Review Article
Constructive Field Theory in Zero Dimension

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Constructive field theory can be considered as a reorganization of perturbation theory in a convergent way. In this pedagogical note we propose to wander through five different methods to compute the number of connected graphs of the zero-dimensional ϕ^4 field theory, in increasing order of sophistication and power.

1. Introduction

New constructive Bosonic field theory methods have been recently proposed which are based on applying a canonical forest formula to repackage perturbation theory in a better way. This allows to compute the connected quantities of the theory by the same formula but summed over trees instead of forests. (Constructive Fermionic field theory is easier and was repackaged in terms of trees much earlier [1–3].) The resulting formulation of the theory is given by a convergent rather than divergent expansion. In short this is because there are much less trees than graphs, but they still capture the vital physical information, which is connectedness.

Combining such a forest formula with the intermediate field method leads to a convenient resummation of ϕ^4 perturbation theory. The main advantage of this formalism over previous cluster and Mayer expansions is that connected functions are captured by a single formula, and for example, a Borel summability theorem for matrix ϕ^4 models can be obtained which scales correctly with the size of the matrix [4]. The resulting method applies to ordinary quantum field theory on commutative space as well [5]. However in this point of view the connected functions still involve functional integrals over many replicas of the intermediate field.

Another even more recent constructive point of view [6] is that a quantum Euclidean Bosonic field theory is a particular positive scalar product on a universal vector space spanned by "marked trees." That scalar product is obtained by applying a tree or forest

formula to the ordinary perturbative expansion of the QFT model under consideration. That formula itself is model-independent and reorganizes perturbation theory differently, by breaking Feynman amplitudes into pieces and putting these pieces into boxes labeled by trees.

In this point of view, constructive bounds reduce essentially to the positivity of the universal Hamiltonian operator. The vacuum is the trivial tree and the correlation functions are given by "vacuum expectation values" of the resolvent of that combinatoric Hamiltonian operator. Model-dependent details such as space-time dimension, interactions and propagators enter the definition of the matrix elements of this scalar product. These matrix elements are just finite sums of finite dimensional Feynman integrals.

We were urged to explain these new ideas in a pedagogical way. This is what we do in this short note on the simplest possible example, namely, the connected graphs of the zero dimensional ϕ^4 theory. This theory corresponds to an ordinary integral on a single variable, and we hope that following the different constructive steps on this simple example will expose the core ideas better. The main picture that emerges is that the essence of constructive theory is about using cleverly trees and replicas.

2. The Forest Formula

A forest on n points is a graph on these points with no cycle (a cycle is usually called a loop by physicists). The graph without lines is a forest called the empty forest. With this convention the reader can check that there are 2 differents, forests on 2 points, 7 different, forests on three points, and 28 on four points.

A tree (also called spanning tree) on n points is a connected forest, that is a graph without cycles which connects all the n points. A famous theorem by Cayley states that there are exactly n^{n-2} different trees on n points.

Consider n of such points, the set of pairs P_n of such points which has $n(n-1)/2$ elements $\ell = (i, j)$ for $1 \leq i < j \leq n$, and a smooth function f of $n(n-1)/2$ variables $x_\ell, \ell \in P_n$. Noting ∂_ℓ for $\partial/\partial x_\ell$, the standard canonical forest formula is [7, 8]

$$f(1,\ldots,1) = \sum_{\mathcal{F}}\left[\prod_{\ell \in \mathcal{F}}\int_0^1 dw_\ell\right]\left(\left[\prod_{\ell \in \mathcal{F}}\partial_\ell\right]f\right)\left[x_\ell^{\mathcal{F}}(\{w_{\ell'}\})\right], \tag{2.1}$$

where

(i) the sum over $\mathcal{F}$ is over forests over the n vertices, including the empty one;

(ii) $x_\ell^{\mathcal{F}}(\{w_{\ell'}\})$ is the infimum of the $w_{\ell'}$ for ℓ' in the unique path from i to j in $\mathcal{F}$, where $\ell = (i, j)$. If there is no such path, $x_\ell^{\mathcal{F}}(\{w_{\ell'}\}) = 0$ by definition;

(iii) the symmetric n by n matrix $X^{\mathcal{F}}(\{w\})$ defined by $X_{ii}^{\mathcal{F}} = 1$ and $X_{ij}^{\mathcal{F}} = x_{ij}^{\mathcal{F}}(\{w_{\ell'}\})$ for $1 \leq i < j \leq n$ is positive.

