

## Research Article

# The Existence of Positive Solution to a Nonlinear Fractional Differential Equation with Integral Boundary Conditions

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The expression and properties of Green's function for a class of nonlinear fractional differential equations with integral boundary conditions are studied and employed to obtain some results on the existence of positive solutions by using fixed point theorem in cones. The proofs are based upon the reduction of problem considered to the equivalent Fredholm integral equation of second kind. The results significantly extend and improve many known results even for integer-order cases.

## 1. Introduction

Fractional calculus is an area having a long history, its infancy dates back to three hundred years, the beginnings of classical calculus. It had attracted the interest of many old famous mathematicians, such as L'Hospital, Leibniz, Liouville, Riemann, Grünward, Letnikov, and so forth [1, 2]. As the old mathematicians expected, in recent several decades fractional differential equations have been found to be a powerful tool in more and more fields, such as materials, physics, mechanics, and engineering [1–5]. For the basic theory and recent development of the subject, we refer the reader to a text by Lakshmikantham et al. [6]. For more details and examples, see [7–24] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages, and many aspects of this theory need to be explored. In [13], Bai and Lü used the fixed point theorems to show the existence and multiplicity of positive solutions to the nonlinear fractional boundary value problem

$$\begin{aligned} D^\alpha u(t) + f(t, u(t)) &= 0, & 0 < t < 1, \\ u(0) = 0, \quad u(1) &= 0, \end{aligned} \tag{1.1}$$

where  $1 < \alpha \leq 2$ ,  $\mathbf{D}^\alpha$  is the standard Riemann-Liouville differentiation, and  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is a given continuous function.

In [15], Zhang showed the existence and multiplicity of positive solutions of the fractional boundary value problem

$$\begin{aligned} \mathbf{D}_{0+}^\alpha u(t) &= f(t, u(t)), \quad 0 < t < 1, \\ u(0) + u'(0) &= 0, \quad u(1) + u'(1) = 0, \end{aligned} \quad (1.2)$$

where  $1 < \alpha \leq 2$  is a real number, and  $\mathbf{D}_{0+}^\alpha$  is the Caputo's fractional derivative. The function  $f$  is continuous on  $[0, 1] \times [0, \infty)$ .

Recently, Ahmad and Nieto [11] investigated some existence results for a nonlinear fractional integrodifferential equation with integral boundary conditions

$$\begin{aligned} {}^c\mathbf{D}^q x(t) + f(t, x(t), (\chi x)(t)) &= 0, \quad 0 < t < 1, \quad 1 < q \leq 2, \\ \alpha x(0) + \beta x'(0) &= \int_0^1 q_1(x(s)) ds, \\ \alpha x(1) + \beta x'(1) &= \int_0^1 q_2(x(s)) ds, \end{aligned} \quad (1.3)$$

where  ${}^c\mathbf{D}^q$  is the Caputo fractional derivative,  $f : [0, 1] \times X \times X \rightarrow X$ , for  $\gamma : [0, 1] \times [0, 1] \rightarrow [0, \infty)$ ,

$$(\chi x) = \int_0^1 \gamma(t, s)x(s), \quad (1.4)$$

$q_1, q_2 : X \rightarrow X$ ,  $\alpha > 0, \beta \geq 0$  are real numbers, and  $X$  is a Banach space.

Being directly inspired by [11, 13, 15], we intend in this paper to study the following boundary value problems of fractional order differential equation

$$\begin{aligned} \mathbf{D}_{0+}^\alpha x(t) + g(t)f(t, x) &= 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x'(1) &= \int_0^1 h(t)x(t) dt, \end{aligned} \quad (1.5)$$

where  $1 < \alpha \leq 2$ ,  $g \in C((0, 1), [0, +\infty))$  and  $g$  may be singular at  $t = 0$  or/and at  $t = 1$ ,  $\mathbf{D}_{0+}^\alpha$  is the standard Riemann-Liouville differentiation,  $h \in L^1[0, 1]$  is nonnegative, and  $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ .

