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Research Article

Nonlocal Boundary Value Problem for Impulsive Differential Equations of Fractional Order

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We study a nonlocal boundary value problem of impulsive fractional differential equations. By means of a fixed point theorem due to O'Regan, we establish sufficient conditions for the existence of at least one solution of the problem. For the illustration of the main result, an example is given.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in various fields, such as physics, mechanics, aerodynamics, chemistry, and engineering and biological sciences, involves derivatives of fractional order. Fractional differential equations also provide an excellent tool for the description of memory and hereditary properties of many materials and processes. In consequence, fractional differential equations have emerged as a significant development in recent years, see [1–3].

As one of the important topics in the research differential equations, the boundary value problem has attained a great deal of attention from many researchers, see [4–11] and the references therein. As pointed out in [12], the nonlocal boundary condition can be more useful than the standard condition to describe some physical phenomena. There are three noteworthy papers dealing with the nonlocal boundary value problem of fractional differential equations. Benchohra et al. [12] investigated the following nonlocal boundary value problem

$$\begin{gathered} {}^cD^\alpha u(t) + f(t,u(t)) = 0, \quad 0 < t < T,\ 1 < \alpha \le 2, \\ u(0) = g(u), \qquad u(T) = u_T, \end{gathered} \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo's fractional derivative.

Zhong and Lin [13] studied the following nonlocal and multiple-point boundary value problem

$$\begin{gathered} {}^{c}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \; 1 < \alpha \leq 2, \\ u(0) = u_0 + g(u), \qquad u'(1) = u_1 + \sum_{i=1}^{m-2} b_i u'(\xi_i). \end{gathered} \tag{1.2}$$

Ahmad and Sivasundaram [14] studied a class of four-point nonlocal boundary value problem of nonlinear integrodifferential equations of fractional order by applying some fixed point theorems.

On the other hand, impulsive differential equations of fractional order play an important role in theory and applications, see the references [15–21] and references therein. However, as pointed out in [15, 16], the theory of boundary value problems for nonlinear impulsive fractional differential equations is still in the initial stages. Ahmad and Sivasundaram [15, 16] studied the following impulsive hybrid boundary value problems for fractional differential equations, respectively,

$$\begin{gathered} {}^{c}D^{q}u(t) + f(t,u(t)) = 0, \quad 1 < q \leq 2, \; t \in J_1 = [0,1] \setminus \{t_1, t_2, \dots, t_p\}, \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \quad t_k \in (0,1), \; k = 1,2,\dots,p, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{gathered} \tag{1.3}$$

$$\begin{gathered} {}^{c}D^{q}u(t) + f(t,u(t)) = 0, \quad 1 < q \leq 2, \; t \in J_1 = [0,1] \setminus \{t_1, t_2, \dots, t_p\}, \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \quad t_k \in (0,1), \; k = 1,2,\dots,p, \\ \alpha u(0) + \beta u'(0) = \int_0^1 q_1(u(s))ds, \qquad \alpha u(1) + \beta u'(1) = \int_0^1 q_2(u(s))ds. \end{gathered} \tag{1.4}$$

Motivated by the facts mentioned above, in this paper, we consider the following problem:

$$\begin{gathered} {}^{c}D^{q}u(t) = f\big(t,u(t),u'(t)\big), \quad 1 < q \leq 2, \; t \in \mathcal{J}_1 = [0,1] \setminus \{t_1, t_2, \dots, t_p\}, \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \quad t_k \in (0,1), \; k = 1,2,\dots,p, \\ \alpha u(0) + \beta u'(0) = g_1(u), \qquad \alpha u(1) + \beta u'(1) = g_2(u), \end{gathered} \tag{1.5}$$

where $J = [0,1]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$, $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$, $k = 1,2,\dots,p$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $\alpha > 0$, $\beta \geq 0$, and $g_1, g_2 : \mathrm{PC}(J,\mathbb{R}) \to \mathbb{R}$ are two continuous functions. We will define $\mathrm{PC}(J,\mathbb{R})$ in Section 2.

To the best of our knowledge, this is the first time in the literatures that a nonlocal boundary value problem of impulsive differential equations of fractional order is considered.

In addition, the nonlinear term $f(t,u(t),u'(t))$ involves $u'(t)$. Evidently, problem (1.5) not only includes boundary value problems mentioned above but also extends them to a much wider case. Our main tools are the fixed point theorem of O'Regan. Some recent results in the literatures are generalized and significantly improved (see Remark 3.6)

The organization of this paper is as follows. In Section 2, we will give some lemmas which are essential to prove our main results. In Section 3, main results are given, and an example is presented to illustrate our main results.

2. Preliminaries

At first, we present here the necessary definitions for fractional calculus theory. These definitions and properties can be found in recent literature.

Definition 2.1 (see [1–3]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \tag{2.1}$$

where the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 (see [1–3]). The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$${}^{c}D^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{n}(s)ds, \tag{2.2}$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the number α, and the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3 (see [1–3]). *Let $\alpha > 0$, then the fractional differential equation ${}^{c}D^{q}u(t) = 0$ has solutions*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$, $n = [q]+1$.

Lemma 2.4 (see [1–3]). *Let $\alpha > 0$, then one has*

$$I_{0+}^{\alpha}\, {}^{c}D^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$, $n = [q]+1$.