One can easily check that for $n = 2$, this is nothing but the fundamental theorem of calculus: $H(1) = H(0) + \int_0^1 dh\ H'(h)$. For higher values of n, on the other hand, the outcome
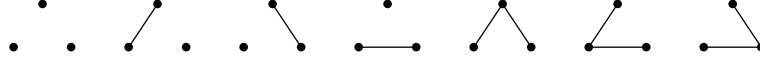
Figure 1: The complete graph over 3 vertices and its 7 forests.

of (2.1) is genuinely nontrivial, as the case $n = 3$ already demonstrates, where we get a sum of seven terms matching the seven forests of Figure 1:

$$
\begin{aligned}
H(1,1,1) = {} & H(0,0,0) + \int_0^1 dh_1 \partial_1 H(h_1,0,0) + \int_0^1 dh_2 \; \partial_2 H(0,h_2,0) \\
& + \int_0^1 dh_3 \partial_3 H(0,0,h_3) + \int_0^1 dh_1 \int_0^1 dh_2 \; \partial_{12}^2 H(h_1,h_2,\min(h_1,h_2)) \\
& + \int_0^1 dh_1 \int_0^1 dh_3 \partial_{13}^2 H(h_1,\min(h_1,h_3),h_3) \\
& + \int_0^1 dh_2 \int_0^1 dh_3 \partial_{23}^2 H(\min(h_2,h_3),h_2,h_3).
\end{aligned}
\tag{2.2}
$$

A particular variant of this formula (2.1) is in fact better suited to direct application to the parametric representation of Feynman amplitudes. It consists in changing variables $x \to 1 - x$ and rescaling to $[0,1] \to [0,\infty]$ of the range of the variables. One gets that if f is smooth with well-defined limits for any combination of x_ℓ tending to ∞,

$$
f(0,\ldots,0) = \sum_{\mathcal{F}} \left[\prod_{\ell \in \mathcal{F}} \int_0^\infty ds_\ell \right] \left(\left[\prod_{\ell \in \mathcal{F}} - \partial_\ell \right] f \right) \left[x_\ell^{\mathcal{F}}(\{s_{\ell'}\}) \right],
\tag{2.3}
$$

where

  (i) the sum over $\mathcal{F}$ is like above;

  (ii) $x_\ell^{\mathcal{F}}(\{s_{\ell'}\})$ is the *supremum* of the $s_{\ell'}$ for ℓ' in the unique path from i to j in $\mathcal{F}$, where $\ell = (i,j)$. If there is no such path, $x_\ell^{\mathcal{F}}(\{s_{\ell'}\}) = \infty$ by definition. This is because the change of variables exchanged inf and sup.

To distinguish these two formulas we call w the parameters of the first one (like "weakening") since the formula involves infima, and s the parameters of the second one (like "strengthening" or "supremum") since the formula involves suprema.

Scale analysis is the key to renormalization, and scales can be conveniently defined in quantum field theory through the parametric representation of the propagator $H^{-1} = \int_0^\infty e^{-\alpha H} d\alpha$. The strengthening formula (2.3) rather than the weakening formula is therefore particularly adapted to scale analysis and renormalization.

Finally notice that various extensions of these formulas should be useful to study further the theory. A quite general theory of such formulas is given in [9].

