

Research Article

Symmetric Three-Term Recurrence Equations and Their Symplectic Structure

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We revive the study of the symmetric three-term recurrence equations. Our main result shows that these equations have a natural symplectic structure, that is, every symmetric three-term recurrence equation is a special discrete symplectic system. The assumptions on the coefficients in this paper are weaker and more natural than those in the current literature. In addition, our result implies that symmetric three-term recurrence equations are completely equivalent with Jacobi difference equations arising in the discrete calculus of variations. Presented applications of this study include the Riccati equation and inequality, detailed Sturmian separation and comparison theorems, and the eigenvalue theory for these three-term recurrence and Jacobi equations.

1. Introduction

In this paper, we consider the symmetric three-term recurrence equation

$$S_{k+1}x_{k+2} - T_{k+1}x_{k+1} + S_k^T x_k = 0, \quad k \in [0, N-1]_{\mathbb{Z}}, \quad (\text{T})$$

where $x_k \in \mathbb{R}^n$ for $k \in [0, N+1]_{\mathbb{Z}}$, the real $n \times n$ matrices S_k and T_k are defined on $[0, N]_{\mathbb{Z}}$ with T_k being symmetric and S_k being invertible. The discrete intervals are defined by $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$. Traditionally, the recurrence equation (T) is studied in the literature; see, for example, [1, Chapter 5] or [2–4], as a generalization of the Jacobi difference equation

$$\Delta \left(R_k \Delta x_k + Q_k^T x_{k+1} \right) = P_k x_{k+1} + Q_k \Delta x_k, \quad k \in [0, N-1]_{\mathbb{Z}}, \quad (\text{J})$$

where $\Delta x_k := x_{k+1} - x_k$ is the forward difference and where the matrices $P_k, Q_k, R_k \in \mathbb{R}^{n \times n}$ for $k \in [0, N]_{\mathbb{Z}}$ with P_k and R_k being symmetric. Jacobi equation (J) arises in the discrete calculus of variations as the Euler equation for the second variation; see, for example, [1, Section 4.2] or [5]. When the forward differences in (J) are expanded, then (J) becomes the three-term recurrence equation (T) in which the matrices $S_k := R_k + Q_k^T$ are invertible for all $k \in [0, N]_{\mathbb{Z}}$ and $T_k := R_k + R_{k-1} + Q_{k-1} + Q_{k-1}^T + P_{k-1}$ are symmetric; see [1, Section 3.6] or Proposition 2.4. In the same reference, it is shown that Jacobi difference equations (J) and recurrence equations (T) can be embedded into discrete symplectic systems (see Section 2 for the details)

$$x_{k+1} = \mathbb{A}_k x_k + \mathbb{B}_k u_k, \quad u_{k+1} = \mathbb{C}_k x_k + \mathbb{D}_k u_k, \quad k \in [0, N]_{\mathbb{Z}}, \quad (\mathbb{S})$$

where for $k \in [0, N]_{\mathbb{Z}}$

$$\mathbb{S}_k^T \mathcal{J} \mathbb{S}_k = \mathcal{J}, \quad \mathbb{S}_k := \begin{pmatrix} \mathbb{A}_k & \mathbb{B}_k \\ \mathbb{C}_k & \mathbb{D}_k \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (1.1)$$

that is, the $2n \times 2n$ matrices \mathbb{S}_k are symplectic. However, the transition from (J) and (T) into (S) in reference [1] requires that *both* S_k and R_k be invertible. This invertibility assumption essentially means that these equations are first transformed into a linear Hamiltonian system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \quad k \in [0, N]_{\mathbb{Z}}, \quad (\mathbb{H})$$

for which it is required that $I - A_k = R_k^{-1} S_k$ be invertible so that the solutions of (H) exist in the backward time. And then the linear Hamiltonian system (H) is written as the symplectic system (S).

Recently in [6], the authors proposed to study the Jacobi equations (J) as discrete symplectic systems (S) in a direct way which bypasses the Hamiltonian system (H). This new approach requires that only the matrices S_k be invertible while the matrices R_k are allowed to be singular, which yields more general results for (J) obtained, for example, through the theory of symplectic systems (S). In the present paper, we continue in this direction and we show that the three-term recurrence equations (T) naturally possess a symplectic structure (Theorem 3.1 and Corollary 3.2). More precisely, we show that symmetric three-term recurrences (T) and Jacobi equations (J) and symplectic systems (S) (with \mathbb{B}_k invertible) are completely equivalent. Therefore, the general theory of discrete symplectic systems recently developed, for example, in [7–18] can be applied to obtain, in particular, the Riccati equations and inequalities, and the oscillation and Sturmian theorems (including multiplicities of focal points) for the symmetric three-term recurrence equations (T).

The paper is divided as follows. In the next section, we present an overview of the known transformations between the equations (T), (J), and system (S). In Section 3 we prove the main results about the symplectic structure of the recurrence equation (T). In Sections 4–6, we present recent results from the theory of discrete symplectic systems (S) adopted for the setting of recurrence equations (T). These results include the Riccati equations and inequalities as being a part of the Reid roundabout theorems in Section 4, the Sturmian separation and comparison theorems in Section 5, and the oscillation theorems and Rayleigh principle in Section 6. In Section 7, we make some final comments about the results of this paper.

2. Known Results

We first present known transformations between the three-term recurrence equation (T), Jacobi equation (J), and the symplectic system (S). We adopt the following notation.

Notation 2.1 (Three-term recurrence (T)). The matrices S_k, T_k , and the vectors x_k in (T) have the following properties: $S_k, T_k \in \mathbb{R}^{n \times n}$ are defined on $[0, N + 1]_{\mathbb{Z}}$ with S_k invertible and T_k symmetric; $x_k \in \mathbb{R}^n$ are defined on $[0, N + 1]_{\mathbb{Z}}$.

Notation 2.2 (Jacobi equation (J)). The matrices P_k, Q_k, R_k , and the vectors x_k in (J) have the following properties: $P_k, Q_k, R_k \in \mathbb{R}^{n \times n}$ are defined on $[0, N]_{\mathbb{Z}}$, P_k and R_k are symmetric, and the matrix $S_k := R_k + Q_k^T$ is invertible; $x_k \in \mathbb{R}^n$ for k in $[0, N + 1]_{\mathbb{Z}}$.

Note that the coefficients T_0, S_{N+1} , and T_{N+1} are not explicitly needed in (T) and the coefficient P_N is not needed in (J). However, it will be convenient to use them when we transform (T) into (J) or system (S) and vice versa. For example, we can now define $x_{N+2} := S_{N+1}^{-1}(T_{N+1}x_{N+1} - S_N^T x_N)$, so that the recurrence in (T) is satisfied also at $t = N$.

Notation 2.3 (Symplectic system (S)). The matrices A_k, B_k, C_k, D_k , and the vectors x_k, u_k in (S) have the following properties: $A_k, B_k, C_k, D_k \in \mathbb{R}^{n \times n}$ are defined on $[0, N]_{\mathbb{Z}}$ and satisfy (1.1); $x_k, u_k \in \mathbb{R}^n$ are defined for k in $[0, N + 1]_{\mathbb{Z}}$.