Second, we define

$\mathrm{PC}(J,\mathbb{R}) = \{x : J \to \mathbb{R};\ x \in C((t_k,t_{k+1}],\mathbb{R}),\ k = 0,1,\ldots,p+1$ and $x(t_k^+), x(t_k^-)$ exist with $x(t_k^-) = x(t_k),\ k = 1,\ldots,p\}$.

$PC^1(J,\mathbb{R}) = \{x \in PC(J,\mathbb{R});\ x'(t) \in C((t_k,t_{k+1}],\mathbb{R}),\ k = 0,1,\ldots,p+1,\ x'(t_k^+),\ x'(t_k^-)$ exist, and x' is left continuous at t_k, $k = 1,\ldots,p\}$. Let $\mathcal{C} = PC^1(J,\mathbb{R})$; it is a Banach space with the norm $\|x\| = \sup_{t\in J}\{\|x(t)\|_{PC}, \|x'(t)\|_{PC}\}$, where $\|x\|_{PC} = \sup_{t\in J}|x(t)|$.

Like Definition 2.1 in [16], we give the following definition.

Definition 2.5. A function $u \in \mathcal{C}$ with its Caputo derivative of order q existing on $\mathcal{J}_1$ is a solution of (1.5) if it satisfies (1.5).

To deal with problem (1.5), we first consider the associated linear problem and give its solution.

Lemma 2.6. *Assume that*

$$J_i = \begin{cases} [t_0,t_1], & i = 0, \\ (t_i,t_{i+1}], & i = 1,2,\ldots,p, \end{cases}$$
$$\mathcal{X}(t) = \begin{cases} 0, & t \in J_0, \\ 1, & t \,\overline{\in}\, J_0. \end{cases} \tag{2.5}$$

For any $\sigma \in C[0,1]$, the solution of the problem

$$\begin{gathered} {}^cD^q u(t) = \sigma(t), \quad 1 < q \le 2,\ t \in \mathcal{J}_1 = [0,1] \setminus \{t_1,t_2,\ldots,t_p\}, \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \quad t_k \in (0,1),\ k = 1,2,\ldots,p, \\ \alpha u(0) + \beta u'(0) = g_1(u), \qquad \alpha u(1) + \beta u'(1) = g_2(u) \end{gathered} \tag{2.6}$$

is given by

$$\begin{aligned} u(t) = &\int_{t_i}^{t} \frac{(t-s)^{q-1}\sigma(s)}{\Gamma(q)}ds \\ &+ \left(\frac{\beta}{\alpha} - t\right)\Bigg[\int_{t_p}^{1} \frac{(1-s)^{q-1}\sigma(s)}{\Gamma(q)}ds + \frac{\beta}{\alpha}\int_{t_p}^{1} \frac{(1-s)^{q-2}\sigma(s)}{\Gamma(q-1)}ds \\ &\qquad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}\sigma(s)}{\Gamma(q)}ds + I_k\big(u(t_k^-)\big)\right) \\ &\qquad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha} + 1 - t_k\right)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}\sigma(s)}{\Gamma(q-1)}ds + J_k\big(u(t_k^-)\big)\right)\Bigg] \end{aligned}$$

$$+ \mathcal{X}(t) \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}\sigma(s)}{\Gamma(q)} ds + I_k(u(t_k^-)) \right)$$
$$+ \mathcal{X}(t) \sum_{0<t_k<t} (t - t_k) \left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}\sigma(s)}{\Gamma(q-1)} ds + J_k(u(t_k^-)) \right)$$
$$+ \frac{1}{\alpha^2} [\alpha g_1(u) + \mathcal{X}(t)(\alpha t - \beta)(g_2(u) - g_1(u))], \quad \textit{for } t \in J_i, \; i = 0, 1, \ldots, p. \tag{2.7}$$

Proof. By Lemmas 2.3 and 2.4, the solution of (2.6) can be written as

$$u(t) = I_{0^+}^q \sigma(t) - b_0 - b_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - b_0 - b_1 t, \quad t \in [0, t_1], \tag{2.8}$$

where $b_0, b_1 \in \mathbb{R}$. Taking into account that ${}^cD^q I_{0^+}^q u(t) = u(t)$, $I_{0^+}^q I_{0^+}^p u(t) = I_{0^+}^{p+q} u(t)$ for $p, q > 0$, we obtain

$$u'(t) = \int_0^t \frac{(t-s)^{q-2}\sigma(s)}{\Gamma(q-1)} ds - b_1. \tag{2.9}$$

Using $\alpha u(0) + \beta u'(0) = g_1(u)$, we get

$$u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + b_1 \left(\frac{\beta}{\alpha} - t \right) + \frac{1}{\alpha} g_1(u), \quad t \in [0, t_1]. \tag{2.10}$$

If $t \in (t_1, t_2]$, then we have

$$u(t) = \int_{t_1}^t \frac{(t-s)^{q-1}\sigma(s)}{\Gamma(q)} ds - c_0 - c_1(t - t_1), \tag{2.11}$$

where $c_0, c_1 \in \mathbb{R}$. In view of the impulse conditions $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1^-))$, $\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = J_1(u(t_1^-))$, we have

$$u(t) = \int_{t_1}^t \frac{(t-s)^{q-1}\sigma(s)}{\Gamma(q)} ds + \int_0^{t_1} \frac{(t_1 - s)^{q-1}\sigma(s)}{\Gamma(q)} ds + b_1 \left(\frac{\beta}{\alpha} - t \right) + \frac{1}{\alpha} g_1(u) + I_1(u(t_1^-))$$
$$+ (t - t_1) \left[\int_0^{t_1} \frac{(t_1 - s)^{q-2}\sigma(s)}{\Gamma(q-1)} ds + J_1(u(t_1^-)) \right], \quad t \in (t_1, t_2]. \tag{2.12}$$