3. Borel Summability

Borel summability of a power series $\sum_n a_n \lambda^n$ means existence of a function f with two properties [10]:

> (i) analyticity in a disk tangent to the imaginary axis at the origin lying on the right-hand side of that disk, hence defined by a condition $\mathfrak{R}\lambda^{-1} > R^{-1}$,

> (ii) plus remainder estimates uniform in that disk:

$$\left| f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n \right| \leq K^N \ N! \ |\lambda|^{N+1}. \tag{3.1}$$

Given any power series $\sum_n a_n \lambda^n$, there is at most one such function f. When there is one, it is called the Borel sum, and it can be computed from the series to arbitrary accuracy.

Therefore Borel summability is a perfect substitute for ordinary analyticity when a function is expanded at a point on the frontier of its analyticity domain. Borel summability is just a rigidity which plays the same role than analyticity: it selects a *unique* map between a certain class of functions and a certain class of power series. Within that class, all the information about the function is therefore captured in the much more compact list of its Taylor coefficients.

Very early, both functional integrals and the Feynman perturbative series were introduced to study quantum field theory, but it was realized that the corresponding power series were generically divergent. When a link between both approaches can be established, it is usually through Borel summability or a variant thereof. This is why Borel summability is important for quantum field theory.

4. Φ^4 Constructive Theory in Zero Dimension

In this section we propose to test the evolution of ideas in constructive theory on the simple example of a single-variable ordinary integral which represents the ϕ^4 field theory in zero dimensions.

The normalization or partition function of that theory is the ordinary integral:

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}}. \tag{4.1}$$

It is obvious that $F(\lambda)$ is well defined for $\operatorname{Re} \lambda \geq 0$ with $|F(\lambda)| \leq 1$, and analytic for $\operatorname{Re} \lambda > 0$. It has a Taylor series at the origin $F \simeq \sum_n a_n(-\lambda)^n$ with $a_n = (4n)!!/n!$. Using a Taylor expansion with integral remainder it is also very easy to prove that F is Borel summable. But in physics we are interested in computing connected quantities, hence in the function $G(\lambda) = \log F(\lambda)$. We can therefore formulate the simplest of all Bosonic constructive field theory problems as follows.

Problem. How to compute $G(\lambda)$ and prove its Borel summability in the most explicit and efficient manner?

We shall review how different methods of increasing sophistication answer this question:

(i) composition of series (XIXth century),

(ii) a la Feynman (1950),

(iii) "Classical Constructive," à la Glimm-Jaffe-Spencer (1970s–2000s)

(iv) with loop vertices (2007),

(v) with tree vector space (2008).

4.1. Composition of Series

The first remark is that we know the explicit power series for F, and it starts with 1, so that $F = 1+H$; we know the explicit series for $\log(1+x)$ so we can substitute H for x and reexpand and we get an explicit formula for the coefficient b_n of the Taylor series of G:

$$F = 1 + H, \quad H = \sum_{p \geq 1} a_p(-\lambda)^p, \; a_p = \frac{(4p)!!}{p!}$$

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},$$

$$G = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H(\lambda)^n}{n} = \sum_{k \geq 1} b_k(-\lambda)^k, \tag{4.2}$$

$$b_k = \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_1,\ldots,p_n \geq 1 \\ p_1+\cdots+p_n=k}} \prod_j \frac{(4p_j)!!}{p_j!}.$$

To summarize, this method leads to an explicit formula for b_k, but of little use. Borel summability of G is unclear. Even the sign of b_k is unclear.

4.2. A la Feynman

Feynman understood that a_n can be represented as a sum of terms associated to drawings, the famous Feynman graphs. But the real mathematical power of this idea is that it allows a quick computation of logarithms: they are simply given by the same sum but over connected drawings!

The theory of combinatoric species is a rich mathematical orchestration of this intuition; see [11]. A species is roughly a structure on finite sets of "points," and combinatorics consists in counting the number a_n of elements in that species on n points. A fundamental tool to this effect is the generating power series of the species which is $\sum_n (a_n/n!)\lambda^n$. Let us say that a species *proliferate* if its generating power series has zero radius of convergence. A big problem of the current formulation of perturbative quantum field theory is that the species of ordinary Feynman graphs proliferate, whereas trees do not. The main problem of constructive theory is therefore to replace Feynman graphs by trees.