The following two known results are verified by straightforward calculations. They can be found in [1, Section 3.6].

Proposition 2.4 (Jacobi (J) to three-term recurrence (T)). *Assume that $P_k, Q_k, R_k, S_k := R_k + Q_k^T$ satisfy the conditions in Notation 2.2 and set $R_{N+1} := I$. Then the Jacobi equation (J) is the symmetric three-term recurrence equation (T), whose coefficients*

$$S_k := R_k + Q_k^T, \quad T_{k+1} := R_{k+1} + R_k + Q_k + Q_k^T + P_k, \quad k \in [0, N]_{\mathbb{Z}} \quad (2.1)$$

satisfy the conditions in Notation 2.1.

Note that the choice of $R_{N+1} := I$ in Proposition 2.4 is arbitrary, that is, any matrix $R_{N+1} \in \mathbb{R}^{n \times n}$ will do the job.

Proposition 2.5 (Three-term recurrence (T) to Jacobi (J), R_k invertible). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Let R_k be any symmetric and invertible matrices for $k \in [0, N + 1]_{\mathbb{Z}}$. Then the symmetric three-term recurrence equation (T) is the Jacobi equation (J), whose coefficients*

$$R_k, Q_k := S_k^T - R_k, \quad P_k := T_{k+1} - S_k^T - S_k + R_k - R_{k+1}, \quad k \in [0, N]_{\mathbb{Z}} \quad (2.2)$$

satisfy the conditions in Notation 2.2.

Remark 2.6. When the matrix S_k is also symmetric, we may take $R_k := S_k$ in Proposition 2.5. It follows that the recurrence equation (T) is then transformed into the Jacobi equation (J), whose coefficients

$$R_k := S_k, \quad Q_k := 0, \quad P_k := T_{k+1} - S_{k+1} - S_k, \quad k \in [0, N]_{\mathbb{Z}} \quad (2.3)$$

satisfy the conditions in Notation 2.2. Note that in this case the matrix $R_k = S_k$ is invertible.

The invertibility condition on R_k in Proposition 2.5 or Remark 2.6 means that the resulting Jacobi equation can be written as a linear Hamiltonian system (H), which in turn can be written as a symplectic system (S). This is shown in [1, Example 3.17].

Proposition 2.7 (Jacobi (J) to symplectic (S), R_k invertible). *Assume that $P_k, Q_k, R_k, S_k := R_k + Q_k^T$ satisfy the conditions in Notation 2.2 with R_k being invertible. Then the Jacobi equation (J) is the symplectic system (S), whose coefficients*

$$\begin{aligned} \mathbb{A}_k &:= S_k^{-1} R_k, & \mathbb{C}_k &:= (P_k - Q_k R_k^{-1} Q_k^T) S_k^{-1} R_k, & k \in [0, N]_{\mathbb{Z}}, \\ \mathbb{B}_k &:= S_k^{-1}, & \mathbb{D}_k &:= (P_k - Q_k R_k^{-1} Q_k^T) S_k^{-1} + I + Q_k R_k^{-1}, & k \in [0, N]_{\mathbb{Z}}, \end{aligned} \quad (2.4)$$

with $u_k := S_k x_{k+1} - R_k x_k$ on $[0, N]_{\mathbb{Z}}$ and $u_{N+1} := (P_N + Q_N + S_N) x_{N+1} - S_N^T x_N$ satisfy the conditions in Notation 2.3.

It is interesting to observe that by using the identity $Q_k^T = S_k - R_k$ one can eliminate the inverse of R_k in the coefficients (2.4) to obtain the coefficients in Proposition 2.8 below. This was actually the motivation for the investigation of Jacobi systems as discrete symplectic systems in [6]. In this latter reference, the authors showed that it is possible to treat Jacobi equation (J) directly as a symplectic system (S) by bypassing the Hamiltonian system (H); see [6, Corollary 5.2].

Proposition 2.8 (Jacobi (J) to symplectic (S)). *Assume that $P_k, Q_k, R_k, S_k := R_k + Q_k^T$ satisfy the conditions in Notation 2.2. Then the Jacobi equation (J) is the symplectic system (S), whose coefficients*

$$\begin{aligned} \mathbb{A}_k &:= S_k^{-1} R_k, & \mathbb{C}_k &:= P_k S_k^{-1} R_k - Q_k S_k^{-1} Q_k^T, & k \in [0, N]_{\mathbb{Z}}, \\ \mathbb{B}_k &:= S_k^{-1}, & \mathbb{D}_k &:= (P_k + Q_k + Q_k^T + R_k) S_k^{-1}, & k \in [0, N]_{\mathbb{Z}}, \end{aligned} \quad (2.5)$$

with $u_k := S_k x_{k+1} - R_k x_k$ on $[0, N]_{\mathbb{Z}}$ and $u_{N+1} := (P_N + Q_N + S_N) x_{N+1} - S_N^T x_N$ satisfy the conditions in Notation 2.3. Moreover, the resulting symplectic system (S) is Hamiltonian if and only if the matrix R_k is invertible.

The resulting symplectic system in Proposition 2.8 has $\mathbb{B}_k = S_k^{-1}$ invertible. This turns out to be a characterizing property of symplectic systems (S) corresponding to Jacobi equations (J); see [6, Corollary 5.3].

Proposition 2.9 (Symplectic (S) to Jacobi (J)). *Assume that $\mathbb{A}_k, \mathbb{B}_k, \mathbb{C}_k, \mathbb{D}_k$ satisfy the conditions in Notation 2.3 with \mathbb{B}_k being invertible. Then the symplectic system (S) is the Jacobi equation (J), whose coefficients*

$$R_k := \mathbb{B}_k^{-1} \mathbb{A}_k, \quad Q_k := (I - \mathbb{A}_k^T) \mathbb{B}_k^{T-1}, \quad P_k := (\mathbb{D}_k - I) \mathbb{B}_k^{-1} + (\mathbb{A}_k^T - I) \mathbb{B}_k^{T-1}, \quad k \in [0, N]_{\mathbb{Z}}, \quad (2.6)$$

with $S_k = R_k + Q_k^T = \mathbb{B}_k^{-1}$ satisfy the conditions in Notation 2.2.

Next we turn our attention back to the symmetric three-term recurrence equations (T). By combining the transformations in Propositions 2.5 and 2.8 we get the following.