Repeating the process in this way, the solution $u(t)$ for $t \in (t_k, t_{k+1}]$ can be written as

$$
\begin{aligned}
u(t) &= \int_{t_k}^{t} \frac{(t-s)^{q-1}\sigma(s)}{\Gamma(q)}ds + b_1\left(\frac{\beta}{\alpha} - t\right) + \frac{1}{\alpha}g_1(u) \\
&\quad + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}\sigma(s)}{\Gamma(q)}ds + I_k\left(u\left(t_k^-\right)\right)\right) \\
&\quad + \sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}\sigma(s)}{\Gamma(q-1)}ds + J_k\left(u\left(t_k^-\right)\right)\right), \quad t \in (t_k, t_{k+1}].
\end{aligned}
\tag{2.13}
$$

Applying the boundary condition $\alpha u(1) + \beta u'(1) = g_2(u)$, we find that

$$
\begin{aligned}
b_1 &= \int_{t_p}^{1} \frac{(1-s)^{q-1}\sigma(s)}{\Gamma(q)}ds \\
&\quad + \frac{\beta}{\alpha}\int_{t_p}^{1} \frac{(1-s)^{q-2}\sigma(s)}{\Gamma(q-1)}ds \\
&\quad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}\sigma(s)}{\Gamma(q)}ds + I_k\left(u\left(t_k^-\right)\right)\right) \\
&\quad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha} + 1 - t_k\right)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}\sigma(s)}{\Gamma(q-1)}ds + J_k\left(u\left(t_k^-\right)\right)\right) \\
&\quad + \frac{1}{\alpha}\left(g_1(u) - g_2(u)\right).
\end{aligned}
\tag{2.14}
$$

Substituting the value of b_1 into (2.10) and (2.13), we obtain (2.7). □

Now, we introduce the fixed point theorem which was established by O'Regan in [22]. This theorem will be applied to prove our main results in the next section.

Lemma 2.7 (see [13, 22]). *Denote by $\mathcal{U}$ an open set in a closed, convex set Υ of a Banach space E. Assume that $0 \in \mathcal{U}$. Also assume that $F(\overline{\mathcal{U}})$ is bounded and that $F : \overline{\mathcal{U}} \to \Upsilon$ is given by $F = F_1 + F_2$, in which $F_1 : \overline{\mathcal{U}} \to E$ is continuous and completely continuous and $F_2 : \overline{\mathcal{U}} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \overline{\mathcal{U}}$), then either*

(C_1) *F has a fixed point $u \in \overline{\mathcal{U}}$, or*

(C_2) *there exists a point $u \in \partial\mathcal{U}$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\overline{\mathcal{U}}, \partial\mathcal{U}$ represent the closure and boundary of $\mathcal{U}$, respectively.*

3. Main Results

In order to apply Lemma 2.7 to prove our main results, we first give F, F_1, F_2 as follows. Let $\overline{\Omega}_r = \{u \in \mathcal{C} : \|u\| \le r\},\ r > 0$,

$$
\begin{aligned}
[F_1 u](t) = {} & \int_{t_i}^{t} \frac{(t-s)^{q-1} f(s,x(s),x'(s))}{\Gamma(q)} ds \\
& + \left(\frac{\beta}{\alpha} - t\right)\Bigg[\int_{t_p}^{1} \frac{(1-s)^{q-1} f(s,x(s),x'(s))}{\Gamma(q)} ds + \frac{\beta}{\alpha}\int_{t_p}^{1} \frac{(1-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds \\
& \qquad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1} f(s,x(s),x'(s))}{\Gamma(q)} ds + I_k(u(t_k^-))\right) \\
& \qquad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha} + 1 - t_k\right)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds + J_k(u(t_k^-))\right)\Bigg] \\
& + \mathcal{X}(t)\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1} f(s,x(s),x'(s))}{\Gamma(q)} ds + I_k(u(t_k^-))\right) \\
& + \mathcal{X}(t)\sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds + J_k(u(t_k^-))\right), \\
& \qquad\qquad\qquad \text{for } t \in J_i, \quad i = 0,1,\ldots,p, \\
[F_2 u](t) = {} & \frac{1}{\alpha^2}\left[\alpha g_1(u) + \mathcal{X}(t)(\alpha t - \beta)(g_2(u) - g_1(u))\right], \quad \text{for } t \in J_i,\ i = 0,1,\ldots,p, \\
F = {} & F_1 + F_2.
\end{aligned}
\tag{3.1}
$$

Clearly, for any $t \in J_i,\ i = 0,1,\ldots,p$,

$$
\begin{aligned}
[F_1 u]'(t) = {} & \int_{t_i}^{t} \frac{(t-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds \\
& - \Bigg[\int_{t_p}^{1} \frac{(1-s)^{q-1} f(s,x(s),x'(s))}{\Gamma(q)} ds + \frac{\beta}{\alpha}\int_{t_p}^{1} \frac{(1-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds \\
& \qquad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1} f(s,x(s),x'(s))}{\Gamma(q)} ds + I_k(u(t_k^-))\right) \\
& \qquad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha} + 1 - t_k\right)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds + J_k(u(t_k^-))\right)\Bigg] \\
& + \mathcal{X}(t)\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2} f(s,x(s),x'(s))}{\Gamma(q-1)} ds + J_k(u(t_k^-))\right), \\
[F_2 u]'(t) = {} & \frac{1}{\alpha}\left[\mathcal{X}(t)(g_2(u) - g_1(u))\right].
\end{aligned}
\tag{3.2}
$$

Now, we make the following hypotheses.