In our case the drawings are the Wick contractions of ϕ^4 vacuum graphs, that is graphs on n vertices with coordination 4 at each vertex (loops, called *tadpoles* by physicists, being allowed):

$$F = 1 + H, \quad H = \sum_{p \geq 1} a_p(-\lambda)^p, \quad a_p = \frac{1}{p!}\#\{\text{vacuum graphs on } p \text{ vertices}\}, \tag{4.3}$$

$$G = \sum_{k=1}^{\infty}(-\lambda)^k b_k, \quad b_k = \frac{1}{k!}\#\{\text{vacuum connected graphs on } k \text{ vertices}\}. \tag{4.4}$$

We can easily compute in this way that $b_1 = 3$, $b_2 = 48$, $b_3 = 1584\ldots$.

Usually in the quantum field theory literature there are painful discussions on what is a Feynman graph and what is its combinatoric weight or "symmetry factor." This is important to make the shortest possible list of independent Feynman amplitudes that one has to compute in practice. But conceptually it is much better to consider a graph as a set of "Wick contractions," that is a set of pairing of fields, so that no "symmetry factors" are ever discussed. (Fields correspond to half-lines, also called flags in the mathematics literature and the physicists point of view that flags, not lines, are the fundamental elements in graph theory is slowly making its way in the mathematics literature [12, 13].)

Borel summability remains unclear. But as a first fruit of the idea of Feynman graphs clearly, we now see explicitly that $b_k \geq 0$: we know that G has an alternate power series.

4.3. Classical Constructive

The standard method in Bosonic constructive field theory is to first break up the functional integrals over a discretization of space-time and then test the couplings between the corresponding functional integrals (cluster expansion), which results in the theory being written as a polymer gas with hardcore constraints. For that gas to be dilute at small coupling, the normalization of the free functional integrals must be factored out. Finally the connected functions are computed by expanding away the hardcore constraint through a so-called Mayer expansion [14–17].

In the zero-dimensional case there is no need to discretize the single point of space-time; hence it seems that the first step, namely, the cluster expansion is trivial. This is correct except for the fact that what remains from this step is to factorize the "free functional" integral; so a single first-order Taylor expansion with remainder around $\lambda = 0$ corresponds to the zero-dimensional cluster expansion. But after that the Mayer expansion is completely non-trivial and very instructive, because a single point has indeed hardcore constraint with itself!

The first step (cluster expansion) is therefore

$$F = 1 + H, \ H = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x^4 e^{-\lambda t x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} \tag{4.5}$$

The more interesting Mayer expansion in this case consists in introducing many copies or "replicas" of H:

$$\forall i \quad H = H_i = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x_i^4 e^{-\lambda t x_i^4 - x_i^2/2} \frac{dx_i}{\sqrt{2\pi}}, \tag{4.6}$$

Defining $\epsilon_{ij} = 0$ for all i, j we can write the apparently stupid formula:

$$F = 1 + H = \sum_{n=0}^{\infty} \prod_{i=1}^{n} H_i(\lambda) \prod_{1 \le i < j \le n} \epsilon_{ij}. \tag{4.7}$$

But defining $\eta_{ij} = -1$, $\epsilon_{ij} = 1 + \eta_{ij} = 1 + x_{ij}\eta_{ij}|_{x_{ij}=1}$ and applying the forest formula lead to

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \prod_{i=1}^{n} H_i(\lambda) \left\{ \prod_{\ell \in \mathcal{F}} \left[\int_0^1 dw_\ell \right] \eta_\ell \right\} \prod_{\ell \notin \mathcal{F}} \left[1 + \eta_\ell x_\ell^{\mathcal{F}}(\{w\}) \right], \tag{4.8}$$

which allows easily to take the logarithm

$$G = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \prod_{i=1}^{n} H_i(\lambda) \left\{ \prod_{\ell \in \mathcal{T}} \left[\int_0^1 dw_\ell \right] \eta_\ell \right\} \prod_{\ell \notin \mathcal{T}} \left[1 + \eta_\ell x_\ell^{\mathcal{T}}(\{w\}) \right], \tag{4.9}$$

where the second sum runs over trees! In this way we obtain the following.