Proposition 2.10 (Three-term recurrence (T) to symplectic (S), R_k invertible). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Let R_k be any symmetric and invertible matrices for $k \in [0, N+1]_{\mathbb{Z}}$. Then the symmetric three-term recurrence equation (T) is the symplectic system (S), whose coefficients*

$$\begin{aligned} \mathbb{A}_k &:= S_k^{-1} R_k, & \mathbb{C}_k &:= (T_{k+1} - R_{k+1}) S_k^{-1} R_k - S_k^T, & k \in [0, N]_{\mathbb{Z}}, \\ \mathbb{B}_k &:= S_k^{-1}, & \mathbb{D}_k &:= (T_{k+1} - R_{k+1}) S_k^{-1}, & k \in [0, N]_{\mathbb{Z}}, \end{aligned} \quad (2.7)$$

with $u_k := S_k x_{k+1} - R_k x_k$ on $[0, N]_{\mathbb{Z}}$ and $u_{N+1} := (T_{N+1} - R_{N+1}) x_{N+1} - S_N^T x_N$ satisfy the conditions in Notation 2.3.

3. Main Results

The need to have R_k invertible in Proposition 2.10 is artificial, because R_k is not furnished by the three-term recurrence equation (T) and furthermore, R_k^{-1} is not even present in equations (2.7) which define the coefficients of the corresponding symplectic system (S). However, the invertibility of R_k is a requirement inherited from Proposition 2.5 that was derived in [1, Section 3.6]. Therefore, an important question naturally surfaces: is it possible to obtain the result of Propositions 2.5 and 2.10 without any assumption on R_k ?

The following new result provides an answer to the above question, that is, it shows that the recurrence equations (T) are naturally special cases of symplectic systems (S) for any choice of matrices R_k and without any assumption on the invertibility of T_k .

Theorem 3.1 (Three-term recurrence (T) to symplectic (S), R_k arbitrary). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Let R_k be any symmetric matrices for $k \in [0, N+1]_{\mathbb{Z}}$. Then the symmetric three-term recurrence equation (T) is the symplectic system (S), whose coefficients are given by (2.7) with $u_k := S_k x_{k+1} - R_k x_k$ on $[0, N]_{\mathbb{Z}}$ and $u_{N+1} := (T_{N+1} - R_{N+1}) x_{N+1} - S_N^T x_N$ and they satisfy the conditions in Notation 2.3.*

Proof. Given (T) with the data as in Notation 2.1, we set $u_k := S_k x_{k+1} - R_k x_k$ for $k \in [0, N]_{\mathbb{Z}}$ and $u_{N+1} := (T_{N+1} - R_{N+1}) x_{N+1} - S_N^T x_N$. Then

$$\begin{aligned} x_{k+1} &= S_k^{-1} R_k x_k + S_k^{-1} u_k = \mathbb{A}_k x_k + \mathbb{B}_k u_k, & k \in [0, N]_{\mathbb{Z}}, \\ u_{k+1} &= S_{k+1} x_{k+2} - R_{k+1} x_{k+1} \stackrel{(T)}{=} (T_{k+1} - R_{k+1}) x_{k+1} - S_k^T x_k \\ &= \mathbb{C}_k x_k + \mathbb{D}_k u_k, & k \in [0, N-1]_{\mathbb{Z}}. \end{aligned} \quad (3.1)$$

Therefore, with the coefficients \mathbb{A}_k , \mathbb{B}_k , \mathbb{C}_k , and \mathbb{D}_k defined by (2.7), we have that the pair (x, u) satisfies the first equation in system (S) for all $k \in [0, N]_{\mathbb{Z}}$ and the second equation in (S) for all $k \in [0, N-1]_{\mathbb{Z}}$. However, the definition of u_{N+1} yields that the second equation in (S) holds also at $k = N$. It remains to show that the matrix \mathbb{S}_k , defined in (1.1) through \mathbb{A}_k , \mathbb{B}_k , \mathbb{C}_k , and \mathbb{D}_k in (2.7), is symplectic. We have after easy calculations that for every $k \in [0, N]_{\mathbb{Z}}$,

$$\mathbb{S}_k^T \mathcal{J} \mathbb{S}_k = \begin{pmatrix} \mathbb{A}_k & \mathbb{B}_k \\ \mathbb{C}_k & \mathbb{D}_k \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \mathbb{A}_k & \mathbb{B}_k \\ \mathbb{C}_k & \mathbb{D}_k \end{pmatrix} = \mathcal{J}, \quad (3.2)$$

where we used the symmetry of R_k . The proof is complete. \square

Note that the matrices R_k in the above theorem are arbitrary, and hence, one can choose them to simplify the formulas of the coefficients in (2.7). One choice standing out is when $R_k = T_k$. In this case, the result of Theorem 3.1 reduces to the following.

Corollary 3.2 (Three-term recurrence (T) to symplectic (S)). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Then the symmetric three-term recurrence equation (T) is the symplectic system (S), whose coefficients*

$$\mathbb{A}_k := S_k^{-1} T_k, \quad \mathbb{B}_k := S_k^{-1}, \quad \mathbb{C}_k := -S_k^T, \quad \mathbb{D}_k := 0, \quad k \in [0, N]_{\mathbb{Z}}, \quad (3.3)$$

with $u_k := S_k x_{k+1} - T_k x_k$ on $[0, N]_{\mathbb{Z}}$ and $u_{N+1} := -S_N^T x_N$ satisfy the conditions in Notation 2.3.

The above result has an important consequence. By using Proposition 2.9, we can now transform *any* symmetric three-term recurrence equation (T) into a Jacobi equation (J) by the procedure described in Corollary 3.2 and Proposition 2.9. However, compared with the result in Proposition 2.5, we do not need the matrices R_k to be invertible, but they can be arbitrary.

Corollary 3.3 (Three-term recurrence (T) to Jacobi (J)). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Let R_k be any symmetric matrices for $k \in [0, N+1]_{\mathbb{Z}}$. Then the symmetric three-term recurrence equation (T) is the Jacobi equation (J), whose coefficients are given by equations (2.2) and they satisfy the conditions in Notation 2.2. When $R_k := T_k$, the coefficients of (J) reduce to*

$$R_k := T_k, \quad Q_k := S_k^T - T_k, \quad P_k := T_k - S_k - S_k^T, \quad k \in [0, N]_{\mathbb{Z}}. \quad (3.4)$$

Note that if the matrices R_k are invertible, then Corollary 3.3 is a consequence of Proposition 2.5. This is also the case for the second part of the Corollary 3.3 if T_k are invertible.

When combining the transformations in Propositions 2.9 and 2.4, we get the following result.

Corollary 3.4 (Symplectic (S) to three-term recurrence (T)). *Assume that $\mathbb{A}_k, \mathbb{B}_k, \mathbb{C}_k, \mathbb{D}_k$ satisfy the conditions in Notation 2.3 with \mathbb{B}_k being invertible and set $\mathbb{A}_{N+1} = \mathbb{B}_{N+1} := I$. Then the symplectic system (S) is the symmetric three-term recurrence equation (T), whose coefficients*

$$S_k := \mathbb{B}_k^{-1}, \quad T_{k+1} := \mathbb{B}_{k+1}^{-1} \mathbb{A}_{k+1}^{-1} + \mathbb{D}_k \mathbb{B}_k^{-1}, \quad k \in [0, N]_{\mathbb{Z}}, \quad (3.5)$$

with $T_0 = \mathbb{B}_0^{-1} \mathbb{A}_0$ satisfy the conditions in Notation 2.1.

