follows that $\mathbb{O}$ is irregular for $\Omega_\delta$.*

Obviously, "if" parts of both Lemmas 2.1 and 2.2 are true without assuming that the boundary points $z \in \mathcal{S}\Omega\backslash\{\mathbb{O}\}$ are regular points.

## 3. Proof of the irregularity

First, we prove the irregularity assertion of Theorem 1.1 by assuming that $\rho(t)$ is differentiable for $t < 0$ and

$$\frac{t\rho'(t)}{\rho(t)} = O(1) \quad \text{as } t \uparrow 0. \tag{3.1}$$

Under these conditions, we construct an irregularity barrier $\underline{u}$, exactly as it was done in [6] for the case $N = 1$. Consider the function

$$v(x,t) = -\rho(t)\exp\left(-\frac{|x|^2}{4t}\right) + 1, \tag{3.2}$$

which is positive in $\Omega_\delta$ and vanishes on $\mathcal{S}\Omega_\delta$. Since $0 \leq v \leq 1$ in $\overline{\Omega}_\delta$, we have

$$\lim_{t\uparrow 0} v(0,t) = \limsup_{z\to 0,\, z\in\Omega_\delta} v = 1. \tag{3.3}$$

Hence, $v$ satisfies all the conditions of the irregularity barrier besides subparabolicity. We have

$$Dv = \left( -\rho'(t) - \frac{N\rho(t)}{2t} \right) \exp\left( -\frac{|x|^2}{4t} \right). \tag{3.4}$$

Since $\rho(t) \downarrow 0$ as $t \uparrow 0$, $Dv > 0$ and accordingly, it is a superparabolic function. We consider a function $w$ with the following properties

$$Dw = -Dv, \qquad w(x,t) < 0 \quad \text{in } \Omega_\delta \tag{3.5}$$

$$|w(0,t)| \leq \frac{1}{2} \quad \text{for } -\delta < t < 0. \tag{3.6}$$

Clearly, the function $\underline{u}(x,t) = w(x,t) + v(x,t)$ would be a required irregularity barrier. As a function $w$, we choose a particular solution of the equation from (3.5):

$$w(x,t) = -\frac{1}{(4\pi)^{N/2}} \int_{\Omega_\delta \setminus \Omega_t} \frac{\exp\left(-|x-y|^2/4(t-\tau)\right)}{(t-\tau)^{N/2}} Dv(y,\tau)\, dy\, d\tau. \tag{3.7}$$

Since $Dv > 0$ in $\Omega_\delta$, $w$ is negative and we only need to check that for sufficiently small $\delta$, (3.6) is satisfied. From (3.1) it follows that

$$|Dv| < C_1 \left| \frac{\rho(t)}{t} \right| \exp\left( -\frac{|x|^2}{4t} \right), \tag{3.8}$$

where $C_1 = C + N/2$ and $C$ is a constant due to (3.1). Hence,

$$|w(0,t)| < \frac{C_1}{(4\pi)^{N/2}} \int_{-\delta}^{t} \frac{\rho(\tau)}{|\tau|(t-\tau)^{N/2}} \int_{B((4\tau\log\rho(\tau))^{1/2})} \exp\left( -\frac{|y|^2 t}{4(t-\tau)\tau} \right) dy\, d\tau, \tag{3.9}$$

where $B(R) = \{y \in \mathbb{R}_N : |y| < R\}$. Changing the variable in the second integral, we have

$$|w(0,t)| < \frac{C_1}{(4\pi)^{N/2}} \int_{-\delta}^{t} \frac{\rho(\tau)}{|\tau|(t-\tau)^{N/2}} \left( \frac{\tau}{t} \right)^{N/2} \int_{B((4t\log\rho(\tau))^{1/2})} \exp\left( -\frac{|y|^2}{4(t-\tau)} \right) dy\, d\tau. \tag{3.10}$$

We split $\int_{-\delta}^{t}$ into two parts as $\int_{2t}^{t} + \int_{-\delta}^{2t}$ and estimate the first part as follows:

$$\begin{aligned} &\left| \int_{2t}^{t} \frac{\rho(\tau)}{\tau(t-\tau)^{N/2}} \left( \frac{\tau}{t} \right)^{N/2} \int_{B((4t\log\rho(\tau))^{1/2})} \exp\left( -\frac{|y|^2}{4(t-\tau)} \right) dy\, d\tau \right| \\ &\quad < 2^N \left| \int_{2t}^{t} \frac{\rho(\tau)}{\tau} \left( \frac{\tau}{t} \right)^{N/2} \int_{\mathbb{R}^N} \exp\left(-|y|^2\right) dy\, d\tau \right| \\ &\quad < 2^N (2\pi)^{N/2} \int_{2t}^{t} \frac{\rho(\tau)}{|\tau|} d\tau. \end{aligned} \tag{3.11}$$

From the convergence of the integral (1.5), it follows that the right-hand side of (3.11) converges to zero as $t \uparrow 0$. We also have

$$\begin{aligned}
&\left| \int_{-\delta}^{2t} \frac{\rho(\tau)}{\tau(t-\tau)^{N/2}} \left(\frac{\tau}{t}\right)^{N/2} \int_{B((4t\log\rho(\tau))^{1/2})} \exp\left(-\frac{|y|^2}{4(t-\tau)}\right) dy\, d\tau \right| \\
&\quad < \omega_N \left| \int_{-\delta}^{2t} \frac{\rho(\tau)}{\tau} \left(\frac{\tau}{t}\right)^{N/2} \left(\frac{2}{-\tau}\right)^{N/2} \left(4t\log\rho(\tau)\right)^{N/2} d\tau \right| \\
&\quad = 2^{3N/2}\omega_N \int_{-\delta}^{2t} \frac{\rho(\tau)\left|\log\rho(\tau)\right|^{N/2}}{|\tau|} d\tau,
\end{aligned} \tag{3.12}$$

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. Hence, from the convergence of the integral (1.5), it follows that $|w(0,t)| < 1/2$ for $-\delta < t < 0$ if $\delta$ is sufficiently small.