In the case of  $h(t) \equiv 0$ , for all  $t \in [0, 1]$ , boundary value problem (1.5) reduces to the problem studied by Kaufmann and Mboumi [19]. In [19], the authors used the fixed point theorems to show sufficient conditions for the existence of at least one and at least three positive solutions to problem (1.5). For the case of  $\alpha = 2$ , boundary value problem (1.5) is related to a boundary value problems of integer-order differential equation. Feng et al. [25] considered the existence and multiplicity of positive solutions to boundary value problem (1.5) by applying the fixed point theory in a cone for strict set contraction operators.

The organization of this paper is as follows. We will introduce some lemmas and notations in the rest of this section. In Section 2, we present the expression and properties of Green's function associated with boundary value problem (1.5). In Section 3, we give some preliminaries about operator. In particular, we state fixed point theory in cones. In Section 4, the main results of boundary value problem (1.5) will be stated and proved. In Section 5, we offer some interesting discussion of the associated boundary value problem (1.5). Finally, conclusions in Section 6 close the paper.

The fractional differential equations-related notations adopted in this paper can be found, if not explained specifically, in almost all literature related to fractional differential equations. The readers who are unfamiliar with this area can consult for example [1–6] for details.

*Definition 1.1* (see [4]). The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0, \quad (1.6)$$

where  $\alpha > 0$  is called Riemann-Liouville fractional integral of order  $\alpha$ .

*Definition 1.2* (see [4]). For a function  $f(x)$  given in the interval  $[0, 1)$ , the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad (1.7)$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integer part of number  $\alpha$ , is called the Riemann-Liouville fractional derivative of order  $\alpha$ .

**Lemma 1.3** (see [13]). *Assume that  $u \in C(0, 1) \cap L(0, 1)$  with a fractional derivative of order  $\alpha > 0$  that belongs to  $u \in C(0, 1) \cap L(0, 1)$ . Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}, \quad (1.8)$$

for some  $C_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, N$ , where  $N$  is the smallest integer greater than or equal to  $\alpha$ .

## 2. Expression and Properties of Green's Function

In this section, we present the expression and properties of Green's function associated with boundary value problem (1.5).

**Lemma 2.1.** *Assume that  $\int_0^1 h(t)t^{\alpha-1} dt \neq \alpha - 1$ . Then for any  $y \in C[0, 1]$ , the unique solution of boundary value problem*

$$\begin{aligned} D_{0+}^{\alpha} x(t) + y(t) &= 0, \quad 0 < t < 1, \\ x(0) &= 0, \quad x'(1) = \int_0^1 h(t)x(t)dt, \end{aligned} \quad (2.1)$$

is given by

$$x(t) = \int_0^1 G(t, s)y(s)ds, \quad (2.2)$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \quad (2.3)$$

$$G_1(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.4)$$

$$G_2(t, s) = \frac{t^{\alpha-1}}{\alpha - 1 - \int_0^1 h(t)t^{\alpha-1}dt} \int_0^1 h(t)G_1(t, s)dt. \quad (2.5)$$

*Proof.* By Lemma 1.3, we can reduce the equation of problem (2.1) to an equivalent integral equation

$$x(t) = -I_{0+}^{\alpha}y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}. \quad (2.6)$$

By  $x(0) = 0$ , there is  $c_2 = 0$ , and

$$x'(t) = -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2}y(s)ds + c_1(\alpha-1)t^{\alpha-2}. \quad (2.7)$$

By (2.7) and  $x'(1) = \int_0^1 h(t)x(t)dt$ , we have

$$\int_0^1 h(t)x(t)dt = -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2}y(s)ds + c_1(\alpha-1), \quad (2.8)$$

which yields that

$$c_1 = \frac{1}{\alpha-1} \int_0^1 h(t)x(t)dt + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2}y(s)ds. \quad (2.9)$$

Therefore, the unique solution of BVP (2.1) is

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + t^{\alpha-1} \left\{ \frac{1}{\alpha-1} \int_0^1 h(t)x(t) dt + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \right\} \\ &= \int_0^1 G_1(t,s)y(s) ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^1 h(t)x(t) dt, \end{aligned} \tag{2.10}$$

where  $G_1(t, s)$  is defined by (2.4).