In the last part of this section, we pay attention to the quadratic functionals associated with the equations (T), (J), and system (S). In particular, we show that these functionals transform exactly in the same way as their corresponding systems. Consider the quadratic functionals

$$\mathcal{F}_T(x) := \sum_{k=0}^N \left\{ x_{k+1}^T T_{k+1} x_{k+1} - x_k^T S_k x_{k+1} - x_{k+1}^T S_k^T x_k \right\}, \quad (3.6)$$

$$\mathcal{F}_J(x) := \sum_{k=0}^N \left\{ x_{k+1}^T P_k x_{k+1} + 2x_{k+1}^T Q_k x_k + \Delta x_k^T R_k \Delta x_k \right\}, \quad (3.7)$$

$$\mathbb{F}(x, u) := \sum_{k=0}^N \left\{ x_k^T C_k^T A_k x_k + 2x_k^T C_k^T B_k u_k + u_k^T D_k^T B_k u_k \right\}, \quad (3.8)$$

where $x = \{x_k\}_{k=0}^{N+1}$ satisfies $x_0 = 0 = x_{N+1}$. In addition, in functional \mathbb{F} the pair (x, u) solves the first equation in system (S), that is, $x_{k+1} = A_k x_k + B_k u_k$ for $k \in [0, N]_{\mathbb{Z}}$. The following result is from [6, Proposition 3.7].

Proposition 3.5 (Quadratic functionals for (J) and (S)). *Assume that*

- (i) either $P_k, Q_k,$ and R_k satisfy the conditions in Notation 2.2 with $A_k, B_k, C_k, D_k,$ and u_k being given by (2.5) of Proposition 2.8
- (ii) or $A_k, B_k, C_k,$ and D_k satisfy the conditions in Notation 2.3 with B_k being invertible and P_k, Q_k, R_k are given by (2.6).

Then $\mathcal{F}_J(x) = \mathbb{F}(x, u)$ for every $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$.

As a consequence of the results in Theorem 3.1 and Corollary 3.4, and in Proposition 2.4 and Corollary 3.3, we get the transformations between the functionals \mathcal{F}_T and \mathbb{F} , and \mathcal{F}_T and \mathcal{F}_J .

Proposition 3.6 (Quadratic functionals for (T) and (S)). *Assume that*

- (i) either S_k, T_k satisfy the conditions in Notation 2.1 with $A_k, B_k, C_k, D_k,$ and u_k being given by (2.7) of Theorem 3.1
- (ii) or $A_k, B_k, C_k,$ and D_k satisfy the conditions in Notation 2.3 with B_k being invertible and T_k, S_k are given by (3.5).

Then $\mathcal{F}_T(x) = \mathbb{F}(x, u)$ for every $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$.

Proposition 3.7 (Quadratic functionals for (J) and (T)). *Assume that*

- (i) either $P_k, Q_k,$ and R_k satisfy the conditions in Notation 2.2 with T_k, S_k being given by (2.1)
- (ii) or T_k, S_k satisfy the conditions in Notation 2.1 with S_k being invertible and P_k, Q_k, R_k are given by (3.4).

Then $\mathcal{F}_T(x) = \mathcal{F}_J(x)$ for every $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$.

4. Applications in Reid Roundabout Theorems

In the previous section, we proved that symmetric three-term recurrence equations (T) and Jacobi difference equations (J) are completely equivalent, that is, any result for one equation (T) or (J) can be translated via the transformations in Proposition 2.4 and Corollary 3.3 to a result for the other equation. This equivalence is carried over via discrete symplectic systems (S) with \mathbb{B}_k being invertible, utilizing the transformations in Propositions 2.8 and 2.9 (for the passage between equation (J) and system (S)) and the transformations in Theorem 3.1 and Corollary 3.4 (for the passage between equation (T) and system (S)). In this section, we give some applications of this equivalence. For example, we derive the Riccati equation and Riccati inequality which are naturally associated with the symmetric three-term recurrence equations (T)—the results which have not been known in the literature for (T).

Consider the quadratic functional $\mathcal{F}_T(x)$ defined in (3.6) subject to sequences $x = \{x_k\}_{k=0}^{N+1}$ satisfying $x_0 = 0 = x_{N+1}$. Note that due to $x_{N+1} = 0$, the functional \mathcal{F}_T does not depend on the matrix T_{N+1} , as we mentioned at the beginning of Section 2. We say that the functional \mathcal{F}_T is positive definite if $\mathcal{F}_T(x) > 0$ for every $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$ and $x \neq 0$. We say that \mathcal{F}_T is nonnegative if $\mathcal{F}_T(x) \geq 0$ for every $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$.

The positivity of the functional \mathcal{F}_T was first characterized in [2, Theorem 4]; see also [1, Theorem 5.13] and [4, Corollary 2], in terms of the properties of the so-called conjoined bases of (T). These are the $n \times n$ matrix solutions $X = \{X_k\}_{k=0}^{N+1}$ of (T) such that $X_k^T S_k X_{k+1}$ is symmetric and $\text{rank}(X_k^T \ X_{k+1}^T) = n$ for some (and hence for any) index $k \in [0, N]_{\mathbb{Z}}$. A special conjoined basis \widehat{X} of (T), determined by the initial conditions $\widehat{X}_0 = 0$ and $\widehat{X}_1 = S_0^{-1}$, is called the principal solution of (T).

Proposition 4.1 (Reid roundabout theorem—positivity). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Then the following statements are equivalent.*

- (i) *The functional \mathcal{F}_T is positive definite.*
- (ii) *The principal solution \widehat{X} of (T) has \widehat{X}_k invertible for all $k \in [1, N+1]_{\mathbb{Z}}$ and satisfies $\widehat{X}_k^T S_k \widehat{X}_{k+1} > 0$ for all $k \in [1, N]_{\mathbb{Z}}$.*
- (iii) *There exists a conjoined basis X of (T) such that X_k is invertible for all $k \in [0, N+1]_{\mathbb{Z}}$ and satisfying $X_k^T S_k X_{k+1} > 0$ for all $k \in [0, N]_{\mathbb{Z}}$.*

Proof. See [1, Theorem 5.13] or [4, Corollary 2]. □

A similar result holds for the nonnegativity of \mathcal{F}_T .

