

Research Article

Stochastic Theories and Deterministic Differential Equations

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We discuss the concept of “hydrodynamic” stochastic theory, which is not based on the traditional Markovian concept. A Wigner function developed for friction is used for the study of operators in quantum physics, and for the construction of a quantum equation with friction. We compare this theory with the quantum theory, the Liouville process, and the Ornstein-Uhlenbeck process. Analytical and numerical examples are presented and compared.

1. Introduction

Stochastic theories model systems which develop in time and space in accordance with probabilistic laws. (The space is not necessarily the familiar Euclidean space for everyday life. We distinguish between cases which are discrete and continuous in time or space. See Doob [1] or Taylor and Karlin [2] for a mathematical definition of stochastic processes, which is not replicated here. Briefly, the usual situation is to have a set of random variables $\{X_t\}$ defined for all values of the real number t (say time), which could be discrete or continuous. The outcome of a random variable is a state value (often a real number). The set of random variables are called a stochastic process, which is completely determined if the joint distribution of the set of random variables $\{X_t\}$ is known. A realization of the stochastic process is an assignment to each t in the set $\{X_t\}$, a value of X_t .) Essential in stochastic theories is how randomness is accounted for. For Markov [3] processes (in the narrowest sense, a stochastic process has the Markov property if the probability of having state $X_{t+\Delta t}$ at time $t+\Delta t$, conditioned on having the particular state x_t at time t , is equal to the conditional probability

of having that same state $X_{t+\Delta t}$ but conditioned on its value for *all* previous times before t . See Feller [4] for a broader definition.), which are an important class of stochastic processes, a recurrence relation is used such that the state value of $X_{t+\Delta t}$ at time $t + \Delta t$ is given by the state value at time t , plus a state value of a random variable at time t . (By “counting up” the different realizations (tracks) in the state space the joint distribution can be constructed. Although counting up all different realizations in general constructs the joint probability, the inverse does not hold. Hence the joint probability of the set of random variables $\{X_t\}$ does not lead to a unique recurrence relation.) A random “disturbance” in a Markov process may possibly influence all subsequent values of the realization. The influence may decrease rapidly as the time point moves into the future. (A Markov process may be deterministic, that is, all values of the process at time $t' > t$ are determined when the value is given at time t . Or a process may be nondeterministic, that is, a knowledge of the process at time t is only probabilistically useful in specifying the process at time $t' > t$.) This paper considers a so-called “hydrodynamic” approach to account for randomness. We specify constitutive relations in an equation set akin to what is used in hydrodynamic formulations of gas flow (e.g., [5]).

Consider the variables as position and velocity for illustration, but the method applies generally. By integrating the equation for the joint distribution for two stochastic variables with respect to the second variable (velocity), the well-known equation for the conservation of probability in space is found. This equation, which is only the conservation of probability, can be used without referring to any stochastic theory. The equation includes the so-called current velocity. It is well known that in Boltzmann kinetic theory or in most Langevin models, the total derivative of the current velocity is equal to the classical force minus a term that is proportional to $1/\rho_X(x, t)$ times the space derivative of $\text{Var}(Y_t/X_t)\rho_X(x, t)$, where $\text{Var}(Y_t/X_t)$ is the variance of Y_t at time t given the position of X_t , and $\rho_X(x, t)$ is the density of X_t [6]. Now, the equation for the conservation of probability in space is a first partial differential equation. As a second equation, set the total derivative of the current velocity equal to the classical force minus a term that is proportional to $1/\rho_X(x, t)$ times the space derivative of $-\rho_X(x, t) \text{Var}(Y_t/X_t)$ as in Boltzmann’s kinetic theory or in Langevin models. Thus randomness can be accounted for by constitutive relations for $\text{Var}(Y_t/X_t)$ without postulating a relation for a joint or quasi joint distribution (Larsen 1978, [5]). For the Liouville process realizations in the position- velocity space (phase space) cannot cross. In addition, for a conservative classical force, all realizations that start at the same position will have a unique velocity at a given position when applying the Liouville process, which implies $\text{Var}(Y_t/X_t) = 0$. The equation for the total derivative of the current velocity, which now equals the classical force, can be integrated in space to give the familiar Hamilton-Jacobi equation in classical mechanics as a special case. More generally, the total derivative of the current velocity of the Liouville and the Ornstein-Uhlenbeck [7] processes (assuming uncorrelated Gaussian noise) has also been analyzed when assuming initial conditions in position and velocity that are independent and Gaussian distributed. It has been shown that $\text{Var}(Y_t/X_t)$ is independent of x , but time dependent for the free particle or for the harmonic oscillator [8–10]. We believe that this hydrodynamic method can be useful when experimental data pertain to the variables in the equation set, and there is no direct experimental access to microscopic dynamics.

We follow the idea that stochastic processes could in some way be used to understand quantum mechanics [11–13] (Kaniadakis [14]) by studying the Liouville process and the Ornstein-Uhlenbeck process more carefully in relation to the well-known and so-called “operator ordering problem” in quantum physics [15–21]. A relation for the operator

ordering of p^2q^2 is important for quantization of the kinetic energy in a curved space. We use an approach different from that usually presented in the literature (see also Moxnes and Hausken [22].)

Quantum mechanics based on the Schrödinger equation makes it difficult to describe irreversible processes like the decay of unstable particles and measurements processes. The fact that classical and quantum systems must be coupled by a dissipative rather than reversible dynamics follows from the no-go theorem, where it was shown in a general framework that the information of the measured object cannot be transmitted to values of macroscopic observables as long as the dynamics of the total system is reversible in time [23, 24]. (see Appendix B. See also Bell [25], Bell [26], Haag [27], Blanchard and Jadczyk [28], Haag [29], Machida and Namiki [30], Araki [31], Araki [32], Ozawa [24], Olkiewicz [33] for the literature related to the measuring problem.)

Macroscopic systems are usually described either by classical physics of a few classical parameters or by quantum statistical mechanics if the quantum nature is essential. But in the Machida-Namiki model of measurement a new formulation of describing the process of measurement is given. The measured object is microscopic but the measuring apparatus is described macroscopically [24, 30–32]. However, there are examples of macroscopic quantum phenomena where a large number of particles can be described by a few degrees of freedom. In these cases the evolution of the quantum object depends on the classical environment, but also a modification of the dynamics of the classical system through some expectations values appears. Friction arises from the transfer of collective translational kinetic energy into nearly random motion and can formally be considered as resulting from the process of eliminating the microscopic degrees of freedom. This paper does not review the various attempts to solve the difficulties associated with the measuring problem, but we observe that one of the many attempts to overcome the difficulties has led to the development of the so-called collapse theories, that is, to the dynamical reduction program [34]. This theory accepts a modification of the standard evolution law such that micro processes and macro processes are governed by a unique dynamics. The dynamics implies that the micro-macro interaction in a measurement process leads to the wave packed reduction. (See Giancarlo Chirardi (2007) at <http://www.plato.stanford.edu/entries/qm-collapse>, for review of collapse theories and Efinger [35] for a nonlinear unitary framework for quantum state reduction. See Bassi et al. [36] for experiments that could be crucial to check the dynamical reduction models versus quantum mechanics.)

A linear friction term together with an uncorrelated random Gaussian noise term is inherent in most classical Langevin models. The Boltzmann distribution is achieved as a steady-state solution. Notice that when friction and the random term are zero, every solution for the joint density of the type $\rho(t, x, y) = \rho(H)$ is a steady-state solution of the Liouville equation, where H is the Hamiltonian. This shows the importance of linear friction to achieve the Boltzmann distribution as the steady-state asymptotic classical behavior. Interestingly, models accounting for friction have been further developed into so-called quantum Langevin models for quantum noise [37]. The transformation to quantum mechanics is pursued by using the Heisenberg picture of quantum mechanics. That is, the transformation to quantum mechanics is achieved by letting position and momentum be transformed to the corresponding operators. In the Heisenberg picture the concept of a Hamiltonian is not generally necessary, and friction can be incorporated. To study the “measuring problem” this paper does not use the common Heisenberg picture but instead a nonlinear quantum equation accounting for linear friction in the Schrödinger picture. We use a differential equation for the Wigner function including linear friction to establish a quantum equation

accounting for linear friction. The friction describes the interaction with the measuring device [22].

Section 2 considers second-order processes and joint distributions. Section 3 develops a general stochastic theory not based on joint distributions. Section 4 considers the operator problem. Section 5 formulates the quantum equation with friction. Section 6 compares different solutions from the Liouville and the Ornstein-Uhlenbeck processes with different solutions from the quantum equation with and without friction. Section 7 concludes.

2. Some Relations That Follow for Phase-Space Functions

In classical physics or in the phase-space formulation of quantum physics, two dimensional systems are generally described by a phase-space distribution or quasi distribution. In this section we study the following equation for the phase-space function:

$$\begin{aligned} \dot{\bar{\rho}}(t, x, y) \stackrel{\text{mod}}{=} & -D_1(\bar{\rho}(t, x, y)y) - D_2(\bar{\rho}(t, x, y)f_Y(x, y)) \\ & - \beta(t)D_1D_2(\bar{\rho}(t, x, y)) + \frac{1}{2!}D_2^2(g_{2Y}(t, x)\bar{\rho}(t, x, y)) \\ & + \frac{1}{3!}D_2^3(g_{3Y}(t, x)\bar{\rho}(t, x, y)) + \frac{1}{4!}D_2^4(g_{4Y}(t, x)\bar{\rho}(t, x, y)) + \dots, \\ \rho_X(t, x) \stackrel{\text{def}}{=} & \int \bar{\rho}(t, x, y)dy, \rho_Y(t, y) \stackrel{\text{def}}{=} \int \bar{\rho}(t, x, y)dx, E(X_t^n Y_t^m) \stackrel{\text{def}}{=} \int \bar{\rho}(t, x, y)x^n y^m dx dy \end{aligned} \quad (2.1)$$

where “def” means definition and “mod” means model assumptions. $\beta(t)$ is some function. It can be shown that $\beta(t)$ can be related to correlation [38, 39]. However, as it stands in (2.1) such an interpretation is not needed. But for simplicity we call $\beta(t)$ a correlation factor. $g_{2Y}(t, x), g_{3Y}(t, x), g_{4Y}(t, x), \dots$ are some functions, and $\bar{\rho}(t, x, y)$ is a so-called quasi joint distribution which does not have to be positive definite for all parametric functions. Examples of special cases of (2.1) are the Ornstein-Uhlenbeck process [7], the Liouville process, and the differential equation for the Wigner [15] function as defined by Gardiner and Zoller [37, page 126] or Appendix C for further details. Equation (2.1) corresponds to adding terms to the Liouville equation that includes higher-order derivative of the momentum (velocity). Integrating (2.1) gives

$$\dot{\rho}_X(t, x) + D_1(\rho_X(t, x)v_X(t, x)) = 0, \quad (2.2a)$$

$$v_X(t, x) \stackrel{\text{def}}{=} \frac{\int \bar{\rho}(t, x, y)y dy}{\rho_X(t, x)}, \quad v_2(t, x) \stackrel{\text{def}}{=} \frac{\int \bar{\rho}(t, x, y)y^2 dy}{\rho_X(t, x)}, \quad (2.2b)$$

$$\dot{v}_X(t, x) = \frac{v_X(t, x)D_1(\rho_X(t, x)v_X(t, x))}{\rho_X(t, x)} + \frac{1}{\rho_X(t, x)} \int \dot{\bar{\rho}}(t, x, y)y dy, \quad (2.2c)$$

$$\begin{aligned} \frac{1}{2}D_1(v_X(t, x)^2) &= \frac{1}{2}D_1\left(\frac{(\int \bar{\rho}(t, x, y)y dy)^2}{\rho_X(t, x)^2}\right) \\ &= -v_X(t, x)^2 \frac{D_1\rho_X(t, x)}{\rho_X(t, x)} + v_X(t, x) \frac{D_1(\rho_X(t, x)v_X(t, x))}{\rho_X(t, x)}. \end{aligned} \quad (2.2d)$$