Multiplying (2.10) with  $h(t)$  and integrating it, we can see

$$\int_0^1 h(t)x(t) dt = \int_0^1 h(t) \int_0^1 G_1(t,s)y(s) ds dt + \frac{\int_0^1 h(t)t^{\alpha-1} dt}{\alpha-1} \int_0^1 h(t)x(t) dt. \tag{2.11}$$

Therefore,

$$\int_0^1 h(t)x(t) dt = \frac{1}{1 - \int_0^1 h(t)t^{\alpha-1} dt / (\alpha-1)} \int_0^1 h(t) \int_0^1 G_1(t,s)y(s) ds dt. \tag{2.12}$$

Substituting (2.12) into (2.10), we obtain

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s)y(s) ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^1 h(t)x(t) dt \\ &= \int_0^1 G_1(t,s)y(s) ds + \frac{t^{\alpha-1}}{\alpha-1} \frac{1}{1 - \int_0^1 h(t)t^{\alpha-1} dt / (\alpha-1)} \int_0^1 h(t) \int_0^1 G_1(t,s)y(s) ds dt \\ &= \int_0^1 G_1(t,s)y(s) ds + \int_0^1 G_2(t,s)y(s) ds \\ &= \int_0^1 G(t,s)y(s) ds, \end{aligned} \tag{2.13}$$

where  $G(t, s)$ ,  $G_1(t, s)$ , and  $G_2(t, s)$  are defined by (2.3), (2.4), and (2.5), respectively. The proof is complete.  $\square$

From (2.3), (2.4), and (2.5), we can prove that  $G(t, s)$ ,  $G_1(t, s)$ , and  $G_2(t, s)$  have the following properties.

**Proposition 2.2.** *The function  $G_1(t, s)$  defined by (2.4) satisfies the following.*

- (i)  $G_1(t, s) \geq 0$  is continuous for all  $t, s \in [0, 1]$ ,  $G_1(t, s) > 0$ , for all  $t, s \in (0, 1)$ ;
- (ii)  $G_1(t, s) \leq G_1(s, s) = s^{\alpha-1}(1-s)^{\alpha-2}/\Gamma(\alpha)$ , for all  $t \in [0, 1]$ ,  $s \in (0, 1)$ .

*Proof.* (i) It is obvious that  $G_1(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ . For  $0 \leq s < t \leq 1$ ,

$$t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} = (1-s)^{\alpha-1} \left[ \frac{t^{\alpha-1}}{1-s} - \left( \frac{t-s}{1-s} \right)^{\alpha-1} \right] \geq 0. \quad (2.14)$$

So, by (2.4), we have

$$G_1(t, s) \geq 0, \quad \forall t, s \in [0, 1]. \quad (2.15)$$

Similarly, for  $t, s \in (0, 1)$ , we have  $G_1(t, s) > 0$ .

(ii) Since  $\alpha \leq 2$ , for given  $s \in (0, 1)$ ,  $s < t \leq 1$ , we have

$$t \geq \frac{t-s}{1-s}, \quad (2.16)$$

$$t^{\alpha-2} \leq \left( \frac{t-s}{1-s} \right)^{\alpha-2}. \quad (2.17)$$

Therefore, from (2.17) and the definition of  $G_1(t, s)$ , for given  $s \in (0, 1)$ ,  $s < t \leq 1$ , we have

$$\begin{aligned} \frac{\partial G_1(t, s)}{\partial t} &= \frac{\alpha-1}{\Gamma(\alpha)} \left\{ t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2} \right\} \\ &= \frac{\alpha-1}{\Gamma(\alpha)} (1-s)^{\alpha-2} \left\{ t^{\alpha-2} - \left( \frac{t-s}{1-s} \right)^{\alpha-2} \right\} \\ &\leq 0. \end{aligned} \quad (2.18)$$

On the other hand, it is clear that

$$\frac{\partial G_1(t, s)}{\partial t} = \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \geq 0, \quad \forall 0 \leq t \leq s \leq 1. \quad (2.19)$$

Therefore, we have

$$\max_{t \in [0, 1]} G_1(t, s) = G_1(s, s) = \frac{s^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, \quad s \in (0, 1). \quad (2.20)$$

□

Let

$$\mu = \int_0^1 h(t)t^{\alpha-1} dt. \quad (2.21)$$

**Proposition 2.3.** *If  $\mu \in [0, \alpha - 1)$ , then one has*

- (i)  $G_2(t, s) \geq 0$  is continuous for all  $t, s \in [0, 1]$ ,  $G_2(t, s) > 0$ , for all  $t, s \in (0, 1)$ ;
- (ii)  $G_2(t, s) \leq (1/(\alpha - 1 - \mu)) \int_0^1 h(t)G_1(t, s)dt$ , for all  $t \in [0, 1]$ ,  $s \in (0, 1)$ .