Now we need to remove the additional assumptions imposed on $\rho$. To remove the differentiability assumption, consider a function $\rho_1(t)$ such that $\rho_1$ is $C^1$ for $t < 0$, $\rho_1 \downarrow 0$ as $t \uparrow 0$ and $\rho(t) < \rho_1(t) < 2\rho(t)$ for $-\delta \le t < 0$. Then we consider a domain $\Omega_\delta^1$ by replacing $\rho$ with $\rho_1$ in $\Omega_\delta$. Since the integral (1.5) converges for $\rho$, it also converges for $\rho_1$. Therefore, $\mathbb{O}$ is irregular point regarded as a boundary point of $\Omega_\delta^1$. Since $\Omega_\delta^1 \subset \Omega_\delta$ from Lemma 2.3, it follows that $\mathbb{O}$ is irregular point regarded as a boundary point of $\Omega_\delta$.

We now prove that the assumption (3.1) imposed on $\rho$ may be also removed. In fact, exactly this question was considered in [6]. However, there is a point which is not clearly justified in [6] and for that reason, we present a slightly modified proof of this assertion.

Consider a one-parameter family of curves

$$\rho_C(t) = \left|\log(Ct)\right|^{-3}, \quad C < 0, \qquad C^{-1} < t < 0. \tag{3.13}$$

Obviously, for each point $(\rho(t),t)$ on the quarter plane, there exists a unique value

$$C = C(t) = t^{-1}\exp\left(-\rho^{-1/3}(t)\right), \tag{3.14}$$

such that $\rho_C(t)$ passes through the point $(\rho(t),t)$. One cannot say anything about the behavior of $C(t)$ as $t \uparrow 0$. But it is clear that $tC(t) \downarrow 0$ as $t \uparrow 0$. It is also clear that if $C_1 < C_2 < 0$, then $\rho_{C_1}(t) > \rho_{C_2}(t)$ for $C_1^{-1} < t < 0$. It may be easily checked that for any $C < 0$, the function $\rho_C(t)$ satisfies all the conditions which we used to prove the irregularity of $\mathbb{O}$. Accordingly, $\mathbb{O}$ is irregular point regarded as a boundary point of $\Omega_\delta$ with $\rho$ replaced by $\rho_C$. By using Lemma 2.3, we conclude that if for some $C < 0$ and $t_0 < 0$,

$$\rho(t) \le \rho_C(t) \quad \text{for } t_0 \le t \le 0, \tag{3.15}$$

then $\mathbb{O}$ must be irregular regarded as a boundary point of $\Omega_\delta$. Hence, we need only to consider the function $\rho$ with the property that for arbitrary $C < 0$ and $t_0 < 0$, the inequality (3.15) is never satisfied. Since $\rho_C(C^{-1}+0) = +\infty$, it follows that within the interval $(-\delta,0)$, our function $\rho(t)$ must intersect all the functions $\rho_C(t)$ with $C \le -\delta^{-1}$. Therefore, at least for some sequence $\{t_n\}$, we have $C(t_n) \to -\infty$ as $t_n \uparrow 0$. In [6], Petrovsky introduced the set $M$ formed by all values of $t$ $(0 > t > -\delta)$ with the following property (which is called "condition $C$" in [6]): the curve $\rho_C(t)$ which passes through the point $(\rho(t),t)$ cannot intersect the curve $\rho = \rho(t)$ for any smaller value of $t > -\delta$. Denote by $\overline{M}$ the closure of $M$. It is claimed in [6] that "$C(t)$ monotonically decreases as $t \uparrow 0$ and $t \in \overline{M}$; moreover, $C(t)$ takes equal values at the end points of every interval forming the complement of $\overline{M}$."

We construct a function $\rho$ which shows that this assertion is, in general, not true. Consider two arbitrary negative and strictly monotone sequences $\{C_1^{(n)}\}$, $\{C_2^{(n)}\}$, $n = 0,1,2,\dots$ such that $C_1^0 = C_2^0 > -\delta^{-1}$ and

$$C_1^{(n)} \downarrow -\infty, \qquad C_2^{(n)} \uparrow 0 \quad \text{as } n \uparrow \infty. \tag{3.16}$$

We form by induction a new sequence $\{C_n\}$ via sequences $\{C_1^{(n)}\}$ and $\{C_2^{(n)}\}$:

$$\begin{gathered} C_0 = C_2 = C_1^{(0)}, \qquad C_1 = C_1^{(1)}, \qquad C_3 = C_2^{(1)}, \\ C_{4n} = C_{4n-3}, \qquad C_{4n+1} = C_1^{(n+1)}, \qquad C_{4n+2} = C_{4n-1}, \\ C_{4n+3} = C_2^{(n+1)}, \quad n = 1,2,\dots. \end{gathered} \tag{3.17}$$

The sequence $\{C_n\}$ has arbitrarily large oscillations between $-\infty$ and 0 as $n \uparrow \infty$. Our purpose is to construct a function $\rho(t)$, $-\delta < t < 0$ in such a way that the related function $C = C(t)$ will satisfy

$$C(a_n) = C_n, \quad n = 0,1,2,\dots \tag{3.18}$$

at some points $a_n$. We now construct the sequence $\{a_n\}$ by induction:

$$a_0 = -\delta, \quad 0 > a_{n+1} > \max\left(a_n; \frac{C_n}{C_{n+1}} a_n; \frac{1}{(n+1)C_{n+1}}\right), \; n = 0,1,2,\dots. \tag{3.19}$$

Having $\{a_n\}$, we define the values of the function $\rho$ at the end-points of intervals $(a_n, a_{n+1})$, $n = 0,1,2,\dots$, as

$$\rho(a_n) = \left|\log(C_n a_n)\right|^{-3}, \quad n = 0,1,2,\dots. \tag{3.20}$$