This gives that

$$\begin{aligned}
& \frac{1}{\rho_X(t, x)} \int \dot{\bar{\rho}}(t, x, y) y dy = \frac{1}{\rho_X(t, x)} \\
& \quad \times \int \left(-D_1(\bar{\rho}(t, x, y) y) - D_2(\bar{\rho}(t, x, y) f_Y(x, y)) - \beta(t) D_1 D_2(\bar{\rho}(t, x, y)) \right. \\
& \quad \quad \left. + \frac{D_2^2}{2} (g_{2Y} \bar{\rho}(t, x, y)) + \dots \right) y dy \\
& = -\frac{1}{\rho_X(t, x)} D_1 \int \bar{\rho}(t, x, y) y^2 dy + \frac{1}{\rho_X(t, x)} \int \bar{\rho}(t, x, y) f_Y(x, y) dy \\
& \quad + \frac{\beta(t)}{\rho_X(t, x)} D_1 \int \bar{\rho}(t, x, y) dy \\
& = -\frac{1}{\rho_X(t, x)} D_1(\rho_X(t, x) v_2(t, x)) + \tilde{f}_Y(x) + \beta(t) \frac{D_1 \rho_X(t, x)}{\rho_X(t, x)}, \\
& \tilde{f}_Y(x) \stackrel{\text{def}}{=} \frac{\int \bar{\rho}(t, x, y) f_Y(x, y) dy}{\rho_X(t, x)},
\end{aligned} \tag{2.3}$$

and finally that

$$\dot{\rho}_X(t, x) + D_1(\rho_X(t, x) v_X(t, x)) = 0, \tag{2.4a}$$

$$\begin{aligned}
& \dot{v}_X(t, x) + v_X(t, x) D_1 v_X(t, x) \\
& = \frac{v_X(t, x) D_1(\rho_X(t, x) v_X(t, x))}{\rho_X(t, x)} - \frac{1}{\rho_X(t, x)} D_1(\rho_X(t, x) v_2(t, x)) \\
& \quad + \tilde{f}_Y(x) + \beta(t) \frac{D_1 \rho_X(t, x)}{\rho_X(t, x)} - v_X(t, x)^2 \frac{D_1 \rho_X(t, x)}{\rho_X(t, x)} + \frac{v_X(t, x) D_1(\rho_X(t, x) v_X(t, x))}{\rho_X(t, x)} \\
& = \tilde{f}_Y(x) + \frac{D_1(\rho_X(t, x) (v_X^2(t, x) - v_2(t, x) + q\beta(t)))}{\rho_X(t, x)} \\
& = \tilde{f}_Y(x) + \frac{D_1(\rho_X(t, x) \Pi(t, x))}{\rho_X(t, x)},
\end{aligned} \tag{2.4b}$$

$$\Pi(t, x) \stackrel{\text{def}}{=} v_X(t, x)^2 - v_2(t, x) + \beta(t), \quad \text{Var}\left(\frac{Y_t}{X_t}\right) = v_X(t, x)^2 - v_2(t, x). \tag{2.4c}$$

Equation (2.4a) is the familiar conservation of probability density in space. Notice that if $\bar{\rho}(t, x, y)$ is a Dirac delta function in y , the term $(v_X(t, x)^2 - v_2(t, x))$ becomes zero. This is achieved for the Liouville process and a conservative force if the initial values in position are a Dirac delta function, which means that all realizations start from a common position.

Then the Liouville process can be given a more simplified expression through the well known Hamilton-Jacobi equation.

The equations in (2.4a)–(2.4c) are not closed due to $\tilde{f}_Y(x)$ and $v_2(t, x)$ (or alternatively $\text{Var}(Y_t/X_t)$). $\text{Var}(Y_t/X_t)$ can be calculated explicitly for a Gaussian initial distribution by applying the Gaussian uncorrelated noise [9]. It is found that the term is independent of x and that for a free particle for the Ornstein-Uhlenbeck [7] process

$$\begin{aligned} & - \frac{D_1 \left(\rho_X^{\text{OU}}(t, x) \text{Var}^{\text{OU}}(Y_t/X_t) \right)}{\rho_X^{\text{OU}}(t, x)} \\ & = 4 \left(a^2 b^2 + 2a^2 q t + \frac{2}{3} b^2 q t^3 + \frac{1}{3} q^2 t^4 \right) D_1 \left[\left(1 / \left(2 \sqrt{\rho_X^{\text{OU}}(t, x)} \right) \right) D_1^2 \sqrt{\rho_X^{\text{OU}}(t, x)} \right]. \end{aligned} \quad (2.5)$$

The initial distribution is in this case chosen as two independent Gaussian distributions. a^2 is the initial variance in position and b^2 the variance in velocity. $q = g_{2Y}/2$ is the diffusion coefficient that is now assumed to be constant. The Liouville solution follows as a special case when the diffusion coefficient q is set to zero.

3. The Hydrodynamic Method: A Stochastic Theory Not Based on Joint Distributions

By using (2.1), (2.4a)–(2.4c) follows for the Ornstein-Uhlenbeck [7] process, the Liouville process or the quantum theory based on the Wigner function. Generally, Boltzmann kinetics also allows the same mathematical structure [6]. More generally, we formulate a stochastic theory by constitutive equations in the equation for the total derivative of the current velocity, akin to what is used for hydrodynamic theories, to read

$$\dot{\rho}_X(t, x) + D_1(\rho_X(t, x)v_X(t, x)) = 0, \quad (3.1a)$$

$$\dot{v}_X(t, x) + v_X(t, x)D_1 v_X(t, x) = f(x) + \frac{D_1(\rho_X(t, x)\Pi(t, x))}{\rho_X(t, x)}. \quad (3.1b)$$

The stochastic theory is fully described by postulating a so-called “constitutive” relation for $\Pi(t, x)$. A joint or quasi joint phase-space distribution is not used. It follows from (3.1a) and (3.1b) that

$$\begin{aligned} \dot{E}(X_t) &= \int \dot{\rho}_X(t, x)x dx = - \int D_1(\rho_X(t, x)v_X(t, x))x dx = \int \rho_X(t, x)v_X(t, x)dx, \\ \dot{E}(X_t) &= \frac{\partial}{\partial t} \int \rho_X(t, x)v_X(t, x)dx = \int (\dot{\rho}_X(t, x)v_X(t, x) + \dot{v}_X(t, x)\rho_X(t, x))dx \\ &= \int (\dot{v}_X(t, x) + v_X(t, x)D_1 v_X(t, x))\rho_X(t, x)dx \\ &= \int (\rho_X(t, x)f(x) + D_1(\rho_X(t, x)\Pi(t, x)))dx = E(f(x)). \end{aligned} \quad (2.2)$$

Thus we fulfill our crucial equation $\dot{E}(X(t)) = E(f(X(t)))$. As three test examples, we set the constitutive relation in (3.1a) and (3.1b) as

$$\text{Alt1 : } \Pi(t, x) \stackrel{\text{mod}}{=} \Pi^Q(t, x) \stackrel{\text{def}}{=} \frac{1}{4\rho_X^Q(t, x)} \left(D_1^2 \rho_X^Q(t, x) - \frac{(D_1 \rho_X^Q(t, x))^2}{\rho_X^Q(t, x)} \right), \quad (3.3a)$$

$$\Rightarrow \frac{D_1(\rho_X^Q(t, x)\Pi^Q(t, x))}{\rho_X^Q(t, x)} = D_1 \left(\frac{D_1^2 \rho_X^Q(t, x)^{1/2}}{2\rho_X^Q(t, x)^{1/2}} \right), \quad (3.3b)$$

$$\text{Alt2 : } \Pi(t, x) \stackrel{\text{mod}}{=} \Pi(t, x)^{\text{QC}} \stackrel{\text{def}}{=} \Pi^Q(t, x) + \beta(t), \quad (3.3c)$$

$$\Rightarrow \frac{D_1(\rho_X^{\text{QC}}(t, x)\Pi^{\text{QC}}(t, x))}{\rho_X^{\text{QC}}(t, x)} = D_1 \left(\frac{D_1^2 \rho_X^{\text{QC}}(t, x)^{1/2}}{2\rho_X^{\text{QC}}(t, x)^{1/2}} \right) + \beta(t) \frac{D_1 \rho_X^{\text{QC}}(t, x)}{\rho_X^{\text{QC}}(t, x)}, \quad (3.3d)$$

$$\begin{aligned} \text{Alt3 : } \Pi(t, x) &\stackrel{\text{def}}{=} \Pi^{\text{OUC}}(t, x) \stackrel{\text{mod}}{=} \frac{(a^2 b^2 + 2a^2 q t + (2/3)b^2 q t^3 + (1/3)q^2 t^4)}{\rho_X^{\text{OUC}}(t, x)} \\ &\times \left(D_1^2 \rho_X^{\text{OUC}}(t, x) - \frac{(D_1 \rho_X^{\text{OUC}}(t, x))^2}{\rho_X^{\text{OUC}}(t, x)} \right) + \beta(t) \\ &\Rightarrow \frac{D_1(\rho_X^{\text{OUC}}(t, x)\Pi^{\text{OUC}}(t, x))}{\rho_X^{\text{OUC}}(t, x)} = 4 \left(a^2 b^2 + 2a^2 q t + \frac{2}{3} b^2 q t^3 + \frac{1}{3} q^2 t^4 \right) \\ &\times D_1 \left(\frac{D_1^2 \rho_X^{\text{OUC}}(t, x)^{1/2}}{2\rho_X^{\text{OUC}}(t, x)^{1/2}} \right) + \beta(t) \frac{D_1 \rho_X^{\text{OUC}}(t, x)}{\rho_X^{\text{OUC}}(t, x)}, \end{aligned} \quad (3.3e)$$

$$\text{Alt4 : } \Pi(t, x) = \text{const.} = K.$$

Alternative 1 (superscript Q) corresponds to the quantum theory as we will show. Alternative 2 (superscript QC) corresponds to what we call quantum theory with exponential correlation. Alternative 3 (superscript OUC) mimics the classical results for the Ornstein-Uhlenbeck [7] process. Correlation is included through the correlation factor $\beta(t)$ [38, 39]. As an example, alternative 4 gives an ideal fluid where $K\rho_X(t, x)$ is the "pressure".