*Proof.* From the properties of  $G_1(t, s)$ , and the definition of  $G_2(t, s)$ , we can prove that the results of Proposition 2.3 hold. □

**Theorem 2.4.** *If  $\mu \in [0, \alpha - 1)$ , the function  $G(t, s)$  defined by (2.3) satisfies the following.*

- (i)  $G(t, s) \geq 0$  is continuous for all  $t, s \in [0, 1]$ ,  $G(t, s) > 0$ , for all  $t, s \in (0, 1)$ ;
- (ii)  $G(t, s) \leq \Lambda s^{\alpha-1}(1 - s)^{\alpha-2}$ , for all  $t, s \in [0, 1]$ , where

$$\Lambda = \frac{\alpha - 1 - \mu - \int_0^1 h(t)dt}{\Gamma(\alpha)(\alpha - 1 - \mu)}. \tag{2.22}$$

*Proof.* (i) From Propositions 2.2 and 2.3, we obtain that  $G(t, s) \geq 0$  is continuous for all  $t, s \in [0, 1]$ ,  $G(t, s) > 0$ , for all  $t, s \in (0, 1)$ .

(ii) From Proposition 2.2 and (2.3), we have

$$\begin{aligned} G(t, s) &= G_1(t, s) + G_2(t, s) \\ &\leq G_1(s, s) + \frac{t^{\alpha-1}}{\alpha - 1 - \mu} \int_0^1 h(t)G_1(s, s)dt \\ &\leq G_1(s, s) \left( 1 + \frac{1}{\alpha - 1 - \mu} \int_0^1 h(t)dt \right) \\ &\leq G_1(s, s) \frac{\alpha - 1 - \mu + \int_0^1 h(t)dt}{\alpha - 1 - \mu} \\ &\leq \frac{s^{\alpha-1}(1 - s)^{\alpha-2}}{\Gamma(\alpha)} \frac{\alpha - 1 - \mu - \int_0^1 h(t)dt}{\alpha - 1 - \mu} \\ &= \Lambda s^{\alpha-1}(1 - s)^{\alpha-2}, \quad \forall t, s \in [0, 1]. \end{aligned} \tag{2.23}$$

□

*Remark 2.5.* From (i) of Theorem 2.4, we obtain that there exists  $\tau > 0$  such that

$$G(t, s) \geq \tau, \quad \forall t, s \in [\theta, 1 - \theta], \tag{2.24}$$

where  $\theta \in (0, 1/2)$ .

### 3. Preliminaries

In this section, we give some preliminaries for discussing the existence of positive solutions of boundary value problem (1.5).

Let  $J = [0, 1]$ . The basic space used in this paper is  $E = C[0, 1]$ . It is well known that  $E$  is a real Banach space with the norm  $\|\cdot\|$  defined by  $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$ . Let

$$\begin{aligned} K &= \{x \in E : x(t) \geq 0, t \in J\}, \\ K_r &= \{x \in K : \|x\| \leq r\}, \quad \partial K_r = \{x \in K : \|x\| = r\}, \end{aligned} \quad (3.1)$$

where  $r > 0$ .

On the basis of Lemma 3.3 below we will establish in Section 4 the existence of positive solution to the problem (1.5). Here we make the following hypotheses:

- (H<sub>1</sub>)  $g \in C((0, 1), [0, +\infty))$ ,  $g(t) \neq 0$  on any subinterval of  $(0, 1)$  and  $\int_0^1 s^{\alpha-1}(1-s)^{\alpha-2} g(t) dt < +\infty$ ;
- (H<sub>2</sub>)  $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$  and  $f(t, 0) = 0$  uniformly with respect to  $t$  on  $[0, 1]$ ;
- (H<sub>3</sub>)  $\mu \in [0, \alpha - 1)$ , where  $\mu$  is defined by (2.21).