From (3.19) it follows that $\rho(a_n) \downarrow 0$ as $n \uparrow \infty$. Having the values $\{\rho(a_n)\}$, we construct monotonically decreasing function $\rho(t)$ as follows: $\rho$ is $C^1$ for $-\delta \le t < 0$ and if $C_{n+1} < C_n$ (resp., $C_{n+1} > C_n$) then within the interval $[a_n; a_{n+1}]$, $\rho(t)$ intersects each function $x = \rho_C(t)$ with $C_{n+1} \le C \le C_n$ (resp., with $C_n \le C \le C_{n+1}$) just once, and moreover at the intersection point, we have

$$\rho'(t) \ge (\text{resp., } \le) \rho'_C(t). \tag{3.21}$$

Obviously, it is possible to make this construction. Clearly, the related function $C = C(t)$ satisfies (3.18). It has infinitely large oscillations near 0 and for arbitrary $\overline{C}$ satisfying $-\infty \le \overline{C} \le 0$, there exists a sequence $t_n \uparrow 0$ as $n \uparrow \infty$ such that $C(t_n) \to \overline{C}$. One can easily see that according to the definition of the set $M$ given in [6], we have

$$M = \bigcup_{n=0}^{+\infty} \{ (a_{4n}, a_{4n+1}] \cup (a_{4n+2}, a_{4n+3}] \}. \tag{3.22}$$

In view of our definition, we have $C_{4n+1} < C_{4n}$, $C_{4n+3} > C_{4n+2}$, $n = 0, 1, 2, \ldots$. Accordingly, $C(t)$ is neither monotonically increasing nor monotonically decreasing function as $t \uparrow 0$ and $t \in \overline{M}$.

We now give a modified definition of the set $M$. It is easier to define the set $M$ in terms of the function $C(t)$:

$$M = \{ t \in [-\delta, 0) : C_1(t) = C(t) \}, \tag{3.23}$$

where $C_1(t) = \min_{-\delta \le \tau \le t} C(t)$. Denote by $(\overline{M})^c$ the complement of $\overline{M}$. Since $(\overline{M})^c$ is open set, we have

$$(\overline{M})^c = \bigcup_n (t_{2n-1}, t_{2n}). \tag{3.24}$$

From the definition, it follows that $C(t)$ monotonically decreases for $t \in \overline{M}$ and, moreover, we have

$$C(t_{2n-1}) = C(t_{2n}). \tag{3.25}$$

Indeed, we take $t', t'' \in M$ with $t' < t''$. Since $C_1(t') = C(t')$ and $C_1(t'') = C(t'')$, it follows that $C(t'') \le C(t')$. For $t', t'' \in \overline{M}$, the same conclusion follows in view of continuity of $C(t)$. To prove (3.25), first note that since $t_{2n-1}, t_{2n} \in \overline{M}$, we have $C_1(t_{2n-1}) = C(t_{2n-1})$ and $C_1(t_{2n}) = C(t_{2n})$. If (3.25) is not satisfied, then we have $C_1(t_{2n-1}) > C_1(t_{2n})$. Since $C_1$ is continuous function, there exists $\epsilon \in (0, t_{2n} - t_{2n-1})$ such that $C_1(t_{2n} - \epsilon) < C_1(t_{2n-1})$. Let $C_1(t_{2n} - \epsilon) = C(\theta)$. Obviously, $\theta \in (t_{2n-1}, t_{2n} - \epsilon]$ and $C_1(\theta) = C(\theta)$. But this is the contradiction with the fact that $(t_{2n-1}, t_{2n}) \in (\overline{M})^c$. Hence, (3.25) is proved.

If we apply the modified definition of $M$ to the example constructed above, then one can easily see that

$$\begin{gathered} M = \bigcup_{n=0}^{+\infty} [a_{4n}, a_{4n+1}], \qquad (\overline{M})^c = \bigcup_{n=0}^{+\infty} (a_{4n+1}, a_{4(n+1)}), \\ C(a_{4n+1}) = C_1^{(n+1)} = C(a_{4(n+1)}) = C_{4(n+1)-3} = C_{4n+1} \downarrow -\infty \quad \text{as } n \uparrow \infty. \end{gathered} \tag{3.26}$$

Now we define the new function $\rho_1(t)$ as follows:

(a) $\rho_1(t) = \rho(t)$ for $t \in \overline{M}$;

(b) $\rho_1(t) = |\log(C(t_{2n-1})t)|^{-3}$ for $t_{2n-1} < t < t_{2n}$.

Equivalent definition might be given simply by taking $\rho_1(t) = |\log(C_1(t)t)|^{-3}$, $-\delta \le t < 0$. Otherwise speaking, the function $C(t)$ defined for $\rho_1(t)$ via (3.14) coincides with $C_1(t)$. Obviously, $\rho_1$ is continuous function satisfying $\rho_1(t) \ge \rho(t)$ and possibly $\rho_1(t) \ne \rho(t)$ on a numerate number of intervals $(t_{2n-1}, t_{2n})$. This new function may be nondifferentiable at the points $t = t_{2n-1}, t_{2n}$. Therefore, we consider another function $\rho_2(t)$ with the following properties:

(a) $\rho_2$ is $C^1$ for $t < 0$;

(b) $\rho_2(t) \geq \rho_1(t)$;

(c) $\rho_2(t)$ satisfies everywhere weak condition C: the curve $x = \rho_C(t)$ which passes through the point $(\rho_2(t), t)$ may not satisfy the condition $\rho_C(t) < \rho_2(t)$ for any smaller value of $t > -\delta$;

(d) for arbitrary $\epsilon$ with $-\delta < \epsilon < 0$, we have

$$\left| \int_{-\delta}^{\epsilon} \frac{1}{t} \left( \rho_1(t) \left| \log \rho_1(t) \right|^{N/2} - \rho_2(t) \left| \log \rho_2(t) \right|^{N/2} \right) dt \right| < 1. \tag{3.27}$$

Obviously, this function may be constructed. Again, it is easier to express this construction in terms of the related function $C(t)$. Having a function $C_1(t)$, we consider a function $C_2(t)$ which is $C^1$ for $t < 0$, monotonically decreasing, $C_2(t) \leq C_1(t)$ for all $-\delta \leq t \leq 0$ and $tC_2(t) \to 0$ as $t \uparrow 0$. Then we consider a function $\rho_2(t)$ as

$$\rho_2(t) = \left| \log \left( C_2(t) t \right) \right|^{-3}, \quad -\delta < t < 0. \tag{3.28}$$