We now show the well-known results that the constitutive equation (3.3a) leads to the Schrödinger equation. First we set that the current velocity is a gradient, to read $v_X^Q(t, x) = D_1 S(t, x)$. This gives after one integration of (3.1b) a kind of "Hamilton-Jacobi type system", to read

$$\frac{\partial \rho_X^Q(t, x)}{\partial t} + D_1(\rho_X^Q(t, x)v_X^Q(t, x)) = 0, \quad (3.4a)$$

$$\frac{\partial S(t, x)}{\partial t} + \frac{1}{2}(D_1 S(t, x))^2 + V(x) = \left(\frac{D_1^2 \rho_X^Q(t, x)^{1/2}}{2\rho_X^Q(t, x)^{1/2}} \right). \quad (3.4b)$$

We use the traditional mathematical trick and introduce the well-known Madelung decomposition, to read $\psi(t, x) \stackrel{\text{def}}{=} \rho_X^Q(t, x)^{1/2} \text{Exp}(i S(t, x))$. This allows the two nonlinear equations in (3.4a) and (3.4b) to be written as one linear equation for the in general complex $\psi(t, x)$, to read [12]

$$-\frac{1}{2}D_1^2\psi(t, x) + V(x)\psi(t, x) = i\dot{\psi}(t, x) \quad (3.5)$$

which is the Schrödinger equation. (We use units such that the mass $m = 1$ and the reduced Planck constant $\hbar = 1$.) For alternative 2 we achieve that

$$-\frac{1}{2}D_1^2\psi(t, x) + V(x)\psi(t, x) - \beta(t)\text{Ln}(\lambda\psi(t, x)^*\psi(t, x)) = i\dot{\psi}(t, x). \quad (3.6)$$

The constant λ is arbitrary. Equation (3.6) is a kind of Schrödinger equation accounting for correlation. Without correlation $\beta(t) = 0$. For alternative 3 we achieve

$$-\frac{1}{2}h^2(t)D_1^2\psi(t, x) + V(x)\psi(t, x) - \beta(t)\text{Ln}(\lambda\psi(t, x)^*\psi(t, x)) = i\dot{\psi}(t, x), \quad (3.7)$$

$$h^2(t) = 4\left(a^2b^2 + 2a^2qt + \frac{2}{3}b^2qt^3 + \frac{1}{3}q^2t^4\right).$$

Notice that when $\beta(t) = 0$ and $a^2b^2 = 1/4$ we achieve the Schrödinger equation. In ordinary units $a^2b^2 = 1/4$ means that $\text{Var}(X_{t_0}) \text{Var}(Y_{t_0}) = (1/4)\hbar^2/m$. Alternative 4 simply gives (3.6) without the D_1^2 term.

4. Operators

When the equation set (3.1a) and (3.1b) is postulated together with a constitutive equation for $\Pi(t, x)$ for a stochastic theory, the equation set does not show a way of calculating $E(Y_t^n)$, simply because there is, as such, no stochastic variable Y_t (velocity or momentum) in the theory. But this has caused problems in quantum physics. Briefly, Following the traditional concept in quantum physics, we write

$$\theta(p, t) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{1/2}} \int \psi(t, q) \text{Exp}(-iqp) dq, \quad \int \psi(t, q)^* \psi(t, q) dq = 1, \quad \delta(q) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int \text{Exp}(ipq) dq,$$

$$E(p^m) \stackrel{\text{def}}{=} \int p^m \theta(p, t)^* \theta(p, t) dp. \quad (4.1)$$

It follows that

$$\begin{aligned} \int \theta(p, t)^* \theta(p, t) dp &= \left(\frac{1}{2\pi} \right) \int \psi(t, q')^* \text{Exp}(ip(q' - q)) \psi(t, q) dp dq dq' \\ &= \int \psi(t, q')^* \delta(q' - q) \psi(t, q) dq dq' = \int \psi(t, q)^* \psi(t, q) dq = 1. \end{aligned} \quad (4.2)$$

Further we achieve that

$$\begin{aligned} E(p^m) &\stackrel{\text{def}}{=} \int p^m \theta(p, t)^* \theta(p, t) dp = \left(\frac{1}{2\pi} \right) \int p^m \psi(t, q')^* \text{Exp}(ip(q' - q)) \psi(t, q) dp dq dq' \\ &= \int \psi(t, q')^* \left(-i \frac{\partial}{\partial p} \right)^m \text{Exp}(ip(q' - q)) \psi(t, q) dp dq dq' = \int \psi(t, q)^* (-iD_1)^m \psi(t, q) dq \\ &= \int \psi(t, q)^* (p_{\text{op}})^m \psi(t, q) dq, \\ E(q^m) &= \int \theta(t, p)^* (iD_1)^m \theta(t, p) dp = E(q^m) = \int \theta(t, p)^* (q_{\text{op}})^m \theta(t, p) dp, \end{aligned} \quad (4.3)$$

where the p operator raised to the power of m becomes $p_{\text{op}}^m \stackrel{\text{def}}{=} (-iD_1)^m = (-i\partial/\partial q)^m$. We use the definition

$$I(\alpha) = \int \left| i\alpha [x\psi(t, q) - E(q)\psi(t, q)] + \left(\frac{1}{i} \right) D_1 \psi(t, q) - E(p)\psi(t, q) \right|^2 dq \geq 0. \quad (4.4)$$

Expanding the terms in (4.4) gives $\text{Var}(p) \geq -\alpha(1 + \alpha \text{Var}(q))$. The right-hand side has a maximum for $\alpha = -1/(2 \text{Var}(q))$. This gives $\text{Var}(q) \text{Var}(p) \geq 1/4$, which is the uncertainty relation. However, these equations above in this section are well known, purely mathematical and follow as seen directly from the definitions. No physics is involved so far. So why is p associated with velocity (or momentum)? Say that we calculate $\partial/\partial t E(X_t)$. By using the Madelung decomposition we set for any stochastic theory $v_X(t, x) = D_1(1/2i) \text{Ln}(\psi(t, x)/\psi^*(t, x))$. We find when using the conservation of probability

in space that

$$\begin{aligned}\dot{E}(X_t) &= \frac{\partial}{\partial t} \int \rho_X(t, x) x dx = \int \dot{\rho}_X(t, x) x dx = \int x (-D_1(\rho_X(t, x) v_X(t, x))) dx \\ &= \int \psi(t, x) \psi^*(t, x) D_1 \frac{1}{2i} \text{Ln} \left(\frac{\psi(t, x)}{\psi^*(t, x)} \right) dx = \int \psi^* (-iD_1) \psi dx.\end{aligned}\tag{4.5}$$

Thus we find quite generally that $\partial/\partial t E(X_t) = E(p)$. Then the next question is why is $\partial/\partial t E(X_t)$ associated with the expectation of velocity, or more generally $E(Y_t^m) \rightarrow E(p^m)$? The arrow means association in the sense that the right-hand side of the arrow is associated to mean the same as the left-hand side. The velocity is a classical concept when “trajectories” are differentiable. Notice that the formulation of quantum mechanical operators from its classical counterpart is straightforward as long as the classical quantity is either a function of $x(q)$ or $y(p)$, or if it is the sum of such functions. One merely replaces p by the operator of p and q by the operator of q . But if the classical counterpart contains product terms of q and p , then difficulties arise, because the received quantum theory gives no unique way of forming the quantum mechanical operator. The phase-space formulation of quantum mechanics gives a solution to the operator problem. Once a quasi phase-space probability density is chosen, each such function would lead to a unique operator ordering, and any chosen operator ordering for $q^n p^m$ leads to a unique quasi probability distribution [15–21]. These quantum quasi distributions have been widely used in quantum optics and in optical image processing [40–42]. The Margenau-Hill quasi probability density distribution is equivalent to the rule of symmetrization. Thus logically, there are an infinite number of quantum theories, one for each chosen phase-space quasi joint distribution. However, interestingly, for the special case qp a unique operator seems to be given. That means that the operator is independent of the chosen type of quasi joint distribution. This operator is found by the association $E(X_t Y_t) \rightarrow (1/2) \partial E(X_t^2) / \partial t$. In general an association is $E(X_t^n Y_t) \rightarrow (\partial/\partial t)(1/(n+1)) E(X_t^{n+1})$. Thus the operator of $x^n y$ is assumed to be found by simply calculating $(\partial/\partial t)(1/(n+1)) E(X_t^{n+1})$. This is not only associations but actually mathematical deductions if we use a joint or quasi joint distribution of the type in (2.1), to read

$$\begin{aligned}\dot{E}(X_t^n) &= \frac{\partial}{\partial t} \int \bar{\rho}(t, x, y) x^n dx dy = \int \dot{\bar{\rho}}(t, x, y) x^n dx dy \\ &= \int \left(\begin{aligned} &-D_1(\bar{\rho}(t, x, y) y) - D_2(\bar{\rho}(t, x, y) f_Y(x, y)) - \beta(t) D_1 D_2(\bar{\rho}(t, x, y)) \\ &+ \left(\frac{1}{2}\right) D_2^2(\bar{\rho}(t, x, y) g_{2Y}(x, y) + \dots) \end{aligned} \right) x^n dx dy \\ &= nE(X_t^{n-1} Y_t).\end{aligned}\tag{4.6}$$

Thus the Liouville process and the Ornstein-Uhlenbeck [7] process allow taking the time derivative inside the expectation. But notice that the results are based on a specific joint distribution or a quasi joint distribution of the type in (2.1). So $\partial/\partial t E(X_t)$ is found to be the

expectation of the velocity if a joint or quasi joint distribution of the type in (2.1) is postulated. The reason why the time derivative can be taken inside the expectation is that (2.1) does not have terms of the type D_1^n , $n \geq 2$.