Proposition 4.2 (Reid roundabout theorem—nonnegativity). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Then the following statements are equivalent.*

- (i) *The functional \mathcal{F}_T is nonnegative.*
- (ii) *The principal solution \widehat{X} of (T) has \widehat{X}_k invertible for all $k \in [1, N]_{\mathbb{Z}}$ and satisfies $\widehat{X}_k^T S_k \widehat{X}_{k+1} > 0$ for all $k \in [1, N-1]_{\mathbb{Z}}$ and $\widehat{X}_N^T S_N \widehat{X}_{N+1} \geq 0$.*

Proof. See [4, Theorem 2]. □

In the following result, we add three more equivalent conditions to Proposition 4.1 and one more equivalent condition to Proposition 4.2 in terms of solutions of the discrete Riccati matrix equation and inequality corresponding to (T), thus completing the above results to their full standard forms. For symmetric matrices $W = \{W_k\}_{k=0}^{N+1}$, define the Riccati operator

$$R[W]_k := W_{k+1}S_k^{-1}(T_k + W_k) + S_k^T, \quad k \in [0, N]_{\mathbb{Z}}. \tag{4.1}$$

This Riccati operator is obtained from the Riccati operator $\mathbb{R}[W]_k := W_{k+1}(\mathbb{A}_k + \mathbb{B}_k W_k) - (\mathbb{C}_k + \mathbb{D}_k W_k)$ for symplectic system (S); see, for example, [7, 14–16, 19]. That is, if the coefficients of (S) are given by formulas (3.3), then $\mathbb{R}[W]_k = R[W]_k$. The equation $R[W]_k = 0$ for $k \in [0, N]_{\mathbb{Z}}$ is called the *discrete Riccati matrix equation*. If $T_k + W_k$ is invertible, then we may solve the equation $R[W]_k = 0$ for W_{k+1} and obtain the (symmetric) Riccati equation corresponding to the recurrence equation (T), that is,

$$W_{k+1} + S_k^T(T_k + W_k)^{-1}S_k = 0, \quad k \in [0, N]_{\mathbb{Z}}. \tag{RE}$$

The Riccati equation (RE) has been studied in the literature by many authors; see, for example, the references discussed in [20, page 12]. However, its natural connection to the recurrence equation (T) is established for the first time in this paper. In addition, the discrete Riccati inequality $\mathbb{R}[W]_k(\mathbb{A}_k + \mathbb{B}_k W_k)^{-1} \leq 0$ derived in [14, Theorem 1] for symplectic systems (S) yields through Corollary 3.2 a new Riccati inequality

$$W_{k+1} + S_k^T(T_k + W_k)^{-1}S_k \leq 0, \quad k \in [0, N]_{\mathbb{Z}}, \tag{RI}$$

for the recurrence equation (T). Equivalently, the Riccati equation (RE) and inequality (RI) can be obtained from the Riccati equation and inequality for the Jacobi equation

$$\Delta W_k - P_k + (W_k - Q_k)(R_k + W_k)^{-1}(W_k - Q_k^T) = 0, \quad (\leq 0) \quad k \in [0, N]_{\mathbb{Z}}, \tag{4.2}$$

in which the coefficients are given by the formulas in (3.4). Note that discrete Riccati equations obtained from symplectic system (S) corresponding to three-term recurrence equations (T) with R_k being invertible (as in Propositions 2.5 and 2.10) are considered in [20], [1, Section 6.1], and [16, Section 4]. In Theorem 4.3 below, we do not require any condition on R_k . The following result is a complement of Proposition 4.1.

Theorem 4.3 (Reid roundabout theorem—positivity (continued)). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Then each of the conditions (i)–(iii) of Proposition 4.1 is equivalent to any of the following statements.*

- (iv) *There exists a symmetric solution W_k on $[0, N + 1]_{\mathbb{Z}}$ of the Riccati equation (RE) for $k \in [1, N]_{\mathbb{Z}}$ such that $W_0 = 0$ and $T_k + W_k > 0$ for all $k \in [1, N]_{\mathbb{Z}}$.*
- (v) *There exists a symmetric solution W_k on $[0, N + 1]_{\mathbb{Z}}$ of the Riccati equation (RE) such that $T_k + W_k > 0$ for all $k \in [0, N]_{\mathbb{T}}$.*
- (vi) *There exists a symmetric solution W_k on $[0, N + 1]_{\mathbb{Z}}$ of the Riccati inequality (RI) satisfying $T_k + W_k > 0$ for all $k \in [0, N]_{\mathbb{T}}$.*

Proof. With the coefficients of system (S) given by (3.3), the equivalence of conditions (i) and (v) is established in [16, Theorem 7] and the equivalence of (i) and (vi) in [14, Theorem 1]. Condition (ii) of Proposition 4.1 implies condition (iv) by setting $W_0 := 0$, $W_k := S_k \widehat{X}_{k+1} \widehat{X}_k^{-1} - T_k$ for $k \in [1, N]_{\mathbb{Z}}$, and $W_{N+1} := -S_N^T \widehat{X}_N \widehat{X}_{N+1}^{-1}$ (i.e., we set $W_k := \widehat{U}_k \widehat{X}_k^{-1}$ on $[1, N+1]_{\mathbb{Z}}$, where $\widehat{U}_k := S_k \widehat{X}_{k+1} - T_k \widehat{X}_k$ for $k \in [0, N]_{\mathbb{Z}}$ and $\widehat{U}_{N+1} := -S_N^T \widehat{X}_N$ accordingly to the definition of u_k in Theorem 3.1). Then $T_k + W_k = \widehat{X}_k^{T-1} (\widehat{X}_k^T S_k \widehat{X}_{k+1}) \widehat{X}_k^{-1} > 0$ for all $k \in [1, N]_{\mathbb{Z}}$. Hence, condition (iv) is satisfied. Finally, if we assume that condition (iv) holds, then the proof of the positivity of \mathcal{F}_T is similar to the proof of [4, Corollary 3]. \square

Note that condition (vi) of Theorem 4.3 yields that $-W_{k+1} \geq S_k^T (T_k + W_k)^{-1} S_k > 0$, which implies that the matrices W_k in Theorem 4.3(vi) are negative definite for $k \in [1, N+1]_{\mathbb{Z}}$.

The second result of this section is concerned with the nonnegativity of the functional \mathcal{F}_T .

Theorem 4.4 (Reid roundabout theorem—nonnegativity (continued)). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Then each of the conditions (i)–(ii) of Proposition 4.2 is equivalent to the following statement.*

- (iii) *There exists a symmetric solution W_k on $[0, N+1]_{\mathbb{Z}}$ of the Riccati equation (RE) for $k \in [1, N-1]_{\mathbb{Z}}$ such that $W_0 = 0$, $T_k + W_k > 0$ for all $k \in [1, N-1]_{\mathbb{Z}}$, and $T_N + W_N \geq 0$.*

Proof. The proof is similar to implications (ii) \Rightarrow (iv) \Rightarrow (i) from the proof of Theorem 4.3. The details are here therefore omitted. Alternatively, see the proof of [4, Theorem 3]. \square

By comparing the Riccati equation conditions for the positivity and nonnegativity of \mathcal{F}_T in Theorem 4.3(iv) and Theorem 4.4(iii), we can see that the Riccati equation for the positivity of \mathcal{F}_T is satisfied on the *closed* interval (including $k = N$), while the Riccati equation for the nonnegativity of \mathcal{F}_T is satisfied on the *open* interval (excluding $k = N$). This phenomenon resembles the situation in the continuous time setting in [21] or [22, Section 6.2], that is, for Jacobi differential equations or Hamiltonian systems and their corresponding Riccati differential equations.