Monotonicity of $C_2(t)$ is equivalent to the property (c) of $\rho_2$. Finally, (d) will be achieved by choosing $C_2(t)$ close to $C_1(t)$. The rest of the proof coincides with Petrovsky's proof from [6]. First, it is easy to show that $\rho_2(t)$ satisfies (3.1). We have

$$\left| \frac{t\rho_C'(t)}{\rho_C(t)} \right| = \left| \frac{3}{\log(Ct)} \right|, \tag{3.29}$$

and the right-hand side is arbitrarily small for sufficiently small $Ct$. From the property (c) of the function $\rho_2(t)$, it follows that

$$\left| \rho_2'(t) \right| \leq \left| \rho_C'(t) \right| = \left| \frac{3}{t \log^4(Ct)} \right|, \tag{3.30}$$

provided that $C = C_2(t)$ or equivalently (3.28) is satisfied. Hence, we have

$$\left| \frac{t\rho_2'(t)}{\rho_2(t)} \right| = \left| \frac{3}{\log \left( C_2 t \right)} \right|. \tag{3.31}$$

Since $tC_2 \to 0$ as $t \uparrow 0$, the right-hand side is arbitrarily small for small $|t|$.

Consider a domain $\Omega_\delta^2$ by replacing $\rho$ with $\rho_2$ in $\Omega_\delta$. Since $\rho_2(t) \geq \rho_1(t)$, we have $\Omega_\delta^2 \subset \Omega_\delta$. From Lemma 2.3, it follows that if $\mathbb{O}$ is an irregular point regarded as a boundary point of $\Omega_\delta^2$, then it is also irregular point regarded as a boundary point of $\Omega_\delta$.

It remains only to show that the convergence of the integral (1.5) with $\rho$ implies the convergence of the integral (1.5) with $\rho = \rho_2$. In view of the property (d) of $\rho_2$, it is enough to show the convergence of the integral (1.5) with $\rho = \rho_1$. Having a modified definition of the set $M$, the elegant proof given in [6] applies with almost no change. The proof of the irregularity assertion is completed.

## 4. Proof of the regularity

First, we prove the regularity assertion of Theorem 1.1 by assuming that $\rho(t)$ is differentiable for $t < 0$, $\rho(t)$ satisfies (3.1), and

$$\rho(t) = O\Big(\big|\log|t|\big|^{-1}\Big), \quad \text{as } t \uparrow 0. \tag{4.1}$$

As in [6], the proof of the regularity of $\mathbb{O}$ is based on the construction of the one-parameter family of superparabolic functions $\overline{u}_h(x,t)$, $-\delta < h < 0$ with the following properties:

(a) $|1 - \overline{u}_h(x,-\delta)| \le 1/2$ and $|1 - \overline{u}_h(x,-\delta)| \to 0$ uniformly in $x$ as $h \to 0$;

(b) $\overline{u}_h(x,h) \to 0$ uniformly in $x$ as $h \to 0$;

(c) $\overline{u}_h(x,t) \ge 0$ in $\Omega_\delta \backslash \Omega_h$.

The existence of $\overline{u}_h$ with these properties implies the existence of the regularity barrier for $\mathbb{O}$ regarded as a boundary point of $\Omega_\delta$. Indeed, first we can choose a function $\rho_*(t)$ such that $\rho_*(t) < \rho(t)$ for $-\delta \le t < 0$, and moreover $\rho_*$ satisfies all the restrictions imposed on $\rho$. One can easily show that it is possible to choose such a function. Then we consider a domain $\Omega_\delta^*$ by replacing $\rho$ with $\rho_*$ in $\Omega_\delta$. Let $u_*$ be Perron's solution of FBVP in $\Omega_\delta^*$ with boundary function

$$f(x,t) = \begin{cases} \dfrac{1}{2} & \text{if } t = -\delta, \\ 0 & \text{if } t > -\delta. \end{cases} \tag{4.2}$$

For the domain $\Omega_\delta^*$, there exists a one-parameter family of supersolutions $\overline{u}_h^*$ with the same properties as $\overline{u}_h$. Obviously, $\overline{u}_h^*$ is an upper barrier for $u_*$. Accordingly, $u_*$ vanishes continuously at $\mathbb{O}$. From the strong maximum principle it follows that $u_*$ is positive in $\Omega_\delta^*$. Since $\Omega_\delta \subset \Omega_\delta^*$, it follows that $u_*$ is the regularity barrier for $\mathbb{O}$ regarded as a boundary point of $\Omega_\delta$.

We construct $\overline{u}_h$. Consider a function $v$ from (3.2) and let $w$ be some solution of the equation

$$Dw = \frac{N\rho(t)}{2t}\exp\Big(-\frac{|x|^2}{4t}\Big). \tag{4.3}$$

From (3.4), it follows that $v + w$ is a superparabolic function. As a function $w$, we consider the following particular solution of (4.3):

$$w(x,t) = \frac{1}{(4\pi)^{N/2}}\int_{\Omega_\delta\backslash\Omega_t} \frac{\exp\big(-|x-y|^2/4(t-\tau)\big)}{(t-\tau)^{N/2}}\frac{N\rho(\tau)}{2\tau}\exp\Big(-\frac{|y|^2}{4\tau}\Big)dy\,d\tau, \quad (x,t) \in \Omega_\delta. \tag{4.4}$$

We have $w \le 0$. We estimate $w(0,t)$ for small values of $|t|$. We have

$$w(0,t) = \frac{N}{2(4\pi)^{N/2}}\int_{-\delta}^{t}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\int_{B((4\tau\log\rho(\tau))^{1/2})}\exp\Big(-\frac{|y|^2 t}{4(t-\tau)\tau}\Big)dy\,d\tau. \tag{4.5}$$