By using that $v_X(t, x) = (1/2i)Ln(\psi(t, x)/\psi^*(t, x))$ we have

$$\begin{aligned}
E(X_t^n Y_t) &= \frac{\partial}{\partial t} \frac{1}{n+1} \int x^{n+1} \rho_X(t, x) dx = \frac{1}{n+1} \int x^{n+1} (-D_1(\rho_X(t, x) v_X(t, x))) dx \\
&= \int x^n \rho_X(t, x) v_X(t, x) dx = \int x^n \psi(t, x) \psi^*(t, x) D_1 \frac{1}{2i} Ln \left(\frac{\psi(t, x)}{\psi^*(t, x)} \right) dx \\
&= \frac{1}{2} \int (\psi^*(t, x) x^n (-iD_1) \psi(t, x) + \psi^*(t, x) (-iD_1) (x^n \psi(t, x))) dx \\
&= \frac{1}{2} \int \psi^* (x^n p_{op} + p_{op} x^n) \psi dx, \quad p_{op} \stackrel{\text{def}}{=} -iD_1 = -i \frac{\partial}{\partial x}.
\end{aligned} \tag{4.7}$$

The well-known rule of symmetrization states $E(X_t^n X_t^m) \rightarrow (1/2) \int \psi^* (p_{op}^m x^n + x^n p_{op}^m) \psi dx$, the rule of Born-Jordan states $E(X_t^n Y_t^m) \rightarrow (1/(m+1)) \int \psi^* (\sum_{l=0}^m p_{op}^{m-l} x^n p_{op}^l) \psi dx$, while the Weyl rule states $E(X_t^n Y_t^m) \rightarrow (1/2^n) \int \psi^* (\binom{n}{l} \sum_{l=0}^m x^{n-l} p_{op}^m x^l) \psi dx$ [43]. For $m = 1$ we have more explicitly

$$\begin{aligned}
E(X_t^n Y_t) &\rightarrow \frac{1}{2} \int \psi^* (p_{op} x^n + x^n p_{op}) \psi dx, \quad \text{symmetrization,} \\
E(X_t^n Y_t) &\rightarrow \frac{1}{2} \int \psi^* (p_{op} x^n + x^n p_{op}) \psi dx, \quad \text{Born-Jordan,} \\
E(X_t^n Y_t) &\rightarrow \frac{1}{2^n} \int \psi^* \left(x^n p_{op} + n x^{n-1} p_{op} x + \frac{n \times (n-1)}{1 \times 2} x^{n-2} p_{op} x^2 + \dots p_{op} x^n \right) \psi dx, \quad \text{Weyl.}
\end{aligned} \tag{4.8}$$

Only for $n = 0$ and $n = 1$ the rules do give the same answer if $m = 1$. Notice that when $n = 0$, we achieve the same result as in (4.5). In general (4.7) equals the rule of symmetrization and the rule of Born-Jordan, corresponding to the Margenau-Hill [44] quasi density function and the Born-Jordan quasi density function.

A fundamental conceptual problem in quantum physics or when using (3.1a) and (3.1b) is to associate something to $E(X_t^n Y_t^2)$. A solution to this operator problem in quantum mechanics can be used for quantization of the kinetic energy in a curved space. Equation (2.1) (leading to (2.4a)–(2.4c)) gives

$$\begin{aligned}
E(X_t^n Y_t^2) &= \int \bar{\rho}(t, x, y) x^n y^2 dx dy = \int v_2(t, x) \rho_X(t, x) x^n dx \\
&= \int (v_X(t, x)^2 - \Pi(t, x) + \beta(t)) \rho_X(t, x) x^n dx.
\end{aligned} \tag{4.9}$$

This equation can further be developed if the constitutive model for $\Pi^Q(t, x)$ given in (3.3a)–(3.3e) is used. However, logically we can base a quantum theory on the equation set (3.1a) and (3.1b) and (3.3a) together with the association rules $E(X_t Y_t) \rightarrow (1/2)\partial E(X_t^2)/\partial t = \int x v_X(t, x) \rho_X(t, x) dx$, $E(X_t^n Y_t^2) \rightarrow \int (v_X(t, x)^2 - \Pi(t, x) + \beta(t)) \rho_X(t, x) x^n dx$. Using (3.3a) with $\beta(t) = 0$, or (3.3b), implies [45]

$$\begin{aligned} E(X_t^n Y_t^2) &= \int (v_X(t, x)^2 - \Pi(t, x) + \kappa(t)) \rho_X(t, x) x^n dx \\ &= \int \left(v_X(t, x)^2 - \frac{1}{4\rho_X(t, x)} \left(D_1^2 \rho_X(t, x) - \frac{(D_1 \rho_X(t, x))^2}{\rho_X(t, x)} \right) \right) \rho_X(t, x) x^n dx. \end{aligned} \quad (4.10)$$

Using again $v_X(t, x) = D_1(1/2i) \text{Ln}(\psi(t, x)/\psi^*(t, x))$ it follows that

$$\begin{aligned} E(X_t^n Y_t^2) &= -\frac{1}{4} \int \left(\left(\frac{D_1 \psi(t, x)}{\psi(t, x)} \right)^2 - 2 \left(\frac{D_1 \psi(t, x)}{\psi(t, x)} \right) \left(\frac{D_1 \psi(t, x)^*}{\psi(t, x)^*} \right) + \left(\frac{D_1 \psi(t, x)^*}{\psi(t, x)^*} \right)^2 \right) \\ &\quad \times \psi(t, x) \psi(t, x)^* x^n \\ &\quad - \frac{(\psi(t, x)^* D_1 \psi(t, x) + \psi(t, x) D_1 \psi(t, x)^*)^2}{\psi(t, x)^* \psi(t, x)} x^n + x^n D_1^2 \rho_X(t, x) \Big) dx \\ &= \int \left(x^n (D_1 \psi(t, x)^*) (D_1 \psi(t, x)) - \frac{1}{4} x^n D_1^2 \rho_X(t, x) \right) dx. \end{aligned} \quad (4.11)$$

We can develop (4.11) further since

$$\begin{aligned} E(X_t^n Y_t^2) &= \int \left(x^n (D_1 \psi(t, x)^*) (D_1 \psi(t, x)) - \frac{1}{4} x^n D_1^2 \rho_X(t, x) \right) dx \\ &= \frac{1}{4} \int \left(4x^n (D_1 \psi(t, x)^*) (D_1 \psi(t, x)) \right. \\ &\quad \left. - x^n (\psi(t, x) D_1^2 \psi(t, x)^* + 2(D_1 \psi(t, x)^*) (D_1 \psi(t, x)) + \psi(t, x)^* D_1^2 \psi(t, x)) \right) dx \\ &= \frac{1}{4} \int \left(x^n (-\psi(t, x) (D_1^2 \psi(t, x)^*) + 2(D_1 \psi(t, x)^*) x^n (D_1 \psi(t, x)) - \psi(t, x)^* x^n D_1^2 \psi(t, x)) \right) dx \\ &= \frac{1}{4} \int \left(-\psi(t, x)^* D_1^2 (x^n \psi(t, x)) - 2\psi(t, x)^* D_1 (x^n D_1 \psi(t, x)) - \psi(t, x)^* x^n D_1^2 \psi(t, x) \right) dx \\ &= \frac{1}{4} \int \psi^* (p_{\text{op}}^2 x^n + 2p_{\text{op}} x^n p_{\text{op}} + x^n p_{\text{op}}^2) \psi dx. \end{aligned} \quad (4.12)$$

Thus none of the rules stated after (4.7) are generally in agreement with (4.12). However, for $n = 0$ all the rules give the same answer and equal the results in (4.3). For the particular case $n = 2$, (4.12) equals the Weyl rule. For the cases we have examined we have found that

$$\begin{aligned} E(X_t^n Y_t^m) &= \frac{1}{2^m} \int \psi^* \left(\binom{m}{l} \sum_{l=0}^m p_{\text{op}}^{n-l} x^n p_{\text{op}}^l \right) \psi dx, \\ m = 1 &\implies E(X_t^n Y_t^m) = \frac{1}{2} \int \psi^* (p_{\text{op}} x^n + x^n p_{\text{op}}) \psi dx, \\ m = 2 &\implies E(X_t^n Y_t^m) = \frac{1}{4} \int \psi^* (p_{\text{op}}^2 x^n + 2p_{\text{op}} x^n p_{\text{op}} + x^n p_{\text{op}}^2) \psi dx. \end{aligned} \quad (4.13)$$

More explicitly we can write for $m = 2$,

$$E(X_t^n Y_t^2) \longrightarrow \frac{1}{2} \int \psi^* (p_{\text{op}}^2 x^n + x^n p_{\text{op}}^2) \psi dx, \quad \text{symmetrization}, \quad (4.14a)$$

$$E(X_t^n Y_t^2) \longrightarrow \frac{1}{3} \int \psi^* (p_{\text{op}}^2 x^n + p_{\text{op}} x^n p_{\text{op}} + x^n p_{\text{op}}^2) \psi dx, \quad \text{Born-Jordan}, \quad (4.14b)$$

$$E(X_t^n Y_t^2) \longrightarrow \frac{1}{2^n} \int \psi^* \left(x^n p_{\text{op}}^2 + n x^{n-1} p_{\text{op}}^2 x + \frac{n \times (n-1)}{1 \times 2} x^{n-2} p_{\text{op}}^2 x^2 + \dots p_{\text{op}}^2 x^n \right) \psi dx, \quad \text{Weyl}, \quad (4.14c)$$

$$E(X_t^n Y_t^2) = \frac{1}{4} \int \psi^* (p_{\text{op}}^2 x^n + 2p_{\text{op}} x^n p_{\text{op}} + x^n p_{\text{op}}^2) \psi dx \quad (4.13). \quad (4.14d)$$

The rule in (4.13) for $n = 0$, $m = 2$ can also be found more directly. A joint distribution of the type in (2.1) allows, as shown, taking the time derivative inside the expectation. This gives

$$\begin{aligned} E(Y_t^2) &= \frac{1}{2} \frac{\partial^2}{\partial t^2} E(X_t^2) - E(X_t f(X_t)) = \dot{E}(X_t Y_t) - E(X_t f(X_t)) \\ &= \int (\dot{\rho}_X(t, x) v_X(t, x) x + \rho_X(t, x) \dot{v}_X(t, x) x) dx - E(X_t f(X_t)) \\ &= \int (-v_X(t, x) x D_1(\rho_X(t, x) v_X(t, x)) + \rho_X(t, x) \dot{v}_X(t, x) x) dx - E(X_t f(X_t)) \\ &= \int (\rho_X(t, x) v_X(t, x)^2 + x \rho_X(t, x) (\dot{v}_X(t, x) + v_X(t, x) D_1 v_X(t, x)) dx - E(X_t f(X_t)) \\ &= \int (\rho_X(t, x) v_X(t, x)^2 + x \rho_X(t, x) f(x) + x D_1(\rho_X(t, x) \Pi(t, x))) dx - E(X_t f(X_t)) \\ &= \int (\rho_X(t, x) v_X(t, x)^2 + x D_1(\rho_X(t, x) \Pi(t, x))) dx = \int (v_X(t, x)^2 - \Pi(t, x)) \rho_X(t, x) dx. \end{aligned} \quad (4.15)$$

Thus to shortly summarize, based on assumptions (2.1) and (3.3a)–(3.3e) we conclude (4.13) which is a new rule of operator ordering.