(A_1) $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. There exists a nonnegative function $p(t) \in C[0,1]$ with $p(t) > 0$ on a subinterval of $[0,1]$. Also there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that $|f(t,u,v)| \le p(t)\psi(|u|)$ for any $(t,u,v) \in [0,1] \times \mathbb{R} \times \mathbb{R}$.

(A_2) There exist two positive constants l_1, l_2 such that $((\alpha+\beta)/\alpha^2)(l_1+l_2) = L < 1$. Moreover, $g_1(0) = 0$, $g_2(0) = 0$, and

$$\left|g_1(u) - g_1(v)\right| \le l_1\|u - v\|, \quad \left|g_2(u) - g_2(v)\right| \le l_2\|u - v\|, \quad \forall u, v \in \mathcal{C}. \tag{3.3}$$

(A_3) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous. There exists a positive constant M such that

$$|I_k(u)| \le M, \quad |J_k(u)| \le M, \quad k = 1, 2, \ldots, p. \tag{3.4}$$

Let

$$\begin{aligned}
H_1 &= \left(\frac{\beta}{\alpha}+1\right)Mp + \left(\frac{\beta}{\alpha}+1\right)^2 Mp + 2pM,\\
H_2 &= Mp + \left(\frac{\beta}{\alpha}+1\right)Mp + pM,\\
K_1 &= \int_0^1 \frac{(1-s)^{q-1}p(s)}{\Gamma(q)}ds\\
&\quad + \left(\frac{\beta}{\alpha}+1\right)\Bigg[\int_{t_p}^1 \frac{(1-s)^{q-1}p(s)}{\Gamma(q)}ds + \frac{\beta}{\alpha}\int_{t_p}^1 \frac{(1-s)^{q-2}p(s)}{\Gamma(q-1)}ds\\
&\qquad + \sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}p(s)}{\Gamma(q)}ds + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds\Bigg]\\
&\quad + \sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}p(s)}{\Gamma(q)}ds + \sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds,\\
K_2 &= \frac{P}{\Gamma(q)} + \Bigg[\int_{t_p}^1 \frac{(1-s)^{q-1}p(s)}{\Gamma(q)}ds + \frac{\beta}{\alpha}\int_{t_p}^1 \frac{(1-s)^{q-2}p(s)}{\Gamma(q-1)}ds\\
&\qquad + \sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}p(s)}{\Gamma(q)}ds \sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds\Bigg]\\
&\quad + \sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds,
\end{aligned} \tag{3.5}$$

where $P = \max_{s\in[0,1]} p(s)$.

Now, we state our main results.

Theorem 3.1. *Assume that* (A_1), (A_2), *and* (A_3) *are satisfied; moreover,* $\sup_{r\in(0,\infty)}(r/(H+K\psi(r)))>1/(1-L)$, *where* $H=\max\{H_1,H_2\}$, $K=\max\{K_1,K_2\}$, *then the problem* (1.5) *has at least one solution.*

Proof. The proof will be given in several steps.

Step 1. The operator $F_1:\overline{\Omega_r}\to C$ is completely continuous.

Let $M_r=\max_{s\in[0,1]}\{|f(s,x(s),x'(s))|,\ x\in\overline{\Omega_r}\}$. In fact, by (A_1), M_r can be replaced by $P\psi(r)$. For any $u\in\overline{\Omega_r}$, we have

$$\begin{aligned}
|[F_1u](t)| \le{}& \int_{t_i}^{t}\frac{(t-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds\\
&+\left(\frac{\beta}{\alpha}+1\right)\Bigg[\int_{t_p}^{1}\frac{(1-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds\\
&\qquad+\frac{\beta}{\alpha}\int_{t_p}^{1}\frac{(1-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds+\left|I_k(u(t_k^-))\right|\right)\\
&\qquad+\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1-t_k\right)\\
&\qquad\times\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds+\left|J_k(u(t_k^-))\right|\right)\Bigg]\\
&+\mathcal{X}(t)\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds+\left|I_k(u(t_k^-))\right|\right)\\
&+\mathcal{X}(t)\sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds+\left|J_k(u(t_k^-))\right|\right)\\
\le{}& M_r\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds\\
&+\left(\frac{\beta}{\alpha}+1\right)\Bigg[M_r\int_{t_p}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}ds+\frac{\beta}{\alpha}M_r\int_{t_p}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}ds\\
&\qquad+\sum_{0<t_k<1}\left(M_r\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}ds+M\right)\\
&\qquad+\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}M_r}{\Gamma(q-1)}ds+M\right)\Bigg]\\
&+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}M_r}{\Gamma(q)}ds+M\right)\\
&+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}M_r}{\Gamma(q-1)}ds+M\right)
\end{aligned}$$

