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Research Article

Further Extending Results of Some Classes of Complex Difference and Functional Equations

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The main purpose of this paper is to present some properties of the meromorphic solutions of complex difference equation of the form $\sum_{\lambda\in I}\alpha_\lambda(z)(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda,\nu}})/\sum_{\mu\in J}\beta_\mu(z)(\prod_{\nu=1}^{n}f(z+c_\nu)^{m_{\mu,\nu}})=R(z,f(z))$, where $I=\{\lambda=(l_{\lambda,1},l_{\lambda,2},\ldots,l_{\lambda,n})\mid l_{\lambda,\nu}\in\mathbb{N}\cup\{0\},\ \nu=1,2,\ldots,n\}$ and $J=\{\mu=(m_{\mu,1},m_{\mu,2},\ldots,m_{\mu,n})\mid m_{\mu,\nu}\in\mathbb{N}\cup\{0\},\ \nu=1,2,\ldots,n\}$ are two finite index sets, c_ν $(\nu=1,2,\ldots,n)$ are distinct, nonzero complex numbers, $\alpha_\lambda(z)$ $(\lambda\in I)$ and $\beta_\mu(z)$ $(\mu\in J)$ are small functions relative to $f(z)$, $R(z,f(z))$ is a rational function in $f(z)$ with coefficients which are small functions of $f(z)$. We also consider related complex functional equations in the paper.

1. Introduction and Main Results

Let $f(z)$ be a meromorphic function in the complex plane. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, proximity function $m(r,f)$, counting function $N(r,f)$, the first and second main theorems (see, e.g., [1–4]). We also use $\overline{N}(r,f)$ to denote the counting function of the poles of $f(z)$ whose every pole is counted only once. The notation $S(r,f)$ denotes any quantity that satisfies the condition: $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside an exceptional set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ if and only if $T(r,a(z))=S(r,f)$.

Recently, a number of papers (see, e.g., [5–9]) focusing on Malmquist type theorem of the complex difference equations emerged. In 2000, Ablowitz et al. [5] proved some results on the classical Malmquist theorem of the complex difference equations in the complex differential equation by utilizing Nevanlinna theory. They obtained the following two results.

Theorem A. *If the second-order difference equation*

$$f(z+1)+f(z-1)=R(z,f(z))=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{1.1}$$

with polynomial coefficients a_i ($i=1,2,\dots,p$) and b_j ($j=1,2,\dots,q$), admits a transcendental meromorphic solution of finite order, then $d=\max\{p,q\}\le 2$.

Theorem B. *If the second-order difference equation*

$$f(z+1)f(z-1)=R(z,f(z))=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{1.2}$$

with polynomial coefficients a_i ($i=1,2,\dots,p$) and b_j ($j=1,2,\dots,q$), admits a transcendental meromorphic solution of finite order, then $d=\max\{p,q\}\le 2$.

One year later, Heittokangas et al. [7] extended the above two results to the case of higher-order difference equations of more general type. They got the following.

Theorem C. *Let $c_1,c_2,\dots,c_n\in\mathbb{C}\setminus\{0\}$. If the difference equation*

$$\sum_{i=1}^{n}f(z+c_i)=R(z,f(z))=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{1.3}$$

with the coefficients of rational functions a_i ($i=1,2,\dots,p$) and b_j ($j=1,2,\dots,q$) admits a transcendental meromorphic solution of finite order, then $d=\max\{p,q\}\le n$.

Theorem D. *Let $c_1,c_2,\dots,c_n\in\mathbb{C}\setminus\{0\}$. If the difference equation*

$$\prod_{i=1}^{n}f(z+c_i)=R(z,f(z))=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{1.4}$$

with the coefficients of rational functions a_i ($i=1,2,\dots,p$) and b_j ($j=1,2,\dots,q$) admits a transcendental meromorphic solution of finite order, then $d=\max\{p,q\}\le n$.

Laine et al. [9] and Huang and Chen [8], respectively, generalized the above results. They obtained the following theorem.