5. A Quantum Equation Accounting for Friction

Quantum physics does not in general handle friction, but the formulation of the theory in phase space gives such possibilities. The equation for the Wigner function accounting for friction is given by Gardiner and Zoller (see [37, page 126] or Appendix C for further details). The equation is a special case of (2.1), to read

$$\begin{aligned} \dot{\bar{\rho}}^W(t, x, y) = & -D_1(\bar{\rho}^W(t, x, y)y) + D_2((D_1V(x) + \varepsilon y)\bar{\rho}^W(t, x, y)) \\ & + \sum_{k=1}^{\infty} \frac{(i\hbar/2)^{2k}}{(2k+1)!} D_2^{2k+1}(D_1^{2k+1}V(x))\bar{\rho}^W(t, x, y). \end{aligned} \quad (5.1)$$

We set that the classical force is $f(x, y) = -D_1V(x) - \varepsilon y$. The last term in (5.1) is negligible for the harmonic oscillator, a linear potential, or a free particle. However, there is still a subtle difference with the classical results for the Liouville equation since the possible initial conditions are restricted. The reason is that the initial distribution, for say momentum, should be given through the $\theta(t_0, p)$, which is the Fourier transform of $\psi(t_0, p)$. Thus a given $\psi(t_0, q)$ gives a unique distribution for momentum. This also implies $\text{Var}(q) \text{Var}(p) \geq \hbar^2/4$. Inserting $f(x, y) = -D_1V(x) - \varepsilon y$ into (2.4a)–(2.4c) (choosing unit such that $\hbar = 1$) gives

$$\begin{aligned} \dot{\rho}_X(t, x) + D_1(\rho_X(t, x)v_X(t, x)) &= 0, \\ \dot{v}_X(t, x) + v_X(t, x)D_1v_X(t, x) &= f(x) - \varepsilon v_X(t, x) + \frac{D_1(v_X^2(t, x) - v_2(t, x))}{\rho_X(t, x)} \\ &= -D_1V(x) - \varepsilon D_1S(t, x) + D_1\left(\frac{D_1^2\rho_X(t, x)^{1/2}}{2\rho_X(t, x)^{1/2}}\right), \end{aligned} \quad (5.2)$$

where we have assumed, most importantly, the same functional form for $\text{Var}(Y_t/X_t)(t, x) = v_2(t, x) - v_X^2(t, x)$ as when without friction. We have that $v_X(t, x) = D_1S(t, x)$ and again use the Madelung decomposition. It follows after one integration in space that

$$-\frac{1}{2}D_1^2\psi(t, x) + V(x)\psi(t, x) + \varepsilon\frac{1}{2i}\text{Ln}\left(\frac{\psi(t, x)}{\psi^*(t, x)}\right)\psi(t, x) = i\dot{\psi}(t, x). \quad (5.3)$$

This is a Schrödinger equation with a linear friction term. It is easily verified that with correlation (5.3) becomes

$$\begin{aligned} -\frac{1}{2}D_1^2\psi(t, x) + V(x)\psi(t, x) + \varepsilon\frac{1}{2i}\text{Ln}\left(\frac{\psi(t, x)}{\psi^*(t, x)}\right)\psi(t, x) \\ -\beta(t)D_1\text{Ln}(\lambda\psi(t, x)\psi(t, x)^*) = i\dot{\psi}(t, x) \end{aligned} \quad (5.4)$$

which accounts for phase dependency. Further we achieve without correlation

$$\begin{aligned}
\dot{\theta}(p, t) &= \frac{1}{(2\pi)^{1/2}} \int \dot{\psi}(t, x) \text{Exp}(-ixp) dx \\
&= \frac{1}{(2\pi)^{1/2}} \int \left(-\frac{1}{2} D_1^2 \psi(t, x) + V(x) \psi(t, x) + S(t, x) \psi(t, x) \right) \text{Exp}(-ixp) dx \quad (5.5) \\
&= \frac{p^2}{2} \theta(p, t) + V(iD_1) \theta(p, t) + S(t, iD_1) \theta(p, t).
\end{aligned}$$

We solve (5.3) numerically in the next section. Notice that solutions of partial differential equations yield more variety in the solutions than that obtained with nonpartial differential equations since partial differential equations give solutions with an arbitrary number of constants. For other types of nonlinear Schrödinger equations see Weinberg [46] and Doebner and Goldin [47].

6. Simulations and Comparisons of Different Stochastic Approaches

This section compares the Liouville, Ornstein-Uhlenbeck, and quantum solutions for the harmonic oscillator with and without friction. The initial values are

(i)

$$\rho_X^L(t_0, x) = \rho_X^{\text{OU}}(t_0, x), \quad (6.1)$$

(ii)

$$\rho_Y^L(t_0, y) = \rho_Y^{\text{OU}}(t_0, y), \rho^L(t_0, x, y) = \rho^{\text{OU}}(t_0, x, y) = \rho_X^L(t_0, x) \rho_Y^L(t_0, y). \quad (6.2)$$

We set $f(x, y) = -x - \varepsilon y$. For the quantum solution we choose the initial values

(iii)

$$\rho_X^Q(t_0, x) = \rho_X^L(t_0, x) = \rho_X^{\text{OU}}(t_0, x). \quad (6.3)$$

Quantum theory uses no $\rho_Y^Q(t_0, y)$. We choose

(iv)

$$v_X^Q(t_0, x) = v_X^L(t_0, x) = v_X^{\text{OU}}(t_0, x). \quad (6.4)$$

The initial marginal probability densities for the Liouville and the Ornstein-Uhlenbeck processes are as an example assumed to be Gaussian distributions, that is,

$$\begin{aligned}\rho_X^L(t_0, x) &= \rho_X^{\text{OU}}(t_0, x) = \left(2\pi a^2\right)^{-1/2} e^{-(x-x_0)^2/(2a^2)}, \\ \rho_Y^L(t_0, y) &= \rho_Y^{\text{OU}}(t_0, y) = \left(2\pi b^2\right)^{-1/2} e^{-y^2/(2b^2)},\end{aligned}\tag{6.5}$$

which imply $v_X^L(t_0, x) = v_X^{\text{OU}}(t_0, x) = 0$. Notice that many different types of $\rho_Y^L(t_0, y) = \rho_Y^{\text{OU}}(t_0, y)$ lead to the same $v_X^L(t_0, x) = v_X^{\text{OU}}(t_0, x) = v_X^Q(t_0, x) = 0$.

The analytical solution is for the Liouville and the Ornstein-Uhlenbeck processes with a constant diffusion term, and without friction, given by [9]

$$\rho_X(t_0, x) = \frac{e^{-(x-x_0 \cos t)^2/[2(a^2 \cos^2 t + b^2 \sin^2 t + (q/2)t - (q/2) \sin t \cos t]}}{[2\pi(a^2 \cos^2 t + b^2 \sin^2 t + (q/2)t - (q/2) \sin t \cos t)]^{1/2}},\tag{6.6a}$$

$$\rho_Y(t_0, y) = \frac{e^{-y^2/[2(b^2 \cos^2 t + a^2 \sin^2 t + (q/2)t + (q/2) \sin t \cos t]}}{[2\pi(b^2 \cos^2 t + a^2 \sin^2 t + (q/2)t + (q/2) \sin t \cos t)]^{1/2}},\tag{6.6b}$$

$$v_X(t, x) = \frac{(x - x_0 \cos t)(\cos t \sin t (b^2 - a^2) + q \sin t)}{a^2 \cos^2 t + b^2 \sin^2 t + (q/2)t - (q/2) \sin t \cos t} - x_0 \sin t.\tag{6.6c}$$

The Liouville solution is achieved when setting $q = 0$ in (6.6a)–(6.6c). Observe that the variance when $q = 0$ (Liouville) is steady when $a = b$. When also $x_0 = 0$, we achieve additionally a steady-state density distribution.

The analytical solution of the quantum equation without friction is given by using the well-known propagator, to read

$$\begin{aligned}\psi^Q(t, t_0, x) &= \int_{-\infty}^{\infty} u(x, x_0, t, t_0) \psi^Q(t_0, x_0) dx_0, \\ u(x, x_0, t, t_0) &= [2\pi i \sin(t - t_0)]^{-1/2} e^{i[(x^2 + x_0^2) \cos(t - t_0) - 2ixx_0]/(2 \sin(t - t_0))},\end{aligned}\tag{6.7}$$

$$\psi^Q(t_0, x) = \rho_X^Q(t_0, x)^{1/2} e^{iS(t_0, x)}, \quad D_1 S(t_0, x) = v_X^Q(t_0, x).$$

Inserting $v_X^Q(t_0, x_0) = 0$ into (6.7) gives for $x_0 = 0$

$$\begin{aligned} \psi^Q(t, x) &= \int_{-\infty}^{\infty} [2\pi i \sin(t)]^{-1/2} e^{i[(x^2+p^2) \cos(t) - 2ixp]/(2\sin(t))} (2\pi a^2)^{-1/4} e^{-(p)^2/(4a^2)} dp \\ &= (2\pi i \sin t)^{-1/2} (2\pi a^2)^{-1/4} \left\{ \frac{\pi}{[1/(4a^2) - i \cos t / (4\beta a \sin t)]} \right\}^{1/2} \\ &\quad \times e^{-x^2/[4(\beta^2 \sin^2 t - i\beta a \cos t \sin t)] + ix^2 \cos t / (4\beta a \sin t)}, \quad \beta^2 \stackrel{\text{def}}{=} (4a^2)^{-1}, \quad t_0 = 0, \quad x_0 = 0, \\ S(t, x) &= (2i)^{-1} \text{Ln} \left[\frac{\psi(t, x)}{\psi^*(t, x)} \right] \\ &= \frac{(1/2)x^2 \cos t (1/(4a^2) - a^2) \sin t}{[a^2 \cos^2 t + 1/(4a^2) \sin^2 t]}, \\ v_X^Q(t, x) &= D_1 S(t, x), \quad x_0 = 0, \end{aligned} \tag{6.8}$$

and more generally

$$\rho_X^Q(t, x) = \psi^Q(t, x) \psi^{Q*}(t, x) = \frac{e^{-(x-x_0 \cos t)^2/[2(a^2 \cos^2 t + (4a^2)^{-1} \sin^2 t)]}}{[2\pi (a^2 \cos^2 t + (4a^2)^{-1} \sin^2 t)]^{1/2}}. \tag{6.9}$$

Inserting the special case $a = 2^{-1/2}$ gives a steady variance of $a^2 = 1/2$. When also $x_0 = 0$ we also achieve a steady-state density distribution.

The Liouville and Schrödinger solutions are in some special cases equal. Assuming as a special case that $b = 1/(2a) \Rightarrow a^2 b^2 = 1/4$ for the Liouville solution it follows directly from (6.6a)–(6.6c) and (6.9) that the solutions are equal. This is also in agreement with (2.5).