5. Applications in Sturmian Theory

In [1, Sections 5.3 and 5.6], several Sturmian comparison and separation theorems are presented for the symmetric three-term recurrence equations (T). However, these results do not involve the multiplicities of focal points for conjoined bases of (T). Therefore, our next aim is to extend the Sturmian separation and comparison theorems for symmetric three-term recurrence equations (T) in this direction.

Following [18], we say that a conjoined basis X of (T) has a focal point in the point $k+1$ if X_{k+1} is singular and then $\text{def } X_{k+1} := \dim \text{Ker } X_{k+1}$ is its multiplicity, while the conjoined basis X has a focal point in the interval $(k, k+1)_{\mathbb{Z}}$ if the matrix $X_k^T S_k X_{k+1}$ is not nonnegative definite and then $\text{ind } X_k^T S_k X_{k+1}$ is its multiplicity. Here $\text{ind } A$ is defined as the number of negative eigenvalues of the (symmetric) matrix A . The number of focal points in the interval $(k, k+1)_{\mathbb{Z}}$ (including multiplicities) is then $m_k := \text{def } X_{k+1} + \text{ind } X_k^T S_k X_{k+1}$. We will always count the focal points of conjoined bases of (T) including their multiplicities. This definition of multiplicities is motivated by the appearance of the symmetric matrix $X_k^T S_k X_{k+1}$ in the Reid roundabout theorem (Propositions 4.1 and 4.2). Therefore, the conditions (ii) in Propositions 4.1 and 4.2 can be reformulated as follows.

Corollary 5.1 (Reid roundabout theorem—positivity (continued)). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Condition (ii) of Proposition 4.1 has the following equivalent form.*

(ii)' *The principal solution of (T) has no focal points in the interval $(0, N + 1]_{\mathbb{Z}}$.*

Corollary 5.2 (Reid roundabout theorem—nonnegativity (continued)). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. Condition (ii) of Proposition 4.2 has the following equivalent form.*

(ii)' *The principal solution of (T) has no focal points in the interval $(0, N + 1]_{\mathbb{Z}}$.*

From Corollary 5.1 and Proposition 4.1(iii), we easily get the following.

Corollary 5.3 (Sturmian separation theorem). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. If the principal solution of (T) has a focal point in the interval $(0, N + 1]_{\mathbb{Z}}$, then any other conjoined basis of (T) has a focal point in $(0, N + 1]_{\mathbb{Z}}$ as well.*

The above result can be found as a special case of [23, Theorem 3] or [24, Corollary 3.1]. A refinement of the previous result can be deduced from [9, Theorem 1].

Corollary 5.4 (Sturmian separation theorem). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. If there exists a conjoined basis of (T) with no focal points in $(0, N + 1]_{\mathbb{Z}}$, then any other conjoined basis of (T) has at most n focal points in $(0, N + 1]_{\mathbb{Z}}$.*

The corresponding proof of Corollary 5.4 in [9] is based on the construction of a suitable sequence $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$ and $x \neq 0$, for which the value of the functional $\mathcal{F}_T(x) = 0$, thus contradicting Proposition 4.1. Finally, the most general result in this direction is the following.

Theorem 5.5 (Sturmian separation theorem). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. If the principal solution of (T) has m focal points in $(0, N + 1]_{\mathbb{Z}}$, then any other conjoined basis of (T) has at least m and at most $m + n$ focal points in $(0, N + 1]_{\mathbb{Z}}$.*

Proof. This is a special case of [8, Theorem 3.1] for symplectic systems (S), in which we use the coefficients from Corollary 3.2. □

One can see that Corollary 5.4 is a special case of Theorem 5.5 for $m = 0$.

By comparing the numbers of focal points in $(0, N + 1]_{\mathbb{Z}}$ of two conjoined bases of (T) with the number of focal points in $(0, N + 1]_{\mathbb{Z}}$ of the principal solution of (T), we obtain from Theorem 5.5 the classical Sturmian separation theorem; see [8, Theorem 1.1] and [25, page 366].

Corollary 5.6 (Sturmian separation theorem). *Assume that S_k, T_k satisfy the conditions in Notation 2.1. The difference between the numbers of focal points in $(0, N + 1]_{\mathbb{Z}}$ of any two conjoined bases of (T) is at most n .*

The above Sturmian separation theorems are obtained from the comparison of the numbers of focal points in $(0, N + 1]_{\mathbb{Z}}$ of conjoined bases of two (possibly) different recurrence equations (T) or symplectic systems (S). More precisely, in addition to (T) consider the symmetric three-term recurrence equation

$$\tilde{S}_{k+1}x_{k+2} - \tilde{T}_{k+1}x_{k+1} + \tilde{S}_k^T x_k = 0, \quad k \in [0, N - 1]_{\mathbb{Z}}, \quad (\tilde{T})$$

in which, as in Notation 2.1, the matrices \tilde{S}_k are invertible and \tilde{T}_k are symmetric. Following [23, Section 3.2] and [8, Theorem 1.2] we define the symmetric $2n \times 2n$ matrices

$$\mathcal{G}_k := \begin{pmatrix} T_k & -S_k \\ -S_k^T & 0 \end{pmatrix}, \quad \tilde{\mathcal{G}}_k := \begin{pmatrix} \tilde{T}_k & -\tilde{S}_k \\ -\tilde{S}_k^T & 0 \end{pmatrix}. \quad (5.1)$$

The following result from [1, Theorem 5.20] and [23, Theorem 4] is a comparison complement of the separation theorem in Corollary 5.3. It is a direct consequence of Proposition 4.1.

Corollary 5.7 (Sturmian comparison theorem). *Assume that S_k , T_k , \tilde{S}_k , and \tilde{T}_k satisfy the conditions in Notation 2.1. In addition, let*

$$\mathcal{G}_k \geq \tilde{\mathcal{G}}_k \quad \forall k \in [0, N]_{\mathbb{Z}}. \quad (5.2)$$

If the principal solution of (T) has a focal point in $(0, N + 1]_{\mathbb{Z}}$, then any conjoined basis of (\tilde{T}) has a focal point in $(0, N + 1]_{\mathbb{Z}}$ as well.

More precise statements about the numbers of focal points of the principal solution of (\tilde{T}) , respectively, of (T), and the number of focal points of conjoined bases of (T), respectively, of (\tilde{T}) , are contained in the next two results. In particular, Corollary 5.7 is a special case of Theorem 5.8 for $m = 0$.