(i) Convergence is now easy because each H_i contains a different "copy" $\int dx_i$ of the "functional integration" (which of course here is an ordinary integration).

(ii) Borel summability for G follows now easily from the Borel summability of H.

A shortcoming is that "space-time" and functional integrals remains present. Also the cluster step, suitably generalized in non zero dimension by Glimm, Jaffe, Spencer, and followers, heavily relies on locality, hence does not seem to have the potential to work on nonstandard space-times or for nonlocal or matrix-like theories like the Grosse-Wulkenhaar model of noncommutative theory [18] or for the group field theory models of quantum gravity.

4.4. Loop Vertices

The intermediate field representation is a well-known trick to represent a quartic interaction, in terms of a cubic one:

$$\begin{aligned}
F &= \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\sqrt{2\lambda}\sigma x^2 - x^2/2 - \sigma^2/2} \frac{dx}{\sqrt{2\pi}} \frac{d\sigma}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{+\infty} e^{-1/2 \log[1 + i\sqrt{8\lambda}\sigma] - \sigma^2/2} \frac{d\sigma}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{V^n}{n!} d\mu(\sigma), \quad \text{with } V = -\frac{1}{2} \log\left[1 + i\sqrt{8\lambda}\sigma \right],
\end{aligned} \tag{4.10}$$

where $d\mu(\sigma)$ is the Gaussian measure on σ of covariance 1, namely:

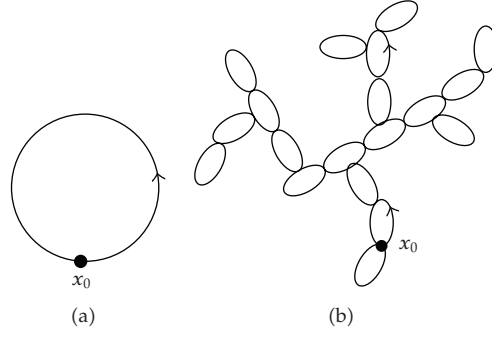$$d\mu(\sigma) = e^{-\sigma^2/2} \frac{d\sigma}{\sqrt{2\pi}}. \tag{4.11}$$

Figure 2: Loop vertices and a tree on them, or cactus.

We can introduce again replicas but in a slightly different way. We duplicate the intermediate field into copies, $V^n(\sigma) \rightarrow \prod_{i=1}^n V_i(\sigma_i)$, one associated to each factor V, also called a "loop vertex." The theory has not changed if we use for all these fields a jointly Gaussian measure with degenerate covariance, $d\mu(\sigma) \rightarrow d\mu_C(\{\sigma_i\})$, $C_{ij} = 1 = x_{ij}|_{x_{ij}=1}$. Remark that the corresponding measure has no density with respect to the Lebesgue measure since

$$d\mu_C(\{\sigma_i\}) = \frac{d\sigma_1}{\sqrt{2\pi}} e^{-\sigma_1^2/2} \prod_{i=2}^n \delta(\sigma_1 - \sigma_i) \, d\sigma_i. \tag{4.12}$$

It is not enough known that a delta function *is* a Gaussian measure!

This does not change the expectation value of any polynomial; hence by the Weierstrass theorem it does not change the theory. But one can now apply the forest formula to the off-diagonal couplings C_{ij}. It gives

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathscr{F}} \left\{ \prod_{\ell \in \mathscr{F}} \left[\int_0^1 dw_\ell \right] \right\} \int \left\{ \prod_{\ell \in \mathscr{F}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right\} d\mu_{C^{\mathscr{F}}}, \tag{4.13}$$

where $C_{ij}^{\mathscr{F}} = x_\ell^{\mathscr{F}}(\{w\})$ if $i < j$, $C_{ii}^{\mathscr{F}} = 1$ and

$$G = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\tau} \left\{ \prod_{\ell \in \tau} \left[\int_0^1 dw_\ell \right] \right\} \int \left\{ \prod_{\ell \in \tau} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i) \right\} d\mu_{C^\tau}, \tag{4.14}$$

where the second sum runs over trees!