Figure 1 shows four different realizations of the Liouville process and the Ornstein-Uhlenbeck process with uncorrelated Gaussian noise when $b = 1/(2a)$, $a^2 = 1/2$, $x_0 = 0$. The realizations are constructed as

$$X_{t+h} = X_t + hY_t, \quad Y_{t+h} = Y_t - hX_t - \varepsilon hY_t + \text{Random}_Y [\text{Gauss}[0, h2q]], \tag{6.10}$$

where ε different from zero gives friction, and $q = 0$ gives the Liouville solution.

Observe for the Liouville process the oscillating behavior without friction and the movement towards zero with friction. The two Ornstein-Uhlenbeck processes reveal a more “scattered” behavior with and without friction, in contrast to the smooth behavior of the two Liouville processes.

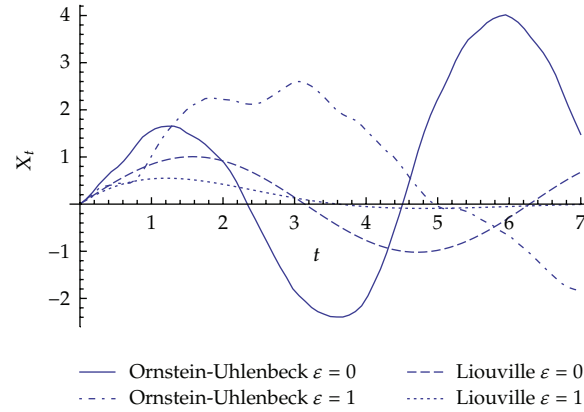


Figure 1: The Liouville and Ornstein-Uhlenbeck realizations as a function of time t .

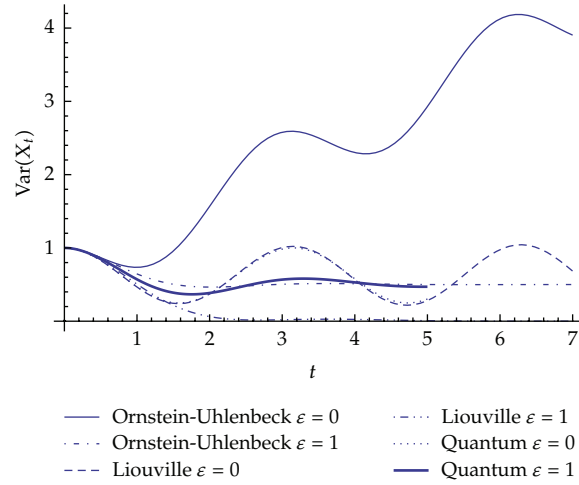


Figure 2: The variance as functions of time t . $E(X_{t_0}) = E(Y_{t_0}) = 0$, $\text{Var}(X_{t_0}) = a^2$, and $\text{Var}(Y_{t_0}) = b^2$, $b = 1/(2a)$, $a = 1$.

Figure 2 shows the variances (standard deviations) as functions of time for four different cases. The expectations are zero at all times in all cases. The Ornstein-Uhlenbeck solutions without friction with $q = \text{constant}$ are given by (6.6a), to read

$$\begin{aligned}\text{Var}(X_t^{\text{OU}}) &= a^2 \text{Cos}^2 t + b^2 \text{Sin}^2 t + \left(\frac{q}{2}\right)t - \left(\frac{q}{2}\right) \text{Sin} t \text{Cos} t, \\ \text{Var}(Y_t^{\text{OU}}) &= b^2 \text{Cos}^2 t + a^2 \text{Sin}^2 t + \left(\frac{q}{2}\right)t + \left(\frac{q}{2}\right) \text{Sin} t \text{Cos} t.\end{aligned}\tag{6.11}$$

The Liouville variance without friction is found by setting $q = 0$ in this formula. The solution for the expectation and variance in the Liouville or Ornstein-Uhlenbeck [7] process

with or without friction can simply be found to be given by the closed equation set (see Appendix A)

$$\begin{aligned} \dot{E}^{\text{OU}}(X_t) &= E^{\text{OU}}(Y_t), & \dot{E}^{\text{OU}}(Y_t) &= -E^{\text{OU}}(X_t) - \varepsilon E^{\text{OU}}(Y_t), \\ \dot{\text{Var}}^{\text{OU}}(X_t) &= 2\text{Cov}^{\text{OU}}(X_t, Y_t), & \dot{\text{Var}}^{\text{OU}}(Y_t) &= -2\text{Cov}^{\text{OU}}(X_t, Y_t) - 2\varepsilon \text{Var}^{\text{OU}}(Y_t) + q, \\ \dot{\text{Cov}}^{\text{OU}}(X_t, Y_t) &= \text{Var}^{\text{OU}}(Y_t) - \text{Var}^{\text{OU}}(X_t) - \varepsilon \text{Cov}^{\text{OU}}(X_t, Y_t). \end{aligned} \quad (6.12)$$

$q = 0$ gives the Liouville solution. This shows that the expectation and variance and covariance develop independently of the chosen form of the initial distribution, and thus the variance equals (6.11) as long as the initial variance and covariance are fixed. A steady-state solution can be found as

$$\begin{aligned} E^{\text{OU}}(X_t) &= E^{\text{OU}}(Y_t) = \text{Cov}^{\text{OU}}(X_t, Y_t) = 0, \\ \text{Var}^{\text{OU}}(Y_t) &= \text{Var}^{\text{OU}}(X_t), \\ \text{Var}^{\text{OU}}(Y_t) &= \frac{q}{(2\varepsilon)}, \end{aligned} \quad (6.13)$$

where $q = 0$ gives the Liouville solution.

Observe in Figure 2 the oscillatory behavior of the Liouville process (and quantum solution which is the same) without friction. The Liouville variance with friction approaches zero which means that all realizations of the Liouville process with friction approach zero with probability 1 as time t approaches infinity, as exemplified in Figure 2. The Ornstein-Uhlenbeck variance without friction approaches infinity as time t approaches infinity. That is, the diffusive term provides a broader density solution as time passes while the expected value in this case equals zero for all t . Adding the friction term to the Ornstein-Uhlenbeck process dampens the broadening of the solution. Studying the variance numerically, the variance approaches the steady-state value $q/(2\varepsilon)=1/2$ as time t approaches infinity, as Figure 2 also shows. Notice that the Ornstein-Uhlenbeck process with friction shows more equality with the quantum solution with friction than the Liouville process with friction.

Figure 3 shows the density $\rho_X(t, x)$ without friction ($\varepsilon = 0$) referred to as Sim1 as a function of time t and position x . The distribution is started to the right with the expectation $x_0 = 1.5$, where $t_0 = 0$. We have chosen values that give steady-state variance.

The variance a^2 in Figure 3 stays stable through time. The Liouville solution is identical to the quantum solution.

Changing from $x_0 = 1.5$ to $x_0 = -1.5$ to generate the mirror solution (i.e., starting to the left rather than to the right), Figure 4 shows the density $\rho_X(t, x)$ without friction referred to as Sim2 when $a^2 = 1/2$, $x_0 = -1.5$, and $\varepsilon = 0$.

Figure 5 shows the average of the two mirror solutions in Figures 3 and 4, referred to as $(\text{Sim1} + \text{Sim2})/2$. This is also a Liouville solution but not quantum solution.

Figure 6 shows the quantum solution as a function of time using the two initial Gaussian distributions. The quantum solutions with friction are found by solving (5.3) without correlation numerically.

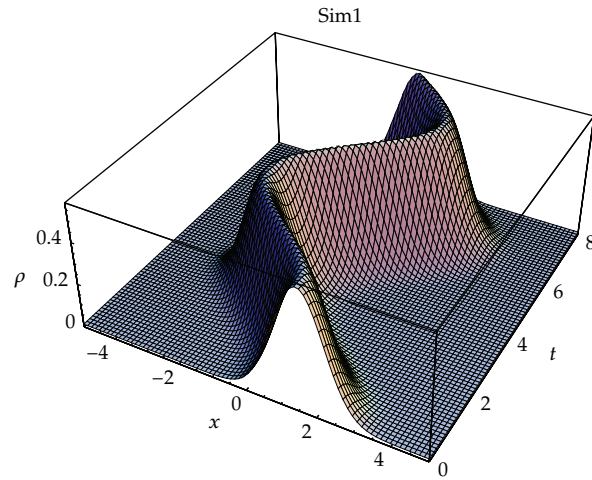


Figure 3: Quantum and Liouville solutions without friction as a function of space x and time t . $\rho_X^Q(t_0, x) = \rho_X^L(t_0, x) = (2\pi a^2)^{-1/2} e^{-(x-x_0)^2/(2a^2)}$, $\rho_Y^L(t_0, y) = (2\pi b^2)^{-1/2} e^{-y^2/(2b^2)}$, $b = 1/(2a)$, $a^2 = 1/2$, $x_0 = 1.5$, and $\varepsilon = 0$.

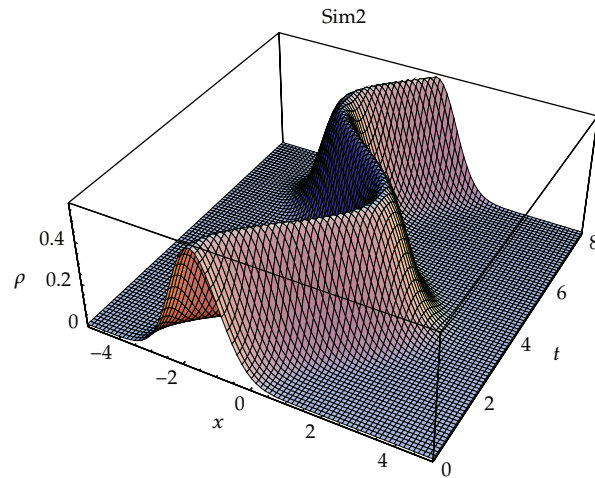


Figure 4: Quantum and Liouville solutions without friction as a function of space x and time t . $\rho_X^Q(t_0, x) = \rho_X^L(t_0, x) = (2\pi a^2)^{-1/2} e^{-(x-x_0)^2/(2a^2)}$, $\rho_Y^L(t_0, y) = (2\pi b^2)^{-1/2} e^{-y^2/(2b^2)}$, $b = 1/(2a)$, $a^2 = 1/2$, $x_0 = -1.5$, and $\varepsilon = 0$.

Observe the difference between the Liouville solution in Figure 5 and the quantum solution in Figure 6. The interference pattern is clearly visible in the quantum solution in Figure 6. Figure 7 shows the quantum solution with the two initial Gaussian distributions with friction with the magnitude of $\varepsilon = 4.0$, found by solving (5.3) without correlation numerically.