Theorem E. *Let $c_1,c_2,\dots,c_n$ be distinct, nonzero complex numbers, and suppose that $f(z)$ is a transcendental meromorphic solution of the difference equation*

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f(z+c_j)\right)=R(z,f(z))=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_q(z)f(z)^q}, \tag{1.5}$$

with coefficients $\alpha_J(z), a_i(z)$ $(i = 0, 1, \ldots, p)$ *and* $b_j(z)$ $(j = 0, 1, \ldots, q)$, *which are small functions relative to* $f(z)$, *where* $\{J\}$ *is a collection of all subsets of* $\{1, 2, \ldots, n\}$. *If the order* $\rho(f)$ *is finite, then* $d = \max\{p, q\} \leq n$.

In the same paper, Laine et al. also obtained Tumura-Clunie theorem about difference equation.

Theorem F. *Suppose that* $c_1, c_2, \ldots, c_n$ *are distinct, nonzero complex numbers and that* $f(z)$ *is a transcendental meromorphic solution of*

$$\sum_{j=1}^{n} \alpha_j(z) f(z + c_j) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{1.6}$$

where the coefficients $\alpha_j(z)$ *are nonvanishing small functions relative to* $f(z)$ *and where* $P(z, f(z))$ *and* $Q(z, f(z))$ *are relatively prime polynomials in* $f(z)$ *over the field of small functions relative to* $f(z)$. *Moreover, we assume that* $q = \deg_f Q > 0$,

$$n = \max\{p, q\} = \max\left\{\deg_f P, \deg_f Q\right\}, \tag{1.7}$$

and that, without restricting generality, Q *is a monic polynomial. If there exists* $\alpha \in [0, n)$ *such that for all* r *sufficiently large,*

$$\overline{N}\left(r, \sum_{j=1}^{n} \alpha_j(z) f(z + c_j)\right) \leq \alpha \overline{N}(r + C, f(z)) + S(r, f), \tag{1.8}$$

where $C := \max\{|c_1|, |c_2|, \ldots, |c_n|\}$, *then either the order* $\rho(f) = +\infty$, *or*

$$Q(z, f(z)) \equiv (f(z) + h(z))^q, \tag{1.9}$$

where $h(z)$ *is a small meromorphic function relative to* $f(z)$.

Remark 1.1. Huang and Chen [8] proved that the Theorem F remains true when the left hand side of (1.6) is replaced by the left hand side of (1.5), meanwhile, the condition (1.8) would be replaced by a corresponding form.

Moreover, Laine et al. [9] also gave the following result.

Theorem G. *Suppose that* f *is a transcendental meromorphic solution of*

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z + c_j)\right) = f(p(z)), \tag{1.10}$$

where $p(z)$ is a polynomial of degree $k \geq 2$, $\{J\}$ is a collection of all subsets of $\{1, 2, \ldots, n\}$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to f and that $n \geq k$. Then

$$T(r, f) = O\left((\log r)^{\alpha+\varepsilon}\right), \tag{1.11}$$

where $\alpha = \log n / \log k$.

In this paper, we consider a more general class of complex difference equations. We prove the following results, which generalize the above related results.

Theorem 1.2. *Let $c_1, c_2, \ldots, c_n$ be distinct, nonzero complex numbers and suppose that $f(z)$ is a transcendental meromorphic solution of the difference equation*

$$\frac{\sum_{\lambda \in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z + c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu \in J} \beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z + c_\nu)^{m_{\mu,\nu}}\right)} = R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_q(z)f(z)^q}, \tag{1.12}$$

with coefficients $\alpha_\lambda(z)(\lambda \in I)$, $\beta_\mu(z)(\mu \in J)$, $a_i(z)$ $(i = 0, 1, \ldots, p)$, and $b_j(z)$ $(j = 0, 1, \ldots, q)$ are small functions relative to $f(z)$, where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ are two finite index sets, denote

$$\sigma_\nu = \max_{\lambda,\mu}\{l_{\lambda,\nu}, m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \ \ \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{1.13}$$

If the order $\rho(f) := \rho$ is finite, then $d = \max\{p, q\} \leq \sigma$.