Define  $T : K \rightarrow K$  by

$$(Tx)(t) = \int_0^1 G(t, s) g(s) f(s, x(s)) ds, \quad (3.2)$$

where  $G(t, s)$  is defined by (2.3).

**Lemma 3.1.** *Let (H<sub>1</sub>)–(H<sub>3</sub>) hold. Then boundary value problems (1.5) has a solution  $x$  if and only if  $x$  is a fixed point of  $T$ .*

*Proof.* From Lemma 2.1, we can prove the results of this Lemma. □

**Lemma 3.2.** *Let (H<sub>1</sub>)–(H<sub>3</sub>) hold. Then  $TK \subset K$  and  $T : K \rightarrow K$  is completely continuous.*

*Proof.* For any  $x \in K$ , by (3.2), we can obtain  $Tx \geq 0$ . Next by similar proof of Lemma 3.1 in [12] and Ascoli-Arzelà theorem one can prove  $T : K \rightarrow K$  is completely continuous. So it is omitted. □

**Lemma 3.3** (see [26]). *Let  $\Omega_1$  and  $\Omega_2$  be two bounded open sets in a real Banach space  $E$ , such that  $0 \in \Omega_1$  and  $\overline{\Omega_1} \subset \Omega_2$ . Let operator  $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$  be completely continuous, where  $P$  is a cone in  $E$ . Suppose that one of the following two conditions is satisfied.*

- (i) *There exists  $x_0 \in P \setminus \{\theta\}$  such that  $x - Ax \neq tx_0$ , for all  $x \in P \cap \partial\Omega_2$ ,  $t \geq 0$ ;  $Ax \neq \mu x$ , for all  $x \in P \cap \partial\Omega_1$ ,  $\mu \geq 1$ .*
- (ii) *There exists  $x_0 \in P \setminus \{\theta\}$  such that  $x - Ax \neq tx_0$ , for all  $x \in P \cap \partial\Omega_1$ ,  $t \geq 0$ ;  $Ax \neq \mu x$ , for all  $x \in P \cap \partial\Omega_2$ ,  $\mu \geq 1$ .*

*Then,  $A$  has at least one fixed point in  $P \cap (\Omega_2 \setminus \overline{\Omega_1})$ .*

#### 4. Existence of Positive Solutions

In this section, we apply Lemma 3.3 to establish the existence of positive solutions for boundary value problems (1.5).

**Theorem 4.1.** *Suppose (H<sub>1</sub>)–(H<sub>3</sub>) and f satisfies the following conditions.*

(H<sub>4</sub>) *There exists 0 < δ < 1 such that 0 < lim inf<sub>x→0+</sub> min<sub>t∈[0,1]</sub>(f(t, x)/x<sup>δ</sup>) ≤ +∞;*

(H<sub>5</sub>) *There exists 0 < β < 1 such that 0 ≤ lim sup<sub>x→+∞</sub> max<sub>t∈[0,1]</sub>(f(t, x)/x<sup>β</sup>) < +∞.*

*Then boundary value problems (1.5) has at least one positive solution.*

*Proof.* For applying Lemma 3.3, we construct a function  $w : [0, 1] \rightarrow R$  via

$$w(t) = \begin{cases} 1, & t \in [\theta, 1 - \theta], \\ 0, & t \notin \left[\frac{\theta}{16}, 1 - \frac{15\theta}{16}\right], \\ \frac{16}{15\theta} \left(t - \frac{\theta}{16}\right), & t \in \left[\frac{\theta}{16}, \theta\right], \\ -\frac{16}{\theta} \left(t - 1 + \frac{15\theta}{16}\right), & t \in \left[1 - \theta, 1 - \frac{15\theta}{16}\right]. \end{cases} \tag{4.1}$$

Obviously,  $w$  is a nonnegative continuous function, that is,  $w \in K$ , and  $\|w\| = 1$ .