Theorem 5.8 (Sturmian comparison theorem). *Assume that S_k , T_k , \tilde{S}_k , and \tilde{T}_k satisfy the conditions in Notation 2.1 and let condition (5.2) hold. If the principal solution of (T) has m focal points in $(0, N + 1]_{\mathbb{Z}}$, then any conjoined basis of (\tilde{T}) has at least m focal points in $(0, N + 1]_{\mathbb{Z}}$.*

Proof. This result follows from [8, Theorem 1.3] for symplectic systems (S) with the coefficients from Corollary 3.2. \square

Theorem 5.9 (Sturmian comparison theorem). *Assume that S_k , T_k , \tilde{S}_k , and \tilde{T}_k satisfy the conditions in Notation 2.1 and let condition (5.2) hold. If the principal solution of (\tilde{T}) has m focal points in $(0, N + 1]_{\mathbb{Z}}$, then any conjoined basis of (T) has at most $m + n$ focal points in $(0, N + 1]_{\mathbb{Z}}$.*

Proof. This result is a special case of [8, Theorem 1.2] for symplectic systems (S) with the coefficients from Corollary 3.2. \square

Remark 5.10. For the Jacobi difference equations, that is, for (J) and another equation of the same type

$$\Delta(\tilde{R}_k \Delta x_k + \tilde{Q}_k^T x_{k+1}) = \tilde{P}_k x_{k+1} + \tilde{Q}_k \Delta x_k, \quad k \in [0, N - 1]_{\mathbb{Z}}, \quad (\tilde{J})$$

where \tilde{P}_k , \tilde{Q}_k , \tilde{R}_k satisfy the conditions in Notation 2.2, the matrices \mathcal{G}_k and $\tilde{\mathcal{G}}_k$ have the form

$$\mathcal{G}_k = \begin{pmatrix} R_k & -S_k \\ -S_k^T & R_k + Q_k + Q_k^T + P_k \end{pmatrix}, \quad \tilde{\mathcal{G}}_k = \begin{pmatrix} \tilde{R}_k & -\tilde{S}_k \\ -\tilde{S}_k^T & \tilde{R}_k + \tilde{Q}_k + \tilde{Q}_k^T + \tilde{P}_k \end{pmatrix}. \quad (5.3)$$

(i) If $Q_k = \tilde{Q}_k \equiv 0$ (so that $S_k = R_k$ and $\tilde{S}_k = \tilde{R}_k$ are symmetric), then the classical conditions $R_k \geq \tilde{R}_k$ and $P_k \geq \tilde{P}_k$ for all $k \in [0, N]_{\mathbb{Z}}$ are equivalent with condition (5.2). This can be seen from the calculation

$$\begin{pmatrix} c \\ d \end{pmatrix}^T (C_k - \tilde{C}_k) \begin{pmatrix} c \\ d \end{pmatrix} = (c - d)^T (R_k - \tilde{R}_k) (c - d) + d^T (P_k - \tilde{P}_k) d \quad \text{for any } c, d \in \mathbb{R}^n. \quad (5.4)$$

(ii) If $R_k \equiv \tilde{R}_k$ and $Q_k \equiv \tilde{Q}_k$, then the condition $P_k \geq \tilde{P}_k$ for all $k \in [0, N]_{\mathbb{Z}}$ is equivalent with condition (5.2).

6. Applications in Eigenvalue Theory

The Sturmian separation and comparison theorems, in particular Theorems 5.5, 5.8, and 5.9, are proven in [8] by using the Rayleigh principle and the oscillation theorem for symplectic systems, that is, a result connecting the number of focal points in $(0, N + 1]_{\mathbb{Z}}$ of the principal solution of (S) with the number of eigenvalues of an associated discrete symplectic eigenvalue problem. The applications of the theory of Section 3 to these results for the three-term recurrence equations (T) will be presented in this section.

Consider the eigenvalue problem for the symmetric three-term recurrence equation (T) of the form

$$S_{k+1}x_{k+2} - T_{k+1}x_{k+1} + S_k^T x_k = -\lambda W_k x_{k+1}, \quad k \in [0, N - 1]_{\mathbb{Z}}, \quad x_0 = 0 = x_{N+1}, \quad (E)$$

where the $n \times n$ matrices

$$W_k \text{ are symmetric and positive definite for all } k \in [0, N]_{\mathbb{Z}}. \quad (6.1)$$

We will denote the recurrence equation in (E) by (T_λ) , so that $(T_0) = (T)$. By Corollary 3.2, equation (T_λ) can be written as a discrete symplectic system

$$x_{k+1} = \mathbb{A}_k x_k + \mathbb{B}_k u_k, \quad u_{k+1} = \mathbb{C}_k x_k + \mathbb{D}_k u_k - \lambda \mathbb{W}_k x_{k+1}, \quad k \in [0, N]_{\mathbb{Z}}, \quad x_0 = 0 = x_{N+1}, \quad (6.2)$$

with symmetric matrices $\mathbb{W}_k := W_k$. Systems of the form (6.2) are known to be the right ones to study the ‘‘symplectic’’ eigenvalue problems; see [26, Remark 3(iii)]. Note that although the matrix \mathbb{W}_N appears in the second equation of (6.2) for $k = N$, the value of \mathbb{W}_N is irrelevant because it is multiplied by $x_{N+1} = 0$.

Let $\hat{X}(\lambda) = \{\hat{X}_k(\lambda)\}_{k=0}^{N+1}$ be the principal solution of the recurrence equation (T_λ) , that is, $\hat{X}_0(\lambda) \equiv 0$ and $\hat{X}_1(\lambda) = S_0^{-1}$ for all $\lambda \in \mathbb{R}$. This means that the initial conditions of the principal solution $\hat{X}(\lambda)$ do not depend on λ . The principal solution, as well as other (matrix or vector) solutions of (T_λ) , depends in general on λ , so that we will emphasize this dependence in the notation of the solutions. A number $\lambda \in \mathbb{R}$ is an *eigenvalue* of the eigenvalue problem (E) if there exists a nontrivial solution $\{x_k(\lambda)\}_{k=0}^{N+1}$ of (E), or equivalently, if $\det \hat{X}_{N+1}(\lambda) = 0$. In

this case, the number $\text{def } \widehat{X}_{N+1}^T(\lambda) := \dim \text{Ker } \widehat{X}_{N+1}(\lambda)$ is the multiplicity of λ as being an eigenvalue of (E). Denote by, including the multiplicities,

$$\begin{aligned} n_1(\lambda) &:= \text{the number of focal points of } \widehat{X}(\lambda) \text{ in } (0, N+1]_{\mathbb{Z}}, \\ n_2(\lambda) &:= \text{the number of eigenvalues of (E) which are less or equal to } \lambda. \end{aligned} \quad (6.3)$$

Then we have the following oscillation theorem for the three-term recurrence equations (T).

Theorem 6.1 (Oscillation theorem). *Assume that S_k, T_k satisfy the conditions in Notation 2.1 and W_k satisfies (6.1). Then for all $\lambda \in \mathbb{R}$, we have*

$$n_1(\lambda) = n_2(\lambda), \quad n_1(\lambda) = n_1(\lambda+), \quad n_2(\lambda) = n_2(\lambda+), \quad (6.4)$$

where $n_j(\lambda+)$ for $j \in \{1, 2\}$ denotes the right-hand limit of the function $n_j(\cdot)$ at λ .