Changing the variable in the second integral, we have

$$w(0,t)=\frac{N}{2(4\pi)^{N/2}}\int_{-\delta}^{t}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\left(\frac{\tau}{t}\right)^{N/2}\int_{B((4t\log\rho(\tau))^{1/2})}\exp\left(-\frac{|y|^2}{4(t-\tau)}\right)dy\,d\tau. \quad (4.6)$$

We split $(-\delta,t)$ into two parts $(-\delta,t\mu(t))$ and $(t\mu(t),t)$, where $\mu(t)$ is a positive function satisfying $\mu(t)\to+\infty$, $t\mu(t)\to 0$ as $t\uparrow 0$. For a while, we keep the function $\mu(t)$ free on our account. Its choice will be clear during the proof. Consider the integral

$$I=\int_{-\delta}^{t\mu(t)}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\left(\frac{\tau}{t}\right)^{N/2}\int_{B((4t\log\rho(\tau))^{1/2})}\exp\left(-\frac{|y|^2}{4(t-\tau)}\right)dy\,d\tau. \quad (4.7)$$

Since $\mu(t)\to+\infty$, we have $-\delta\le\tau\le t\mu(t)\ll 2t$ and $|t-\tau|>-(1/2)\tau$, if $|t|$ is sufficiently small. Hence,

$$\left|-\frac{|y|^2}{4(t-\tau)}\right|<\left|\frac{4t\log\rho(\tau)}{2\tau}\right|\le\left|\frac{2\log\rho(\tau)}{\mu(t)}\right|. \quad (4.8)$$

To make the right-hand side small, we assume here that $\mu(t)\ge k|\log\rho(t)|$, where $k$ is a sufficiently large positive number. Then we have

$$\left|-\frac{|y|^2}{4(t-\tau)}\right|<\frac{2}{k}\frac{\log\rho(\tau)}{\log\rho(t)}<\frac{2}{k}. \quad (4.9)$$

Thus, for any $\epsilon>0$, we can choose $k$ so large that

$$\exp\left(-\frac{|y|^2}{4(t-\tau)}\right)>1-\epsilon, \quad (4.10)$$

which implies that

$$\begin{aligned}(1-\epsilon)\omega_N&\int_{-\delta}^{t\mu(t)}\frac{\rho(\tau)}{|\tau|}\left(\frac{\tau}{t}\right)^{N/2}\left(\frac{4t\log\rho(\tau)}{t-\tau}\right)^{N/2}d\tau\\ &\le|I|\le\omega_N\int_{-\delta}^{t\mu(t)}\frac{\rho(\tau)}{|\tau|}\left(\frac{\tau}{t}\right)^{N/2}\left(\frac{4t\log\rho(\tau)}{t-\tau}\right)^{N/2}d\tau\end{aligned} \quad (4.11)$$

or

$$\begin{aligned}(1-\epsilon)\omega_N4^{N/2}&\int_{-\delta}^{t\mu(t)}\frac{\rho(\tau)|\log\rho(\tau)|^{N/2}}{|\tau|}\left(\frac{|\tau|}{t-\tau}\right)^{N/2}d\tau\\ &\le|I|\le\omega_N4^{N/2}\int_{-\delta}^{t\mu(t)}\frac{\rho(\tau)|\log\rho(\tau)|^{N/2}}{|\tau|}\left(\frac{|\tau|}{t-\tau}\right)^{N/2}d\tau.\end{aligned} \quad (4.12)$$

Since $-\delta \le \tau \le t\mu(t) \le kt\log\rho(t)$, we have

$$\frac{1}{1-k\log\rho(t)} \le \frac{t}{t-\tau} \le 0, \tag{4.13}$$

which implies that

$$\lim_{t\to 0}\frac{|\tau|}{t-\tau} = 1. \tag{4.14}$$

Hence, we have the following asymptotic relation:

$$\frac{I}{\omega_N 4^{N/2}\int_{-\delta}^{t\mu(t)}\left(\rho(\tau)\,|\log\rho(\tau)|^{N/2}/\tau\right)d\tau} \longrightarrow 1 \quad \text{as } t\uparrow 0. \tag{4.15}$$

Since the integral (1.5) is divergent, we easily get the following asymptotic relation:

$$\frac{w(0,t)}{(N\omega_N/2\pi^{N/2})\int_{-\delta}^{t}\left(\rho(\tau)\,|\log\rho(\tau)|^{N/2}/\tau\right)d\tau} \longrightarrow 1 \quad \text{as } t\uparrow 0, \tag{4.16}$$

provided that the following two integrals remain bounded as $t\uparrow 0$:

$$\begin{gathered} I_1 = \int_{t\mu(t)}^{t}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\left(\frac{\tau}{t}\right)^{N/2}\int_{B((4t\log\rho(\tau))^{1/2})}\exp\left(-\frac{|y|^2}{4(t-\tau)}\right)dy\,d\tau, \\ I_2 = \int_{t\mu(t)}^{t}\frac{\rho(\tau)\,|\log\rho(\tau)|^{N/2}}{|\tau|}d\tau. \end{gathered} \tag{4.17}$$

We split $I_2$ into the sum of two integrals along the intervals $(t\mu(t),Mt)$ and $(Mt,t)$, where $M>1$ is some number which we keep free on our account. For sufficiently small $|t|$, we have

$$\begin{gathered} \int_{Mt}^{t}\frac{\rho(\tau)\,|\log\rho(\tau)|^{N/2}}{|\tau|}d\tau < \int_{Mt}^{t}\frac{d\tau}{|\tau|} = \log M, \\ \int_{t\mu(t)}^{Mt}\frac{\rho(\tau)\,|\log\rho(\tau)|^{N/2}}{|\tau|}d\tau < \int_{t\mu(t)}^{Mt}\frac{\rho^{1/2}(\tau)}{|\tau|} = I_3, \end{gathered} \tag{4.18}$$

and we still need to prove that $I_1$ and $I_3$ remain bounded as $t\uparrow 0$. This will be proved below when we prove the boundedness of the integrals $I_4$ and $I_5$.