The main advantage is that the role of propagators and vertices has been exchanged! The result is a sum over trees on loops, or *cacti*.

Since

$$\frac{\partial^k}{\partial \sigma^k} \log\left[1 + i\sqrt{8\lambda}\sigma\right] = -(k-1)!\left(-i\sqrt{8\lambda}\right)^k \left[1 + i\sqrt{8\lambda}\sigma\right]^{-k}, \tag{4.15}$$

the loop vertices involve denominators or "resolvents" $[1 + i\sqrt{8\lambda}\sigma]^{-1}$ rather than log's. However there is a little subtlety with the first "trivial" term in G for $n = 1$ which is a single vertex loop with value $\log[1 + i\sqrt{8\lambda}\sigma]$. To transform it also into an expression with denominators one should perform a single Taylor expansion step and integrate the σ field through integration by parts:

$$\int d\mu(\sigma) \log\left[1 + i\sqrt{8\lambda}\sigma\right] = \int d\mu(\sigma) \int_0^1 dt \frac{i\sqrt{8\lambda}\sigma}{1 + i\sqrt{8\lambda}t\sigma} = \int d\mu(\sigma) \int_0^1 dt \frac{8\lambda t}{\left[1 + i\sqrt{8\lambda}t\sigma\right]^2}. \qquad (4.16)$$

Once G has been rewritten in this way we have the following.

(i) Convergence is easy because $|[1 + i\sqrt{8\lambda}\sigma]^{-k}| \leq 1$, and because trees do not proliferate.

(ii) Borel summability is easy.

(iii) This method extends to noncommutative field theory and gives correct estimates for matrix-like models.

The drawback is that the method is more difficult when the interaction is of higher degree, for example, ϕ^6, because more intermediate fields are needed. Moreover functional integrals over the intermediate fields are still present.

4.5. Tree QFT

This last method no longer requires functional integral at all! It is in a way the closest to Feynman graphs, hence looks at first sight like a step backwards in constructive theory. It starts exactly like constructive Fermionic theory.

Within a given quantum field model, the forest formula indeed associates a natural amplitude A_T to a tree T. It is the sum of all contributions associated to that tree when one applies the tree formula to the nth order of perturbation theory of *that* model.

In our zero-dimensional case it means that we start with the usual *formal* perturbation theory:

$$F = \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int \sum_{n=0}^{\infty} \frac{V^n}{n!} d\mu(x), \qquad (4.17)$$

where $V = (-\lambda x^4)$ and $d\mu(x)$ is the Gaussian measure of covariance 1. Beware that this interchange of sums and integrals is not licit, contrary to (4.10). That is why the result, namely ordinary perturbation theory diverges! Nevertheless we shall use this perturbation series in a heuristic way; namely, we shall repackage according to a forest formula and use the pieces so obtained to build a (semidefinite) scalar product on a vector space generated by marked trees. This will allow still another rigorous constructive expansion of the function G.

Then we introduce replicas again but on the ordinary vertices:

$$F = \sum_{n=0}^{\infty} \int \frac{1}{n!} \prod_{i=1}^{n} \left(-\lambda x_i^4\right) d\mu(\{x_i\}), \qquad (4.18)$$

where $d\mu$ is the degenerate measure with covariance $C_{ij} = 1$ for all i, j.