$$
\begin{aligned}
&\le M_r\frac{1}{\Gamma(q+1)}+\left(\frac{\beta}{\alpha}+1\right)\left[M_r\frac{1}{\Gamma(q+1)}+\frac{\beta}{\alpha}M_r\frac{1}{\Gamma(q)}+p\left(M_r\frac{1}{\Gamma(q+1)}+M\right)\right.\\
&\qquad\left.+p\left(\frac{\beta}{\alpha}+1\right)\left(M_r\frac{1}{\Gamma(q)+M}\right)\right]\\
&\quad+p\left(M_r\frac{1}{\Gamma(q+1)}+M\right)+p\left(M_r\frac{1}{\Gamma(q)}+M\right),\quad \text{for } t\in J_i,\ i=0,1,\ldots,p,\\
\left|[F_1u]'(t)\right| &\le \int_{t_i}^{t}\frac{(t-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds\\
&\quad+\left[\int_{t_p}^{1}\frac{(1-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds+\frac{\beta}{\alpha}\int_{t_p}^{1}\frac{(1-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds+\left|I_k(u(t_k^-))\right|\right)\\
&\qquad\left.+\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1-t_k\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds+\left|J_k(u(t_k^-))\right|\right)\right]\\
&\quad+\mathcal{X}(t)\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds+\left|J_k(u(t_k^-))\right|\right)\\
&\le M_r\frac{1}{\Gamma(q)}+\left[\int_{t_p}^{1}\frac{(1-s)^{q-1}M_r}{\Gamma(q)}ds+\frac{\beta}{\alpha}\int_{t_p}^{1}\frac{(1-s)^{q-2}M_r}{\Gamma(q-1)}ds\right.\\
&\qquad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}M_r}{\Gamma(q)}ds+M\right)\\
&\qquad\left.+\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1-t_k\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}M_r}{\Gamma(q-1)}ds+M\right)\right]\\
&\quad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}M_r}{\Gamma(q-1)}ds+M\right)\\
&\le M_r\frac{1}{\Gamma(q)}+\left[M_r\frac{1}{\Gamma(q+1)}+\frac{\beta}{\alpha}M_r\frac{1}{\Gamma(q)}+p\left(M_r\frac{1}{\Gamma(q+1)}+M\right)\right.\\
&\qquad\left.+p\left(\frac{\beta}{\alpha}+1\right)\left(M_r\frac{1}{\Gamma(q)+M}\right)\right]\\
&\quad+p\left(M_r\frac{1}{\Gamma(q)}+M\right),\quad \text{for } t\in J_i,\ i=0,1,\ldots,p.
\end{aligned}
\tag{3.6}
$$

These imply that $\|[F_1u](t)\| \le B$, where B is a positive constant, that is, F_1 is uniformly bounded. In addition, for any $u \in \overline{\Omega}_r$, for all $\tau_1, \tau_2 \in J_i$, $\tau_1 < \tau_2$, we can obtain

$$
\begin{aligned}
&|[F_1u](\tau_1) - [F_1u](\tau_2)| \\
&\quad\le \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds \\
&\qquad + \int_0^{\tau_1} \frac{\left((\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\right)\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds \\
&\qquad + (\tau_2 - \tau_1)\Bigg[\int_{t_p}^{1} \frac{(1-s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds \\
&\qquad\qquad + \frac{\beta}{\alpha}\int_{t_p}^{1} \frac{(1-s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds \\
&\qquad\qquad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}\left|f(s,x(s),x'(s))\right|}{\Gamma(q)}ds + \left|I_k(u(t_k^-))\right|\right) \\
&\qquad\qquad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha} + 1 - t_k\right)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds + \left|J_k(u(t_k^-))\right|\right)\Bigg] \\
&\qquad + \sum_{0<t_k<\tau_1}(\tau_2 - \tau_1)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds + \left|J_k(u(t_k^-))\right|\right) \\
&\quad\le M_r\frac{(\tau_2 - \tau_1)^q}{\Gamma(q+1)} + M_r\left[-(\tau_2 - \tau_1)^q + (\tau_2)^q - (\tau_1)^q\right]\frac{1}{\Gamma(q+1)} \\
&\qquad + (\tau_2 - \tau_1)\Bigg[\int_{t_p}^{1} \frac{(1-s)^{q-1}M_r}{\Gamma(q)}ds + \frac{\beta}{\alpha}\int_{t_p}^{1} \frac{(1-s)^{q-2}M_r}{\Gamma(q-1)}ds \\
&\qquad\qquad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}M_r}{\Gamma(q)}ds + M\right) \\
&\qquad\qquad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha} + 1 - t_k\right)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}M_r}{\Gamma(q-1)}ds + M\right)\Bigg] \\
&\qquad + \sum_{0<t_k<\tau_1}(\tau_2 - \tau_1)\left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}M_r}{\Gamma(q-1)}ds + M\right), \\
&\left|[F_1u]'(\tau_1) - [F_1u]'(\tau_2)\right| \\
&\quad\le \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{q-2}\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds \\
&\qquad + \int_{t_i}^{\tau_1} \frac{\left((\tau_1 - s)^{q-2} - (\tau_2 - s)^{q-2}\right)\left|f(s,x(s),x'(s))\right|}{\Gamma(q-1)}ds \\
&\quad\le \frac{(\tau_2 - \tau_1)^{q-1}}{\Gamma(q)}M_r + \frac{(\tau_2 - \tau_1)^{q-1} + (\tau_1 - t_i)^{q-1} - (\tau_2 - t_i)^{q-1}}{\Gamma(q)}M_r.
\end{aligned}
\tag{3.7}
$$