Suppose that there is  $\varepsilon_1 > 0$  such that

$$x - Tx \neq 0 \quad (\forall x \in K, 0 < \|x\| \leq \varepsilon_1), \tag{4.2}$$

if not, then the conclusion holds. The condition (H<sub>4</sub>) and  $f(t, 0) = 0$  imply that there exist  $\sigma > 0, \varepsilon_2 > 0$  such that

$$f(t, x) \geq \sigma x^\delta \quad (0 \leq x \leq \varepsilon_2). \tag{4.3}$$

Let  $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2, (\tau\sigma \int_\theta^{1-\theta} g(s)ds)^{1/(1-\delta)}\}$ , and choose  $0 < r < \varepsilon_3$ . We now show that

$$x - Tx \neq \zeta w \quad (\forall x \in \partial K_r, \zeta \geq 0). \tag{4.4}$$

In fact, if there exist  $x_1 \in \partial K_r, \zeta_1 \geq 0$  such that  $x_1 - Tx_1 = \zeta_1 w$ , then (4.4) implies that  $\zeta_1 > 0$ . On the other hand,  $x_1 = \zeta_1 w + Tx_1 \geq \zeta_1 w$ . So we can choose  $\zeta^* = \sup\{\zeta \mid x_1 \geq \zeta w\}$ , then  $\zeta_1 \leq \zeta^* < +\infty, x_1 \geq \zeta^* w$ . Therefore,

$$\zeta^* = \zeta^* \|w\| \leq \|x_1\| = r < \varepsilon_3 \leq \left(\tau\sigma \int_\theta^{1-\theta} g(s)ds\right)^{1/(1-\delta)}. \tag{4.5}$$

Consequently, for any  $t \in [\theta, 1 - \theta]$ , (2.24) and (4.3) imply

$$\begin{aligned}
 x_1(t) &= \int_0^1 G(t,s)g(s)f(s,x_1(s))ds + \zeta_1 w(t) \\
 &\geq \int_0^1 G(t,s)g(s)\sigma[x_1(s)]^\delta ds + \zeta_1 w(t) \\
 &\geq \int_\theta^{1-\theta} G(t,s)g(s)\sigma(\zeta^*)^\delta [w(s)]^\delta ds + \zeta_1 w(t) \\
 &\geq \tau\sigma(\zeta^*)^\delta \int_\theta^{1-\theta} g(s)ds + \zeta_1 w(t) \\
 &\geq (\zeta^* + \zeta_1)w(t),
 \end{aligned} \tag{4.6}$$

that is,  $x_1(t) \geq (\zeta^* + \zeta_1)w(t)$ ,  $t \in [\theta, 1 - \theta]$ . Noticing the definition of  $w(t)$ , we have

$$x_1(t) \geq (\zeta^* + \zeta_1)w(t), \quad t \in [0, 1], \tag{4.7}$$

which is a contradiction to the definition of  $\zeta^*$ . Hence, (4.4) holds.

Now turning to  $(H_5)$ , there exist  $m > 0$ ,  $\varepsilon_4 > 0$ , for  $t \in [0, 1]$ ,  $x \geq \varepsilon_4$ , such that  $f(t, x) \leq mx^\beta$ . Letting  $l = \max_{0 \leq t \leq 1, 0 \leq x \leq \varepsilon_4} f(t, x)$ , then

$$0 \leq f(t, x) \leq mx^\beta + l, \quad \forall t \in [0, 1], x \in [0, \infty). \tag{4.8}$$

Choosing  $R > \varepsilon_4$  such that

$$\frac{l\Lambda M}{R} + \frac{m\Lambda M}{R^{1-\beta}} < 1, \tag{4.9}$$

where  $M = \int_0^1 s^{\alpha-1}(1-s)^{\alpha-2}g(s)ds$ . Now we prove that

$$Tx \neq \lambda x, \quad \forall x \in \partial K_R, \lambda \geq 1. \tag{4.10}$$

If not, then there exist  $x_0 \in \partial K_R$ ,  $\lambda_0 \geq 1$  such that  $Tx_0 = \lambda_0 x_0$ . By (4.8) and (ii) of Theorem 2.4, then for any  $t \in [0, 1]$ , we have

$$\begin{aligned}
 \lambda_0 x_0(t) &= \int_0^1 G(t,s)g(s)f(s,x_0(s))ds \\
 &\leq (l + m\|x_0\|^\beta)\Lambda \int_0^1 s^{\alpha-1}(1-s)^{\alpha-2}g(s)ds.
 \end{aligned} \tag{4.11}$$

So  $R \leq \lambda_0 R = \lambda_0 \|x_0\| \leq (l + m \|x_0\|^\beta) \Lambda \int_0^1 s^{\alpha-1} (1-s)^{\alpha-2} g(s) ds$ , that is,

$$\frac{l\Lambda M}{R} + \frac{m\Lambda M}{R^{1-\beta}} \geq 1, \tag{4.12}$$

which is a contradiction to (4.9). So, (4.10) holds.