We now estimate $w$ inside $\Omega_\delta$ for small $|t|$ and $x\neq \mathbf{0}$. As before, we split the time integral into the sum of three integrals along the intervals $(-\delta,t\mu(t))$, $(t\mu(t),Mt)$ and $(Mt,t)$. Since

$$\frac{|x-y|^2}{4(t-\tau)}+\frac{|y|^2}{4\tau} = \frac{|x|^2}{4t}+\frac{|\tau x-ty|^2}{4t\tau(t-\tau)}, \tag{4.19}$$

we have

$$\begin{aligned}
I_4 &= \frac{N}{2(4\pi)^{N/2}}\int_{Mt}^{t}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\int_{B((4\tau\log\rho(\tau))^{1/2})}\exp\Big(-\frac{|\tau x-ty|^2}{4t\tau(t-\tau)}\Big)dy\,d\tau\exp\Big(-\frac{|x|^2}{4t}\Big)\\
&= \frac{N\exp(-|x|^2/4t)}{2(4\pi)^{N/2}}\int_{Mt}^{t}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\int_{B((4\tau\log\rho(\tau))^{1/2})}\exp\Big(-\frac{\big|(\tau/t)^{1/2}x-(t/\tau)^{1/2}y\big|^2}{4(t-\tau)}\Big)dy\,d\tau.
\end{aligned} \tag{4.20}$$

Introducing the new variable $(t/\tau)^{1/2}y$ instead of $y$, we have

$$I_4=\frac{N\exp(-|x|^2/4t)}{2(4\pi)^{N/2}}\int_{Mt}^{t}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\Big(\frac{\tau}{t}\Big)^{N/2}\int_{B((4t\log\rho(\tau))^{1/2})}\exp\Big(-\frac{\big|(\tau/t)^{1/2}x-y\big|^2}{4(t-\tau)}\Big)dy\,d\tau. \tag{4.21}$$

We also have

$$\begin{aligned}
&\frac{1}{(t-\tau)^{N/2}}\int_{B((4t\log\rho(\tau))^{1/2})}\exp\Big(-\frac{\big|(\tau/t)^{1/2}x-y\big|^2}{4(t-\tau)}\Big)dy\\
&\quad\le 2^N\int_{\mathbb{R}^N}\exp\big(-|z|^2\big)dz=(4\pi)^{N/2}.
\end{aligned} \tag{4.22}$$

Hence,

$$\begin{aligned}
|I_4| &\le \frac{1}{2}N\exp\Big(-\frac{|x|^2}{4t}\Big)\int_{Mt}^{t}\frac{\rho(\tau)}{|\tau|}\Big(\frac{\tau}{t}\Big)^{N/2}d\tau\\
&\le \frac{NM^{N/2}}{2\rho(t)}\int_{Mt}^{t}\frac{\rho(\tau)}{|\tau|}d\tau\le\frac{1}{2}NM^{N/2}\log M\frac{\rho(Mt)}{\rho(t)}.
\end{aligned} \tag{4.23}$$

From (3.1), it follows that

$$\log\frac{\rho(Mt)}{\rho(t)}=-\int_{Mt}^{t}\frac{\rho'(\tau)}{\rho(\tau)}d\tau\le -C\int_{Mt}^{t}\frac{d\tau}{\tau}=\log M^C, \tag{4.24}$$

where $C$ is a constant due to (3.1). Therefore, we have

$$|I_4|\le\frac{1}{2}NM^{N/2+C}\log M. \tag{4.25}$$

We now estimate the integral

$$\begin{aligned}
I_5 &= \frac{N}{2(4\pi)^{N/2}}\int_{t\mu(t)}^{Mt}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\int_{B((4\tau\log\rho(\tau))^{1/2})}\exp\Big(-\frac{|x-y|^2}{4(t-\tau)}-\frac{|y|^2}{4\tau}\Big)dy\,d\tau\\
&= \frac{N}{2(4\pi)^{N/2}}\int_{t\mu(t)}^{Mt}\frac{\rho(\tau)}{\tau(t-\tau)^{N/2}}\int_{B((4\tau\log\rho(\tau))^{1/2})}\exp\Big(-\frac{t|y|^2+\tau|x|^2-2\tau\langle x,y\rangle}{4\tau(t-\tau)}\Big)dy\,d\tau,
\end{aligned} \tag{4.26}$$

where $\langle x,y\rangle = \sum_{i=1}^{N} x_i y_i$. Assuming that $M>2$, from $\tau \le Mt$, it follows that $t-\tau > (1/2)|\tau|$. Therefore, we have

$$\begin{aligned} -\frac{t|y|^2+\tau|x|^2-2\tau\langle x,y\rangle}{4\tau(t-\tau)} &< \frac{\langle x,y\rangle}{2(t-\tau)} < \frac{|x||y|}{|\tau|} < \frac{4\big(t\log(\rho(t))\tau\log(\rho(\tau))\big)^{1/2}}{|\tau|} \\ &\le 4\left(\frac{t\log\rho(t)}{\tau\log\rho(\tau)}\right)^{1/2} |\log\rho(\tau)|. \end{aligned} \tag{4.27}$$

To estimate $(t\log\rho(t)/\tau\log\rho(\tau))^{1/2}$, we first observe that

$$\frac{d(\tau\log\rho(\tau))}{d\tau} = \log\rho(\tau) + \frac{\tau\rho'(\tau)}{\rho(\tau)}. \tag{4.28}$$

Since the second term is bounded function, it follows that the right-hand side is negative for small $|\tau|$ and accordingly, $\tau\log\rho(\tau)$ is decreasing function. From $\tau \le Mt$, it follows that

$$\frac{t\log\rho(t)}{\tau\log\rho(\tau)} \le \frac{1}{M}\frac{\log\rho(t)}{\log\rho(Mt)}. \tag{4.29}$$

We have already proved that $\rho(Mt) \le M^C\rho(t)$. Therefore, we have

$$1 \ge \frac{\log\rho(Mt)}{\log\rho(t)} \ge \frac{C\log M + \log\rho(t)}{\log\rho(t)} \longrightarrow 1 \quad \text{as } t \uparrow 0. \tag{4.30}$$