Taking into account the uniform continuity of the function $t^q,\ t^{q-1}$ on $[0,1]$, we get that F_1 is equicontinuity on $\overline{\Omega}_r$. By the Lemma 5.4.1 in [23], we have $F_1(\overline{\Omega}_r)$ as relatively compact. Due to the continuity of f, I_k, J_k, it is clear that F_1 is continuous. Hence, we complete the proof of Step 1.

Step 2. $F(\overline{\mathcal{U}})$ is bounded.

From $\sup_{r\in(0,\infty)}(r/(H+K\psi(r))) > 1/(1-L)$, it follows that there exists a positive constant r_0, such that

$$\frac{r_0}{H+K\psi(r_0)} > \frac{1}{1-L}. \tag{3.8}$$

Now, we verify the validity of all the conditions in Lemma 2.6 with respect to the operator $F_1,\ F_2$, and F. Let $\Omega_{r_0} = \mathcal{U}$. From (A_2), we have

$$\begin{aligned} |[F_2u](t)| &\le \frac{1}{\alpha^2}\left[(\alpha(1-t)+\beta)\left|g_1(u)-g_1(0)\right| + \left|\alpha t-\beta\right|\left|g_2(u)-g_2(0)\right|\right] \\ &\le \frac{1}{\alpha^2}(\alpha+\beta)(l_1r_0+l_2r_0), \quad \text{for } t\in J_i, \\ \left|[F_2u]'(t)\right| &= \frac{1}{\alpha}[l_2r_0+l_1r_0], \quad \text{for } t\in J_i,\ i=0,1,\ldots,p. \end{aligned} \tag{3.9}$$

Combining with the property that $F_1(\overline{\mathcal{U}})$ is bounded (Step 1), we have F bounded on $\overline{\mathcal{U}}$. Hence, we can assume that $\|F(\overline{\mathcal{U}})\| \le G$, $G>0$ is a constant.

Step 3. F_2 is a nonlinear contraction.

Let $Y = \overline{\Omega}_{r_1},\ r_1 = \max\{G, r_0\},\ E = C$. By (A_2), we obtain $|[F_2u](t)-[F_2v](t)| \le (1/\alpha^2)[|(\alpha(1-t)+\beta)(g_1(u)-g_1(v))| + |(\alpha t-\beta)(g_2(u)-g_2(v))|] \le ((\alpha+\beta)/\alpha^2)(l_1+l_2)\|u-v\|$, and $|[F_2u]'(t)-[F_2v]'(t)| \le (1/\alpha)[l_1+l_2]\|u-v\| \le L\|u-v\|$, for $t\in J_i$. Since $L<1$, we have $\|F_2(u)-F_2(v)\| \le \phi(\|u-v\|)$, that is, F_2 is a nonlinear contraction ($\phi(z)=Lz$).

Step 4. (C_2) in Lemma 2.7 does not occur.

To this end, we perform the argument by contradiction. Suppose that (C_2) holds, then there exist $\lambda\in(0,1),\ u\in\partial\Omega_{r_0}$, such that $u=\lambda Fu$. Hence, we can obtain $\|u\|=r_0$ and

$$\begin{aligned} |u| \le &\int_{t_i}^{t}\frac{(t-s)^{q-1}p(s)\psi(r_0)}{\Gamma(q)}ds \\ &+ \left(\frac{\beta}{\alpha}+1\right)\Bigg[\int_{t_p}^{1}\frac{(1-s)^{q-1}p(s)\psi(r_0)}{\Gamma(q)}ds + \frac{\beta}{\alpha}\int_{t_p}^{1}\frac{(1-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds \\ &\qquad + \sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}p(s)\psi(r_0)}{\Gamma(q)}ds + M\right) \\ &\qquad + \sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1-t_k\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds + M\right)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&+\mathcal{X}(t)\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}p(s)\psi(r_0)}{\Gamma(q)}ds+M\right)\\
&+\mathcal{X}(t)\sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds+M\right)\\
&+\frac{1}{\alpha^2}(\alpha+\beta)(l_1r_0+l_2r_0)\\
\le{}&\frac{1}{\alpha^2}(\alpha+\beta)(l_1+l_2)r_0+\left[\left(\frac{\beta}{\alpha}+1\right)Mp+\left(\frac{\beta}{\alpha}+1\right)^2Mp+2pM\right]\\
&+\psi(r_0)\left\{\int_0^1\frac{(1-s)^{q-1}p(s)}{\Gamma(q)}ds\right.\\
&\qquad+\left(\frac{\beta}{\alpha}+1\right)\left[\int_{t_p}^1\frac{(1-s)^{q-1}p(s)}{\Gamma(q)}ds+\frac{\beta}{\alpha}\int_{t_p}^1\frac{(1-s)^{q-2}p(s)}{\Gamma(q-1)}ds\right.\\
&\qquad\qquad+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}p(s)}{\Gamma(q)}ds\\
&\qquad\qquad\left.+\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds\right]\\
&\qquad\left.+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}p(s)}{\Gamma(q)}ds+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds\right\}\\
\le{}&Lr_0+H_1+K_1\psi(r_0),
\end{aligned}
$$