Proof. We refer to [10, Theorem 2] in which we use the coefficients from Corollary 3.2. Note that the number m in [10, Theorem 2] is here zero, because of our assumption (6.1) and the result of [8, Lemma 4.5]. \square

Remark 6.2. Among other properties of the eigenvalue problem (E), we mention those which are characteristic for the self-adjoint systems. These results, obtained from the symplectic systems theory, can be found in [10, Proposition 2] and [8, Theorem 4.7]. For discrete Jacobi equations (J) with $Q_k \equiv 0$, these properties are shown in [27, Theorem 4.1] or [28, Theorem 3.1].

- (i) The eigenvalues of (E) are real and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the inner product

$$\langle x, y \rangle_W := \sum_{k=0}^N x_{k+1}^T W_k y_{k+1}, \quad x = \{x_k\}_{k=0}^{N+1}, \quad y = \{y_k\}_{k=0}^{N+1}. \quad (6.5)$$

If $\langle x, y \rangle_W = 0$, then we say that x and y are orthogonal and write $x \perp y$.

- (ii) The total number of eigenvalues of (E) including their multiplicities is $r \leq nN$.
- (iii) Every $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$ can be expanded in terms of the (orthonormal system of) eigenfunctions $x^{[1]}, \dots, x^{[r]}$, that is, $x = \sum_{i=1}^r c_i x^{[i]}$, where $c_i = \langle x, x^{[i]} \rangle_W$ for $i = 1, \dots, r$.

Our final application is the Rayleigh principle describing the variational properties of the eigenvalues of (E). Let $\lambda_1 \leq \dots \leq \lambda_r$ be the eigenvalues of (E), where each eigenvalue appears repeatedly according to its multiplicity, and let $x^{[1]}, \dots, x^{[r]}$ be the corresponding orthonormal eigenfunctions, that is, $\langle x^{[i]}, x^{[j]} \rangle_W = \delta_{ij}$. We set $\lambda_0 := -\infty$ and $\lambda_{r+1} := +\infty$.

Theorem 6.3 (Rayleigh principle). *Assume that S_k, T_k satisfy the conditions in Notation 2.1 and W_k satisfies (6.1). Then for every $m \in \{0, \dots, r\}$ we have*

$$\lambda_{m+1} = \min \left\{ \frac{\mathcal{F}_T(x)}{\langle x, x \rangle_W}, \text{ where } x = \{x_k\}_{k=0}^{N+1} \neq 0, x_0 = 0 = x_{N+1}, x \perp x^{[1]}, \dots, x^{[m]} \right\}. \quad (6.6)$$

(If $m = 0$, then the above orthogonality condition is empty.)

Proof. We refer to [8, Theorem 4.6] in which we use the coefficients from Corollary 3.2. □

7. Concluding Remarks

In this section we make some final comments related to the topics of this paper.

Remark 7.1. In [7, Theorem 1(ix)], the notion of “no backward focal points” in $[k, k + 1]_{\mathbb{Z}}$ was introduced for conjoined bases of discrete symplectic systems (S). This yields another characterization of the positivity or nonnegativity of the functional \mathcal{F}_T in terms of the nonexistence of these backward focal points in the interval $[0, N + 1]_{\mathbb{Z}}$ or in the interval $(0, N + 1)_{\mathbb{Z}}$ for the principal solution \bar{X} of (T) at $N + 1$. This principal solution is given by the initial conditions $\bar{X}_{N+1} = 0$ and $\bar{X}_N = S_N^{T-1}$. In this respect, the results presented in Sections 5 and 6 can be transformed into this theory of backward focal points. However, this direction will not be pursued further on in this paper. Some recent results for symplectic systems (S) related to this notion of backward focal points can be found in [11–13, 29].

Remark 7.2. Based on the equivalence of the symmetric three-term recurrence equation (T) and Jacobi equation (J) in Proposition 2.4 and Corollary 3.3, *all the results* presented in Sections 4–6 for equations (T) *remain valid* also for Jacobi equations (J), for which the invertibility of $S_k := R_k + Q_k^T$ is assumed (as in Notation 2.2), while the invertibility of R_k is not imposed.

Remark 7.3. In [6, Corollaries 5.11 and 5.12], we showed the equivalence of the Jacobi equation (J) and the Jacobi equation

$$\Delta \left[\left(\underline{R}_k - \underline{Q}_k \right) \Delta x_k + \left(\underline{Q}_k^T - \underline{P}_k \right) x_k \right] = \underline{P}_k x_k + \underline{Q}_k \Delta x_k, \quad k \in [0, N - 1]_{\mathbb{Z}}. \quad (\underline{J})$$

Both Jacobi equations (J) and (J) arise in the discrete calculus of variations problems—(J) in problems with shift x_{k+1} in the state variable, while (J) in problems with no shift x_k in the state variable; see [30]. Therefore, all the results in this paper are valid also for the Jacobi equations of the type (J). In this setting, it is assumed that the matrices $\underline{S}_k := \underline{R}_k - \underline{Q}_k$ are invertible.

For completeness, the transformations between the Jacobi equations (J) and (J) are displayed in Corollaries 7.5 and 7.6 below. For details see [6, Corollaries 5.11 and 5.12].

Notation 7.4 (Jacobi equation (J)). The matrices $\underline{P}_k, \underline{Q}_k, \underline{R}_k$, and the vectors x_k in (J) have the following properties: $\underline{P}_k, \underline{Q}_k, \underline{R}_k \in \mathbb{R}^{n \times n}$ are defined on $[0, N]_{\mathbb{Z}}$, \underline{P}_k and \underline{R}_k are symmetric, and the matrix $\underline{S}_k := \underline{R}_k - \underline{Q}_k$ is invertible; $x_k \in \mathbb{R}^n$ are defined on $[0, N + 1]_{\mathbb{Z}}$.

Corollary 7.5 (Jacobi (J) to Jacobi (J)). Assume that \underline{P}_k , \underline{Q}_k , \underline{R}_k , and \underline{S}_k satisfy the conditions in Notation 7.4. Then the Jacobi equation (J) is the Jacobi equation (J), whose coefficients

$$P_k := \underline{P}_k, \quad Q_k := \underline{Q}_k - \underline{P}_k, \quad R_k := \underline{R}_k - \underline{Q}_k - \underline{Q}_k^T + \underline{P}_k \quad (7.1)$$

with $S_k = \underline{S}_k$ satisfy the conditions in Notation 2.2.

Corollary 7.6 (Jacobi (J) to Jacobi (J)). Assume that P_k , Q_k , R_k , and S_k satisfy the conditions in Notation 2.2. Then the Jacobi equation (J) is the Jacobi equation (J), whose coefficients

$$\underline{P}_k := P_k, \quad \underline{Q}_k := P_k + Q_k, \quad \underline{R}_k := R_k + Q_k + Q_k^T + P_k \quad (7.2)$$

with $\underline{S}_k = S_k$ satisfy the conditions in Notation 7.4.

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