Hence,

$$\lim_{t\uparrow 0} \frac{1}{M}\frac{\log\rho(t)}{\log\rho(Mt)} = \frac{1}{M}, \tag{4.31}$$

and for arbitrary small $\epsilon > 0$, we have

$$\frac{t\log\rho(t)}{\tau\log\rho(\tau)} \le \frac{1}{M-\epsilon}, \quad \text{for } t\mu(t) < \tau < Mt, \tag{4.32}$$

if $|t|$ is sufficiently small. Finally, we have

$$-\frac{t|y|^2+\tau|x|^2-2\tau\langle x,y\rangle}{4\tau(t-\tau)} < -\frac{4}{(M-\epsilon)^{1/2}}\log\rho(\tau) \quad \text{for } t\mu(t) < \tau < Mt. \tag{4.33}$$

It follows that

$$\begin{aligned} |I_5| &< \frac{N\omega_N}{2(4\pi)^{N/2}} \int_{t\mu(t)}^{Mt} \frac{\rho^{1-4(M-\epsilon)^{-1/2}}(\tau)}{|\tau|\big(-(1/2)\tau\big)^{N/2}} \big(4\tau\log\rho(\tau)\big)^{N/2} d\tau \\ &= \frac{N\omega_N}{2(\pi/2)^{N/2}} \int_{t\mu(t)}^{Mt} \frac{\rho^{1-4(M-\epsilon)^{-1/2}}(\tau)\,|\log\rho(\tau)|^{N/2}}{|\tau|} d\tau. \end{aligned} \tag{4.34}$$

At this point we make precise choice of the number $M$. Since for arbitrary $\epsilon > 0$,

$$\rho^{\epsilon}(\tau)\,|\log\rho(\tau)|^{N/2} \longrightarrow 0, \quad \text{as } \tau \longrightarrow 0, \tag{4.35}$$

we can reduce the boundedness of $|I_5|$ to the boundedness of $|I_3|$ if we choose $M$ such that

$$1 - \frac{4}{(M-\epsilon)^{1/2}} > \frac{1}{2} \quad \text{or } M > 64+\epsilon. \tag{4.36}$$

As in [6], we fix the value $M = 65$. Hence, for sufficiently small $|t|$, we have

$$|I_5| < \frac{N\omega_N}{2(\pi/2)^{N/2}} \int_{t\mu(t)}^{65t} \frac{\rho^{1/2}(\tau)}{|\tau|} d\tau. \tag{4.37}$$

Applying (4.1), we have

$$\begin{aligned}\int_{t\mu(t)}^{65t} \frac{\rho^{1/2}(\tau)}{|\tau|} d\tau &< C\int_{t\mu(t)}^{65t} \frac{d\tau}{|\tau|\,|\log|\tau||^{1/2}} \\ &= 2C\Big(|\log|65t||^{1/2} - |\log|t\mu(t)||^{1/2}\Big) \\ &= \frac{2C(\log\mu(t) - \log 65)}{|\log|65t||^{1/2} + |\log|t\mu(t)||^{1/2}},\end{aligned} \tag{4.38}$$

where $C$ is a constant due to (4.1). Accordingly,

$$\lim_{t\uparrow 0} \int_{t\mu(t)}^{65t} \frac{\rho^{1/2}(\tau)}{|\tau|} d\tau = 0, \tag{4.39}$$

provided that

$$\lim_{t\uparrow 0} \frac{\log\mu(t)}{|\log|65t||^{1/2}} = 0. \tag{4.40}$$

We finally estimate the integral

$$\begin{aligned} I_6 &= \frac{N}{2(4\pi)^{N/2}} \int_{-\delta}^{t\mu(t)} \frac{\rho(\tau)}{\tau(t-\tau)^{N/2}} \int_{B((4\tau\log\rho(\tau))^{1/2})} \exp\left(-\frac{|x-y|^2}{4(t-\tau)} - \frac{|y|^2}{4\tau}\right) dy\, d\tau \\ &= \frac{N}{2(4\pi)^{N/2}} \int_{-\delta}^{t\mu(t)} \frac{\rho(\tau)}{\tau(t-\tau)^{N/2}} \int_{B((4\tau\log\rho(\tau))^{1/2})} \exp\left(-\frac{|y|^2 t}{4(t-\tau)\tau}\right) \exp\left(-\frac{|x|^2 - 2\langle x, y\rangle}{4(t-\tau)}\right) dy\, d\tau. \end{aligned} \tag{4.41}$$

We are going to prove that this integral is close to the corresponding integral for $w(0,t)$:

$$\frac{N}{2(4\pi)^{N/2}} \int_{-\delta}^{t\mu(t)} \frac{\rho(\tau)}{\tau(t-\tau)^{N/2}} \int_{B((4\tau\log\rho(\tau))^{1/2})} \exp\left(-\frac{|y|^2 t}{4(t-\tau)\tau}\right) dy\, d\tau. \tag{4.42}$$

For that purpose, we have to show that the term $\exp(-(|x|^2 - 2\langle x,y\rangle)/4(t-\tau))$ is close to 1 for small $|t|$. If $|x| \geq |y|$, then we have

$$\left|\frac{|x|^2 - 2\langle x,y\rangle}{4(t-\tau)}\right| < \frac{3|x||x|}{2|\tau|} < 6\left|\frac{t\log\rho(t)}{\tau}\right| \leq 6\left|\frac{t\log\rho(t)}{t\mu(t)}\right| \leq \frac{6}{k}, \tag{4.43}$$

and the right-hand side is small if $k$ is sufficiently large. If $|x| \leq |y|$, then we have

$$\begin{aligned}\left|\frac{|x|^2 - 2\langle x,y\rangle}{4(t-\tau)}\right| &< \frac{3|x||y|}{2|\tau|} < \left|\frac{3\big(4t\log\rho(t)\big)^{1/2}\big(4\tau\log\rho(\tau)\big)^{1/2}}{2\tau}\right| \\ &\leq 6\left(\frac{t\log\rho(t)\log\rho(\tau)}{t\mu(t)}\right)^{1/2} \leq 6\left(\frac{\log^2\rho(t)}{\mu(t)}\right)^{1/2}.\end{aligned} \tag{4.44}$$