Applying the tree formula to that covariance gives in the same vein than before

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \left\{ \prod_{\ell \in \mathcal{F}} \left[\int_0^1 dw_\ell \right] \right\} \int \left\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_{i(\ell)}} \frac{\partial}{\partial x_{j(\ell)}} \prod_{i=1}^{n} \left(-\lambda x_i^4 \right) \right\} d\mu_{C^{\mathcal{F}}}, \qquad (4.19)$$

where $C_{ij}^{\mathcal{F}} = x_\ell^{\mathcal{F}}(\{w\})$ if $i < j$, $C_{ii}^{\mathcal{F}} = 1$, so that *formally*

$$G = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\tau} \left\{ \prod_{\ell \in \tau} \left[\int_0^1 dw_\ell \right] \right\} \int \left\{ \prod_{\ell \in \tau} \frac{\partial}{\partial x_{i(\ell)}} \frac{\partial}{\partial x_{j(\ell)}} \prod_{i=1}^{n} \left(-\lambda x_i^4 \right) \right\} d\mu_{C^{\tau}}. \qquad (4.20)$$

The zero-dimensional ϕ^4 tree amplitude for a tree τ is therefore nothing but

$$A_\tau = \left\{ \prod_{\ell \in \tau} \left[\int_0^1 dw_\ell \right] \right\} \int \left\{ \prod_{\ell \in \tau} \frac{\partial}{\partial x_{i(\ell)}} \frac{\partial}{\partial x_{j(\ell)}} \prod_{i=1}^{n} \left(-\lambda x_i^4 \right) \right\} d\mu_{C^{\tau}}. \qquad (4.21)$$

Remark that it is zero for trees with degree more than four at a vertex.

It seems that little has been achieved at the constructive level by rewriting Feynman graphs simply in terms of an underlying tree, like in Fermionic theories. But there is a hidden *convexity* in A_τ which hides the nonperturbative stability of the underlying theory.

It has indeed be shown in [6] that one can construct a scalar product over an abstract infinite dimensional vector space $\mathcal{E}$ with a spanning basis e_T labeled by marked trees, which are trees with a mark on a particular leaf (i.e, vertex of degree 1). On $\mathcal{E}$ there is a natural external gluing operation which sends (T, T') onto the (unmarked) tree $T \star T'$ by gluing the two marks.

This operation induces a natural (semidefinite) scalar product and a natural ϕ^4 Hamiltonian operator H. The scalar product $\langle e_T, e_{T'} \rangle$ is simply $A_{T \star T'}$. Roughly speaking the Hamiltonian operator glues all the trees hanging to a single vertex with two marks. It is a Hermitian negative operator on the Hilbert space which is the completion of $\mathcal{E}$ for the scalar product above.

For the ϕ^4 theory one can always restrict to trees with degrees at most 4 at each vertex.

The constructive expression for the connected two-point function

$$G_2 = \frac{1}{F} \int_{-\infty}^{+\infty} x^2 e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} \qquad (4.22)$$

computed with this method is

$$G_2 = \left\langle e_{T_0}, \frac{1}{1 - H} e_{T_0} \right\rangle, \qquad (4.23)$$

where T_0 is the trivial tree with a single line. It is well defined at nonperturbative level because H is Hermitian negative.

This approach does not require any functional integral, since $\mathcal{E}$ is spanned by finite order trees and its scalar product depends only on finite dimensional perturbative computations. It seems the most promising way to study quantum field theory in the future, including noninteger dimensions. However Borel summability is not completely obvious in this expression hence more work is needed.

Also notice that the expressions of the functions F and G within this method should require a slight extension of [6]. The tree formulas (2.1)–(2.3) should be pushed a single Taylor step further to *endotree* formulas; see above. Vacuum-connected graphs should be distributed as a sum over endotrees rather than tree, and the space $\mathcal{E}$ should be enlarged accordingly. The corresponding formalism together with the multiscale version of this approach is to be developed. Still we think that this approach is the most general and promising one for the future of constructive theory, since it is the most abstract and general. Neither functional integrals nor a space time plays a central role, which makes the theory appealing for situations such as noninteger dimensions or the future theory of quantum gravity.

Acknowledgments

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