By (ii) of Lemma 3.3, (4.4) and (4.10) yield that  $T$  has a fixed point  $x \in K_{r,R}$ ,  $r \leq \|x\| \leq R$ . Thus it follows that boundary value problems (1.5) has at least one positive solution  $x$  with  $r \leq \|x\| \leq R$ . The proof is complete.  $\square$

### 5. Discussion

In this section, we offer some interesting discussion associated with boundary value problems (1.5).

Since the proof of the main theorem (Theorem 4.1) in this paper is independent of the expression form of  $G(t, s)$  and only dependent on its continuity and nonnegativity, there are similar conclusions by analogous methods for boundary value problems (1.5) subject to other boundary value conditions, respectively, the following.

(i) We have

$$x(0) = 0, \quad x'(1) = \int_0^1 h(t)x'(t)dt, \tag{5.1}$$

then

$$G(t, s) = G_1(t, s) + G_2^*(t, s), \tag{5.2}$$

where

$$G_2^*(t, s) = \frac{t^{\alpha-1}}{(\alpha-1)\left(1 - \int_0^1 h(t)t^{\alpha-2}dt\right)} \int_0^1 h(t)G'_{1t}(t, s)dt,$$

$$G'_{1t}(t, s) = \begin{cases} \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5.3}$$

Obviously  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ , and it is easy to see that  $G(t, s) \geq 0$ ,  $t, s \in [0, 1]$  by  $\mu^* \in [0, 1)$ , where

$$\mu^* = \int_0^1 h(t)t^{\alpha-2}dt. \tag{5.4}$$

(ii) We have

$$x(0) = 0, \quad x(1) + x'(1) = \int_0^1 h(t)x(t)dt, \quad (5.5)$$

then

$$G(t, s) = G_1^*(t, s) + G_2^{**}(t, s), \quad (5.6)$$

where

$$G_1^*(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2} - \alpha(t-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2}}{\alpha\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (5.7)$$

$$G_2^{**}(t, s) = \frac{t^{\alpha-1}}{\left(\alpha - \int_0^1 h(t)t^{\alpha-1}dt\right)} \int_0^1 h(t)G_1^*(t, s)dt.$$

Obviously  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ , and it is easy to see that  $G(t, s) \geq 0$ ,  $t, s \in [0, 1]$  by  $\mu \in [0, \alpha)$ , where  $\mu$  is defined by (2.21).

(iii) We have

$$x(0) = 0, \quad x(1) + x'(1) = \int_0^1 h(t)x'(t)dt, \quad (5.8)$$

then

$$G(t, s) = G_1^*(t, s) + G_2^{***}(t, s), \quad (5.9)$$

where

$$G_2^{***}(t, s) = \frac{t^{\alpha-1}}{(\alpha-1)\left(\alpha/(\alpha-1) - \int_0^1 h(t)t^{\alpha-2}dt\right)} \int_0^1 h(t)G_{1t}^{*'}(t, s)dt,$$

$$G_{1t}^{*'}(t, s) = \begin{cases} \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1} - (\alpha-1)\alpha(t-s)^{\alpha-2}}{\alpha\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (5.10)$$

Obviously  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ , and it is easy to see that  $G(t, s) \geq 0$ ,  $t, s \in [0, 1]$  by  $\mu^* \in [0, \alpha/(\alpha-1))$ , where  $\mu^*$  is defined in (5.4).

## 6. Conclusions

In this paper, by using the fixed point theorem of cone, we have investigated the existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and have obtained some easily verifiable sufficient criteria which extend previous results. It is worth mentioning that there are still many problems that remain open in this vital field other than the results obtained in this paper: for example, whether or not we can study the fractional differential equations with integral boundary conditions at resonance (see, e.g., [27]), and whether or not we can give a unified approach applicable to many BVPs (see, e.g., [28–31]). More efforts are still needed in the future.

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