$$
\begin{aligned}
|u'|\le{}&\int_{t_i}^t\frac{(t-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds\\
&+\left[\int_{t_p}^1\frac{(1-s)^{q-1}p(s)\psi(r_0)}{\Gamma(q)}ds+\frac{\beta}{\alpha}\int_{t_p}^1\frac{(1-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds\right.\\
&\quad+\sum_{0<t_k<1}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}p(s)\psi(r_0)}{\Gamma(q)}ds+M\right)\\
&\quad\left.+\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1-t_k\right)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds+M\right)\right]\\
&+\mathcal{X}(t)\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)\psi(r_0)}{\Gamma(q-1)}ds+M\right)+\frac{1}{\alpha}(l_1r_0+l_2r_0)\\
\le{}&\frac{1}{\alpha^2}(\alpha+\beta)(l_1+l_2)r_0+\left[Mp+\left(\frac{\beta}{\alpha}+1\right)Mp+pM\right]
\end{aligned}
$$

$$\begin{aligned}&+\psi(r_0)\left\{\frac{P}{\Gamma(q)}+\left[\int_{t_p}^{1}\frac{(1-s)^{q-1}p(s)}{\Gamma(q)}ds+\frac{\beta}{\alpha}\int_{t_p}^{1}\frac{(1-s)^{q-2}p(s)}{\Gamma(q-1)}ds\right.\right.\\&\qquad+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}p(s)}{\Gamma(q)}ds\\&\qquad\left.\times\sum_{0<t_k<1}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds\right]\\&\left.\quad+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}p(s)}{\Gamma(q-1)}ds\right\}\\&\le Lr_0+H_2+K_2\psi(r_0).\end{aligned}\tag{3.10}$$

Therefore, $r_0 \le Lr_0 + H + K\psi(r_0)$. However, it contradicts with (3.8).

Hence, by using Steps 1–4, Lemmas 2.6 and 2.7, F has at least one fixed point $u \in \overline{\Omega}_{r_0}$, which is the solution of problem (1.5). □

Next, we will give some corollaries.

Corollary 3.2. *Assume that* $(A_1), (A_2)$, *and* (A_3) *are satisfied; moreover,* $\limsup_{r\in(0,\infty)}(r/(H + K\psi(r))) = +\infty$, *where* $H = \max\{H_1, H_2\},\ K = \max\{K_1, K_2\}$*; then the problem* (1.5) *has at least one solution.*

Assume that,

(A_1') (sublinear growth), $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous. There exists a nonnegative function $p(t) \in C[0,1]$ with $p(t) > 0$ on a subinterval of $[0,1]$. Also there exists a constant $\gamma \in [0,1)$, such that $|f(t,u,v)| \le p(t)|u|^{\gamma}$ for any $(t,u,v) \in [0,1]\times\mathbb{R}\times\mathbb{R}$.

Corollary 3.3. *Assume that* $(A_1'), (A_2)$, *and* (A_3) *are satisfied, then the problem* (1.5) *has at least one solution.*

Assume that

(B_1) $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous. There exists a nonnegative function $p(t) \in C[0,1]$ with $p(t) > 0$ on a subinterval of $[0,1]$. Also there exists a constant $\gamma \in [0,1)$ such that $|f(t,u)| \le p(t)|u|^{\gamma}$ for any $(t,u) \in [0,1]\times\mathbb{R}$,

(B_2) there exist two positive constants l_1, l_2 such that $((\alpha+\beta)/\alpha^2)(l_1+l_2) = L < 1$. Moreover, $q_1(0) = 0,\ q_2(0) = 0$, and

$$\left|q_1(u)-q_1(v)\right| \le l_1\|u-v\|,\quad \left|q_2(u)-q_2(v)\right| \le l_2\|u-v\|,\quad \forall u,v \in \mathcal{C},\tag{3.11}$$

(B_3) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous. There exists a positive constant M, such that

$$|I_k(u)| \le M,\quad |J_k(u)| \le M,\quad k = 1,2,\ldots,p.\tag{3.12}$$

Corollary 3.4. *Assume that* $(B_1), (B_2)$, *and* (B_3) *are satisfied, then the problem* (1.4) *has at least one solution.*

Assume that

(B_1') $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. There exists a nonnegative function $p(t) \in C[0,1]$ with $p(t) > 0$ on a subinterval of $[0,1]$. $|f(t,u)| \le p(t)$ for any $(t,u) \in [0,1] \times \mathbb{R}$.

Corollary 3.5. *Assume that* $(B_1'), (B_2)$, *and* (B_3) *are satisfied, then the problem* (1.4) *has at least one solution.*

Remark 3.6. Compared with Theorem 3.2 in [16], Corollary 3.5 does not need conditions $\|f(t,u) - f(t,v)\| \le L_1\|u - v\|$, $\|I_k(u) - I_k(V)\| \le L_2\|u - v\|$, and $\|J_k(u) - J_k(V)\| \le L_2\|u - v\|$. Moreover, we only need $((\alpha + \beta)/\alpha^2)(l_1 + l_2) = L < 1$.