We see here that in order to make the right-hand side small, the restriction $\mu(t) \geq k|\log\rho(t)|$ is not enough. We are forced to assume that $\mu(t) \geq k\log^2\rho(t)$, where $k$ is the sufficiently large positive number. Under this condition, we have

$$\left|\frac{|x|^2 - 2\langle x,y\rangle}{4(t-\tau)}\right| < \frac{6}{k^{1/2}} \quad \text{for } |x| \leq |y|. \tag{4.45}$$

Hence, in both cases $|(|x|^2 - 2\langle x,y\rangle)/4(t-\tau)|$ is sufficiently small for small $|t|$, provided that the constant $k$ is chosen large enough. At this point, we make precise choice of the function $\mu(t)$. We take $\mu(t) = k\log^2\rho(t)$ and check that (4.40) is satisfied. We have

$$\frac{|\log\mu(t)|}{\big|\log|65t|\big|^{1/2}} \leq \frac{\log k + 2\log|\log\rho(t)|}{\big|\log|65t|\big|^{1/2}}. \tag{4.46}$$

Applying l'Hopital's rule and (3.1), we derive that

$$\lim_{t\to 0}\frac{\log^2|\log\rho(t)|}{\big|\log|65t|\big|} = \lim_{t\to 0}\frac{2\log|\log\rho(t)|}{|\log\rho(t)|}\frac{t\rho'(t)}{\rho(t)} = 0. \tag{4.47}$$

Therefore, (4.40) is satisfied. Hence, we proved that

$$\frac{w(x,t)}{w(0,t)} \longrightarrow 1 \quad \text{as } t\uparrow 0 \text{ uniformly for all } x \text{ with } (x,t)\in\Omega_\delta. \tag{4.48}$$

Consider a function

$$\overline{u}_h(x,t) = \frac{v(x,t)+w(x,t)}{\sup_{\Omega_\delta\setminus\Omega_h}|w(x,t)|}+1. \tag{4.49}$$

As in [6], we can check that $\overline{u}_h$ satisfies the conditions (a), (b), and (c) formulated at the beginning of the proof. Accordingly, $\mathbb{O}$ is a regular point regarded as a boundary point of $\Omega_\delta$.

(a) We have $|v|<1$ in $\Omega_\delta$, $w(x,-\delta)=0$ and $w(0,t)\to-\infty$ as $t\uparrow 0$. This implies that

$$\overline{u}_h(x,-\delta)\longrightarrow 1 \quad \text{as } h\downarrow 0 \text{ uniformly for } x. \tag{4.50}$$

(b) We have

$$\begin{gathered}\frac{w(0,t)}{(N\omega_N/2\pi^{N/2})\int_{-\delta}^t\big(\rho(\tau)\,|\log\rho(\tau)|^{N/2}/\tau\big)d\tau}\longrightarrow 1 \quad \text{as } t\uparrow 0,\\ \int_{-\delta}^t\frac{\rho(\tau)\,|\log\rho(\tau)|^{N/2}}{\tau}d\tau\longrightarrow-\infty \quad \text{as } t\uparrow 0,\\ \frac{w(x,t)}{w(0,t)}\longrightarrow 1 \quad \text{as } t\uparrow 0 \text{ uniformly for all } x \text{ with } (x,t)\in\Omega_\delta.\end{gathered} \tag{4.51}$$

From these three conditions, it follows that

$$\overline{u}_h(x,h)\longrightarrow 0 \quad \text{as } h\uparrow 0 \text{ uniformly in } x. \tag{4.52}$$

(c) $\overline{u}_h(x,t)\geq 0$ in $\Omega_\delta\setminus\Omega_h$ since $v\geq 0$ in $\Omega_\delta\setminus\Omega_h$.

We now show that the regularity assertion of Theorem 1.1 is true without additional restrictions imposed on $\rho$. The differentiability assumption may be removed exactly as we did in Section 3. Assumption (4.1) may be removed exactly like Petrovsky did in [6]. Indeed, first of all, from the proof given above, it follows that $\mathbb{O}$ is regular regarded as a boundary point of $\Omega_\delta$ with $\rho(t)=|\log|t||^{-1}$. Therefore, from the Lemma 2.3, it follows that if $\rho(t)$ satisfies $\rho(t)\geq|\log|t||^{-1}$ for all sufficiently small $|t|$, then $\mathbb{O}$ is regular regarded as a boundary point of $\Omega_\delta$. Hence, assuming that (4.1) is not satisfied, we need only to consider functions $\rho(t)$ which has infinitely many intersections with the graph of the function $\rho(t)=|\log|t||^{-1}$ at any small interval $(\epsilon,0)$ with $\epsilon<0$. In [6], it is proved that under this condition the function $\rho_1(t)=\min\{\rho(t);|\log|t||^{-1}\}$ makes the integral $\int^{0-}(\rho_1(t)/t)dt$ divergent. It follows that the integral $\int^{0-}(\rho_1(t)|\log\rho_1(t)|^{N/2}/t)dt$ is also divergent. The function $\rho_1$ satisfies (4.1), and therefore $\mathbb{O}$ is regular regarded as a boundary point of $\Omega_\delta$ with $\rho$ replaced by $\rho_1$. Since $\rho_1\leq\rho$, from Lemma 2.3, it follows that $\mathbb{O}$ is regular regarded as a boundary point of $\Omega_\delta$ as well. Finally, to remove (3.1), we can use the condition (1.4). Indeed, applying l'Hopital's rule, we have

$$0=\lim_{t\to 0}\frac{\log\rho(t)}{\log|t|}=\lim_{t\to 0}\frac{t\rho'(t)}{\rho(t)}. \tag{4.53}$$

Theorem 1.1 is proved.

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