Observe how the two Gaussian distributions in Figure 7 coalesce when time t increases, in the sense that the density eventually approaches a narrow Gaussian distribution with the variance equal to the initial variance $a^2 = 1/2$. This effect is caused by the strong nonlinearity of the quantum friction equation. Reducing the magnitude of the friction from

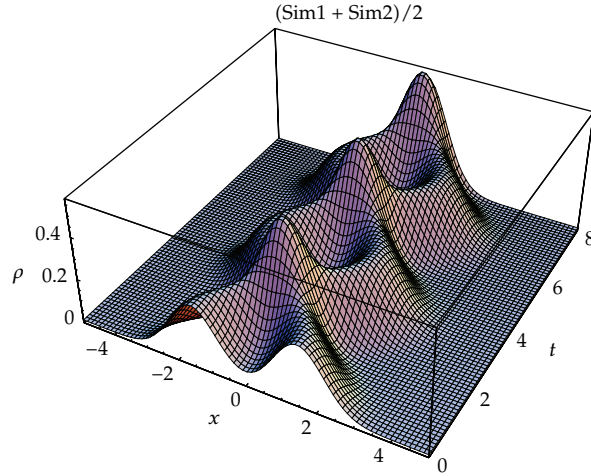


Figure 5: The Liouville solution without friction as a function of space x and time t . $\rho_X^L(t_0, x) = (1/2)[(2\pi a^2)^{-1/2}e^{-(x-x_{01})^2/(2a^2)} + (2\pi a^2)^{-1/2}e^{-(x-x_{02})^2/(2a^2)}]$, $\rho_Y^L(t_0, y) = (2\pi b^2)^{-1/2}e^{-y^2/(2b^2)}$, $b = 1/(2a)$, $a^2 = 1/2$, $x_{01} = 1.5$, $x_{02} = -1.5$, and $\varepsilon = 0$.

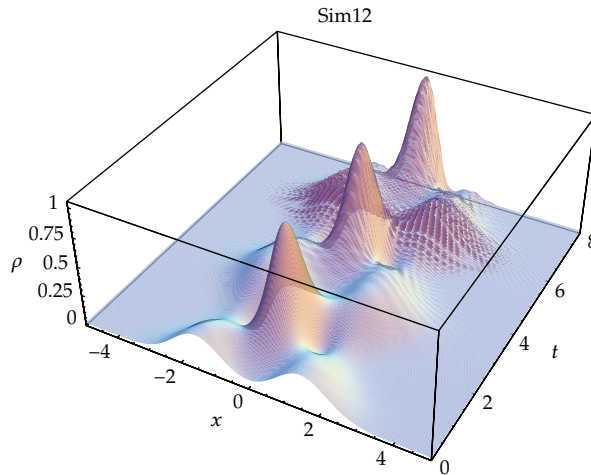


Figure 6: Quantum solution without friction as a function of space x and time t . $\rho_X^Q(t_0, x) = \rho_X^L(t_0, x) = (1/2)[(2\pi a^2)^{-1/2}e^{-(x-x_{01})^2/(2a^2)} + (2\pi a^2)^{-1/2}e^{-(x-x_{02})^2/(2a^2)}]$, $a^2 = 1/2$, $x_{01} = 1.5$, $x_{02} = -1.5$, and $\varepsilon = 0$.

$\varepsilon = 4.0$ to an arbitrarily small but positive number (e.g., $\varepsilon = 0.01$) still causes the density $\rho_X(t, x)$ to eventually reach a Gaussian distribution with $a^2 = 1/2$ and expectation zero after a sufficiently long time period t . We do not observe the typical interference pattern in Figure 7. Figure 7 stands in stark contrast to Figure 6 without friction, where there is an alternating narrowing and broadening of the density through time. Figure 8 shows the difference between the Gaussian distribution with $a^2 = 1/2$ and expectation zero and the quantum equation with friction. We clearly see that the difference vanishes.

Generally we find that with friction the initial Gaussian solutions approach the Gaussian steady-state solution with $a^2 = 1/2$. Both the quantum solution with friction and the Ornstein-Uhlenbeck process with friction approach a steady-state different from zero.

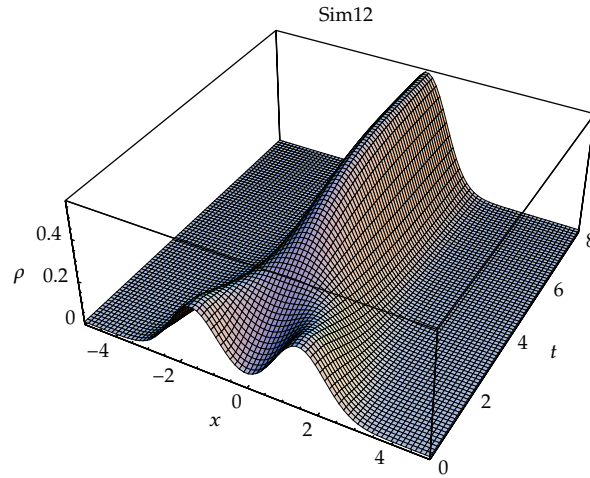


Figure 7: Quantum solution with friction as a function of space x and time t . $\rho_X^Q(t_0, x) = \rho_X^I(t_0, x) = (1/2)[(2\pi a^2)^{-1/2}e^{-(x-x_{01})^2/(2a^2)} + (2\pi a^2)^{-1/2}e^{-(x-x_{02})^2/(2a^2)}]$, $a^2 = 1/2$, $x_{01} = 1.5$, $x_{02} = -1.5$, and $\varepsilon = 4.0$.

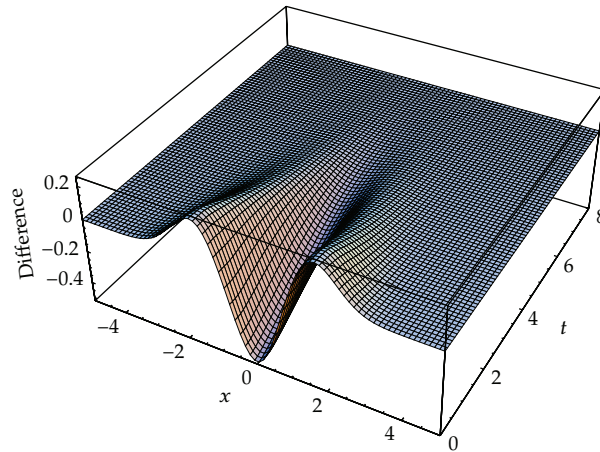


Figure 8: The difference between the quantum solution with friction and a steady-state Gaussian distribution with expectation zero and variance $a^2 = 1/2$ as a function of space x and time t .

7. Conclusion

The paper discusses a new type of stochastic theory not based on a traditional Markovian property, which is named hydrodynamic stochastic theories. The quantum theory is shown to be a special kind of such a stochastic theory. The operator problem is studied, and a quantum equation with friction is studied in relation to the measuring problem in quantum physics. Different numerical solutions are compared.

Appendices

A. The Derivation of the Differential Equation for Expectation and Variance

The Fokker-Planck equation is

$$\begin{aligned} \dot{\rho}^{\text{OU}}(t, x, y) &= -D_1(\rho^{\text{OU}}(t, x, y)y) - D_2(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + q D_2^2(\rho^{\text{OU}}(t, x, y)), \\ f &= f_Y(x, y) = -x - \varepsilon y \end{aligned} \quad (\text{A.1})$$

which implies

$$\begin{aligned} \dot{E}^{\text{OU}}(X_t) &= \int x \dot{\rho}^{\text{OU}}(t, x, y) dx dy \\ &= \int x \left(-D_1(\rho^{\text{OU}}(t, x, y)y) - D_2(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + q D_2^2(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\ &= \int y \rho^{\text{OU}}(t, x, y) dx dy = E^{\text{OU}}(Y_t), \end{aligned}$$

$$\begin{aligned} \dot{E}^{\text{OU}}(Y_t) &= \int y \dot{\rho}^{\text{OU}}(t, x, y) dx dy \\ &= \int y \left(-D_1(\rho^{\text{OU}}(t, x, y)y) - D_2(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + q D_2^2(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\ &= \int \rho^{\text{OU}}(t, x, y) (f_Y(x, y)) dx dy = -E^{\text{OU}}(X_t) - \varepsilon E^{\text{OU}}(Y_t), \end{aligned}$$

$$\begin{aligned} \dot{E}^{\text{OU}}(X_t^2) &= \int x^2 \dot{\rho}^{\text{OU}}(t, x, y) dx dy \\ &= \int x^2 \left(-D_1(\rho^{\text{OU}}(t, x, y)y) - D_2(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + q D_2^2(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\ &= \int 2x \left(\rho^{\text{OU}}(t, x, y)y \right) dx dy = 2E^{\text{OU}}(X_t Y_t), \end{aligned}$$

$$\begin{aligned} \dot{E}^{\text{OU}}(Y_t^2) &= \int y^2 \dot{\rho}^{\text{OU}}(t, x, y) dx dy \\ &= \int y^2 \left(-D_1(\rho^{\text{OU}}(t, x, y)y) - D_2(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + q D_2^2(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\ &= \int \left(2y(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + q(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\ &= \int \left(2y(-x - \varepsilon y)(\rho^{\text{OU}}(t, x, y)) + q(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\ &= -2E^{\text{L}}(X_t Y_t) - 2\varepsilon E^{\text{L}}(Y_t^2) + q, \end{aligned}$$

$$\begin{aligned}
\dot{E}^{\text{OU}}(X_t Y_t) &= \int xy \dot{\rho}^{\text{OU}}(t, x, y) dx dy \\
&= \int xy \left(-D_1(\rho^{\text{OU}}(t, x, y)y) - D_2(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) + qD_2^2(\rho^{\text{OU}}(t, x, y)) \right) dx dy \\
&= \int \left(y(\rho^{\text{OU}}(t, x, y)y) + x(\rho^{\text{OU}}(t, x, y)f_Y(x, y)) \right) dx dy \\
&= E^{\text{OU}}(Y_t^2) - E^{\text{OU}}(X_t^2) - \varepsilon E^{\text{OU}}(X_t Y_t)
\end{aligned} \tag{A.2}$$