Example 3.7. Consider the following problem:

$$\begin{gathered} {}^cD^{3/2}u(t) = \theta u^2 \sin^2(u'(t)), \quad 0 < t < 1,\ t \in \mathcal{J}_1 = [0,1] \setminus \left\{\frac{1}{2}\right\}, \\ \Delta u\left(\frac{1}{2}\right) = \frac{\|u\|^2}{1 + \|u\|^2}, \qquad \Delta u'\left(\frac{1}{2}\right) = \frac{\|u\|^2}{2 + \|u\|^2}, \\ u(0) + u'(0) = \int_0^1 \frac{|u(s)|}{8 + |u(s)|} ds, \qquad u(1) + u'(1) = \int_0^1 \frac{|u(s)|}{8 + |u(s)|} ds, \end{gathered} \tag{3.13}$$

where $\theta > 0$. Here, $\alpha = \beta = 1,\ p = 1,\ q = 3/2$. Let $p(s) \equiv \theta = P$ and $\psi(u) = u^2$, then we can see that (A_1) holds. Choosing $l_1 = l_2 = 1/8,\ L = 1/2$, we can easily obtain that (A_2) holds. Let $M = 1$, then we have that (A_3) also holds. Moreover, $H = 8,\ K = ((4 + 26\sqrt{2})/3\sqrt{\pi})\theta$. Hence, we get $\sup_{r\in(0,\infty)}(r/(H + K\psi(r))) = 1/2\sqrt{(8\theta(4 + 26\sqrt{2}))/3\sqrt{\pi}} > 1/(1 - L) = 2$ for any given $0 < \theta < 3\sqrt{\pi}/(128(4 + 26\sqrt{2}))$. Therefore, By Theorem 3.1, the above problem (3.13) has at least one solution for $0 < \theta < 3\sqrt{\pi}/(128(4 + 26\sqrt{2}))$.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2006.

[2] "The fractional calculus and its applications," in *Lecture Notes in Mathematics*, B. Ross, Ed., vol. 475, Springer, Berlin, Germany, 1975.

[3] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.

[4] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[5] D. Jiang and C. Yuan, "The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 2, pp. 710–719, 2010.

[6] M. Benchohra, J. R. Graef, and S. Hamani, "Existence results for boundary value problems with non-linear fractional differential equations," *Applicable Analysis*, vol. 87, no. 7, pp. 851–863, 2008.

[7] V. D. Gejji, "Positive solutions of a system of non-autonomous fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 1, pp. 56–64, 2005.

[8] E. R. Kaufmann and E. Mboumi, “Positive solutions of a boundary value problem for a nonlinear fractional differential equation,” *Electronic Journal of Qualitative Theory of Differential Equations*, no. 3, p. 11, 2008.
[9] S. Zhang, “Positive solutions for boundary-value problems of nonlinear fractional differential equations,” *Electronic Journal of Differential Equations*, no. 36, p. 12, 2006.
[10] Z. Bai, “On positive solutions of a nonlocal fractional boundary value problem,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.
[11] C. F. Li, X. N. Luo, and Y. Zhou, “Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations,” *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.
[12] M. Benchohra, S. Hamani, and S. K. Ntouyas, “Boundary value problems for differential equations with fractional order and nonlocal conditions,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2391–2396, 2009.
[13] W. Zhong and W. Lin, “Nonlocal and multiple-point boundary value problem for fractional differential equations,” *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1345–1351, 2010.
[14] B. Ahmad and S. Sivasundaram, “On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order,” *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 480–487, 2010.
[15] B. Ahmad and S. Sivasundaram, “Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations,” *Nonlinear Analysis. Hybrid Systems*, vol. 3, no. 3, pp. 251–258, 2009.
[16] B. Ahmad and S. Sivasundaram, “Existence of solutions for impulsive integral boundary value problems of fractional order,” *Nonlinear Analysis. Hybrid Systems*, vol. 4, no. 1, pp. 134–141, 2010.
[17] X. Zhang, X. Huang, and Z. Liu, “The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay,” *Nonlinear Analysis. Hybrid Systems*, vol. 4, no. 4, pp. 775–781, 2010.
[18] Y. Tian and Z. Bai, “Existence results for the three-point impulsive boundary value problem involving fractional differential equations,” *Computers & Mathematics with Applications*, vol. 59, no. 8, pp. 2601–2609, 2010.
[19] R. P. Agarwal and B. Ahmad, “Existence ofsolutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations,” to appear in *Dynamics of Continuous Discrete and Impulsive Systems. Series A*.
[20] S. Abbas and M. Benchohra, “Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order,” *Nonlinear Analysis. Hybrid Systems*, vol. 4, no. 3, pp. 406–413, 2010.
[21] G. Wang, B. Ahmad, and L. Zhang, “Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order,” *Nonlinear Analysis. Theory, Methods and Applications*, vol. 74, no. 3, pp. 792–804, 2011.
[22] D. O’Regan, “Fixed-point theory for the sum of two operators,” *Applied Mathematics Letters*, vol. 9, no. 1, pp. 1–8, 1996.
[23] D. Guo, J. Sun, and Z. Liu, *Functional Methods of Nonlinear Ordinary Differential Equation*, Shandong Science and Technology Press, 2005.