which implies

$$\begin{aligned}
\dot{E}^{\text{OU}}(X_t) &= E^{\text{OU}}(Y_t), & \dot{E}^{\text{OU}}(Y_t) &= -E^{\text{OU}}(X_t) - \varepsilon E^{\text{OU}}(Y_t), \\
\dot{\text{Var}}^{\text{OU}}(X_t) &= \dot{E}^{\text{OU}}(X_t^2) - 2E^{\text{OU}}(X_t)\dot{E}^{\text{OU}}(X_t) = 2E^{\text{OU}}(X_t Y_t) - 2E^{\text{OU}}(X_t)E^{\text{OU}}(Y_t) \\
&= 2\text{Cov}^{\text{OU}}(X_t, Y_t), \\
\dot{\text{Var}}^{\text{OU}}(Y_t) &= \dot{E}^{\text{OU}}(Y_t^2) - 2E^{\text{OU}}(Y_t)\dot{E}^{\text{OU}}(Y_t) \\
&= -2E^{\text{OU}}(X_t Y_t) - 2\varepsilon E^{\text{OU}}(Y_t^2) + q - 2E^{\text{OU}}(Y_t)(-E^{\text{OU}}(X_t) - \varepsilon E^{\text{OU}}(Y_t)) \\
&= -2\text{Cov}^{\text{OU}}(X_t, Y_t) - 2\varepsilon \text{Var}^{\text{OU}}(Y_t) + q, \\
\dot{\text{Cov}}^{\text{OU}}(X_t, Y_t) &= \dot{E}^{\text{OU}}(X_t Y_t) - \dot{E}^{\text{OU}}(X_t)E^{\text{OU}}(Y_t) - E^{\text{OU}}(X_t)\dot{E}^{\text{OU}}(Y_t) \\
&= E^{\text{OU}}(Y_t^2) - E^{\text{OU}}(X_t^2) - \varepsilon E^{\text{OU}}(X_t Y_t) \\
&\quad - E^{\text{OU}}(Y_t)^2 - E^{\text{OU}}(X_t)(-E^{\text{OU}}(X_t) - \varepsilon E^{\text{OU}}(Y_t)) \\
&= \text{Var}^{\text{OU}}(Y_t) - \text{Var}^{\text{OU}}(X_t) - \varepsilon \text{Cov}^{\text{OU}}(X_t, Y_t).
\end{aligned} \tag{A.3}$$

B. The Measuring Problem

A main question is how a quantum mechanical object interacts with a classical object. Friction in general tends to decrease the energy of a system. It is commonly believed that the “measuring problem” is a purely quantum mechanical problem, but also in classical mechanics a conceptual problem exists. Say, as an example, that a particle is moving according to the equation $\dot{x} = f(t, x, \dot{x})$. What is the energy of the particle at time t and how should we measure it? Say that we let the particle interact with a (macroscopic) object, and

that during the interaction, the position of the particle is such that $\dot{x} = f(t, x, \dot{x}) - \dot{x}$ when $t \geq t_m$, and t_m is the start time for the measuring process. The energy at time t_m we define as

$$\begin{aligned} e(t_m) &= - \int_{t_m}^{\infty} (\ddot{x}(u) - f(u, x(u), \dot{x}(u))) \dot{x}(u) du \\ &= - \left[\frac{1}{2} \dot{x}(\infty)^2 - \frac{1}{2} \dot{x}(t_m)^2 \right] + \int_{t_m}^{\infty} f(u, x(u), \dot{x}(u)) \dot{x}(u) du \end{aligned} \quad (\text{B.1})$$

provided that the last integral exists. For the harmonic oscillator we achieve the well-known result $e(t_m) = (1/2)\dot{x}(t_m)^2 + (1/2)x(t_m)^2$.

If we use this kind of logic for a quantum system we need an equation describing the interaction of the quantum system with the measuring device. Thus to overcome the "measuring problem" we suggest to use the nonlinear quantum equation accounting for friction. The equation accounting for friction is studied numerically and compared with other types of solutions. The energy becomes

$$\begin{aligned} \dot{e}(t_m) &= - \int_{t_m}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\psi^*(t, u) \dot{\psi}(t, u)) du dt = - \left[\int_{-\infty}^{\infty} \psi^*(t, u) \dot{\psi}(t, u) du \right]_{t_m}^{\infty} \\ &\Rightarrow e(t_m) \\ &= - \left[\int_{-\infty}^{\infty} \psi^*(t, u) \left(-\frac{1}{2} D^2 \psi(t, u) + V(u) \psi(t, u) + \varepsilon \frac{1}{2i} \text{Ln} \left(\frac{\psi(t, u)}{\psi^*(t, u)} \right) \psi(t, u) \right) du \right]_{t_m}^{\infty}. \end{aligned} \quad (\text{B.2})$$

C. The Harmonic Oscillator

The steady-state solutions of the Schrödinger solution for the harmonic oscillator are stated in almost every book in quantum mechanics, to read when $x_0 = 0$

$$\begin{aligned} \psi_m^Q(t, x) &= \frac{(\pi)^{-1/4} e^{-x^2/2}}{[m!2^m]^{1/2} H_m(x)}, \quad H_m(): \text{ Hermite,} \\ H_0 &= 1, \quad H_1 = 2x, \quad H_2 = 4x^2 - 2, \quad H_3 = 8x^3 - 12x, \dots, \\ \rho_{x_m}^Q(t, x) &= \psi_m^Q(t, x) \psi_m^Q(t, x)^*, \quad v_1^Q(t, x) = 0. \end{aligned} \quad (\text{C.1})$$

Inserting $m = 0$ into (C.1) gives the steady-state solution discussed above for $a^2 = 1/2$ and $x_0 = 0$. An interesting question is whether it is possible to construct other Liouville solutions equivalent to the Schrödinger solutions in (C.1), for instance whether it is possible to find other initial densities $\rho_Y^L(t_0, y)$ such that the Liouville solutions are equivalent to the Schrödinger solutions. First note that every initial distribution for the Liouville density that is independent in position and velocity and that is equal during interchange of x and y is a steady-state solution for the Harmonic oscillator. Thus the Hermit functions are solutions to the Liouville equation also as long as we choose the distribution for position and velocity according to this rule. More generally, we find the same time-dependent solution when

$b = 1/(2a)$ for the Gaussian distribution. To answer this question, notice that the Hermit solutions in (6.10) possess well-known and interesting mathematical properties. The Fourier transform of a Hermit function is the same Hermit function multiplied with i^m , or, more precisely,

$$\widehat{\theta}_m(t_0, y) \stackrel{\text{def}}{=} (2\pi)^{-1/2} \int \psi_m^Q(t_0, x) e^{-ixy} dx \implies \widehat{\theta}_m(t_0, y) = i^m \psi_m(t_0, y). \quad (\text{C.2})$$

Thus a candidate to achieve match would be to choose the initial distribution in the velocity for the Liouville process equal to the Fourier transformed distribution according to

$$\widehat{\theta}(t_0, y) = (2\pi)^{-1/2} \int \psi_m^Q\left(t_0, \frac{x}{a}\right) e^{-ixy} dx, \quad \rho_{Ym}^L(t_0, y) = \widehat{\theta}(t_0, y) \widehat{\theta}(t_0, y)^*. \quad (\text{C.3})$$

We observe that choosing $m = 0$ gives the velocity distribution that has been shown to give agreement with the quantum solution.

We can study this more carefully. The Wigner function is defined by

$$\bar{\rho}^W(t, q, p) \stackrel{\text{def}}{=} \frac{1}{\pi\hbar} \int \text{Exp}\left(\frac{2ipu}{\hbar}\right) \left(\sum w_\lambda \psi_\lambda(t, q-u) \psi_\lambda^*(t, q+u) \right) du, \quad (\text{C.4})$$

where w_λ is the probability of being in the state λ , and $\{\psi_\lambda\}$ is a complete set. For a pure state we achieve

$$\bar{\rho}^W(t, q, p) = \frac{1}{\pi\hbar} \int \text{Exp}\left(\frac{2ipu}{\hbar}\right) \psi_\lambda(t, q-u) \psi_\lambda^*(t, q+u) du. \quad (\text{C.5})$$

We can perform the Fourier transform of $\bar{\rho}^W(q, p, t)$, to read (dropping the \hbar)

$$\begin{aligned} \int \bar{\rho}^W(t, q, p) \text{Exp}(-2ipx') dp &= \frac{1}{\pi} \int \text{Exp}(2ip(u-x')) \left(\sum w_\lambda \psi_\lambda(t, q-u) \psi_\lambda^*(t, q+u) \right) du dp \\ &= \sum w_\lambda \psi_\lambda(t, q-x') \psi_\lambda^*(t, q+x'). \end{aligned} \quad (\text{C.6})$$

We can perform the Fourier transform of the wave function, to read

$$\begin{aligned}
\psi(t, q) &= \int \bar{\rho}^W(t, q, p) dp, & \theta(t, p) &= \int \bar{\rho}^W(t, q, p) dq, \\
\theta(t, p) &= \frac{1}{(2\pi)^{1/2}} \int \psi(t, q') \text{Exp}(-iq'p) dq', & \psi(t, q) &= \int \theta(t, p') \text{Exp}(iqp') dp', \\
\bar{\rho}^W(t, q, p) &= \frac{1}{\pi} \int \text{Exp}(2ipu) \psi^*(t, q+u) \psi(t, q-u) du \\
&= \frac{1}{\pi} \int \frac{1}{2\pi} \int \theta(t, p')^* \text{Exp}(-i(q+u)p' + i(q-u)p'' + 2ipu) \theta(t, p'') dp' dp'' du \\
&= \frac{1}{\pi \hbar} \int \theta(t, p+u)^* \theta(t, p-u) \text{Exp}(-2iqu) du.
\end{aligned} \tag{C.7}$$

Thus $\bar{\rho}^W(t, q, p)$ exhibits the basic symmetry under the interchange between q and p .

Say that the classical potential can be Taylor expanded, to read

$$V(q+y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} V^{(k)}(q). \tag{C.8}$$

By direct time derivation of $\bar{\rho}^W(t, q, p)$ and use of the Schrödinger equation it follows that (introducing \hbar again)

$$\begin{aligned}
\dot{\bar{\rho}}^W(t, q, p) &= \frac{\partial}{\partial q} \left(\bar{\rho}^W(t, q, p) \right) - \frac{\partial}{\partial q} \left(-V'(q) \bar{\rho}^W(t, q, p) \right) \\
&+ \left(\sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\hbar}{2i} \right)^{2n} V^{(k)}(q) \right) \frac{\partial^{2k+1}}{\partial p^{2k+1}} \left(\bar{\rho}^W(t, q, p) \right).
\end{aligned} \tag{C.9}$$

The last term is negligible for the harmonic oscillator, a linear potential, or a free particle. However, there is still a difference with the classical results for the Liouville equation since the possible initial conditions are restricted in (C.9). The reason is that the initial distribution, for say momentum, is given through the $\theta(t_0, p)$, which is the Fourier transform of $\psi(t_0, p)$. Thus a given $\psi(t_0, q)$ gives a unique distribution for momentum. This also implies $\text{Var}(q) \text{Var}(p) \geq \hbar^2/4$.

In the work of Gardiner and Zoller [37] an equation for the Wigner function accounting for linear friction is stated. It is found that the second term on the right-hand side in (C.9) is equal to the term with friction in the Liouville equation, to read $-(\partial/\partial p)((-V'(q) - \varepsilon p) \bar{\rho}^W(t, q, p))$.

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