Corollary 1.3. *Let $c_1, c_2, \ldots, c_n$ be distinct, nonzero complex numbers and suppose that $f(z)$ is a transcendental meromorphic solution of the difference equation*

$$\sum_{\lambda \in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z + c_\nu)^{l_{\lambda,\nu}}\right) = R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_q(z)f(z)^q}, \tag{1.14}$$

with coefficients $\alpha_\lambda(z)(\lambda \in I)$, $a_i(z)$ $(i = 0, 1, \ldots, p)$ and $b_j(z)$ $(j = 0, 1, \ldots, q)$, which are small functions relative to $f(z)$, where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\},\ \nu = 1, 2, \ldots, n\}$ is a finite index set, denote

$$\sigma_\nu = \max_{\lambda}\{l_{\lambda,\nu}\} \quad (\nu = 1, 2, \ldots, n), \quad \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{1.15}$$

If the order $\rho(f) := \rho$ is finite, then $d = \max\{p, q\} \leq \sigma$.

Remark 1.4. In Corollary 1.3, if we take

$$\max_{\lambda,\nu}\{l_{\lambda,\nu}\} = 1, \quad \lambda \in I,\ \nu = 1,2,\ldots,n, \tag{1.16}$$

then Corollary 1.3 becomes Theorem E. Therefore, Theorem 1.2 is a generalization of Theorem E.

Example 1.5. Let $c_1 = \arctan 2,\ c_2 = -\pi/4$. Then it is easy to check that $f(z) = \tan z$ solves the following difference equation:

$$\frac{f(z+c_1)^2 f(z+c_2)}{f(z+c_1) + f(z+c_2)^2} = \frac{f^4 + 4f^3 + 3f^2 - 4f - 4}{2f^4 - 19f^3 + 7f^2 - 5f + 3}. \tag{1.17}$$

Example 1.6. Let $c_1 = \arctan 2$ and $c_2 = \arctan(-2)$. It is easy to check that $f(z) = \tan z$ satisfies the difference equation

$$f(z+c_1)^2 f(z+c_2) + f(z+c_1) f(z+c_2)^2 = \frac{10f^3 - 40f}{16f^4 - 8f^2 + 1}. \tag{1.18}$$

In above two examples, we both have $d = \sigma = 4$ and $\rho(f) = 1 < +\infty$. Therefore, the estimations in Theorem 1.2 and Corollary 1.3 are sharp.

Theorem 1.7. *Suppose that $c_1, c_2, \ldots, c_n$ are distinct, nonzero complex numbers and that $f(z)$ is a transcendental meromorphic solution of*

$$\frac{\sum_{\lambda\in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J} \beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{m_{\mu,\nu}}\right)} = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{1.19}$$

where the coefficients $\alpha_\lambda(z)(\lambda \in I)$, $\beta_\mu(z)(\mu \in J)$ are nonvanishing small functions relative to $f(z)$ and $P(z, f(z))$ and $Q(z, f(z))$ are relatively prime polynomials in $f(z)$ over the field of small functions relative to $f(z)$, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N}\cup\{0\}, \nu = 1,2,\ldots,n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N}\cup\{0\},\ \nu = 1,2,\ldots,n\}$ are two finite index sets, denote

$$\sigma_\nu = \max_{\lambda,\mu}\left\{l_{\lambda,\nu}, m_{\mu,\nu}\right\} \quad (\nu = 1,2,\ldots,n), \quad \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{1.20}$$

Moreover, we assume that $q = \deg_f Q > 0$,

$$\sigma = \max\{p, q\} := \max\left\{\deg_f P, \deg_f Q\right\}, \tag{1.21}$$

and that, without restricting generality, Q is a monic polynomial. If there exists $\alpha \in [0, \sigma)$ such that for all r sufficiently large,

$$\overline{N}\left(r, \frac{\sum_{\lambda \in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu \in J} \beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{m_{\mu,\nu}}\right)}\right) \leq \alpha \overline{N}(r+C, f(z)) + S(r,f), \tag{1.22}$$

$$\sum_{\nu=1}^{n} \sigma_\nu \overline{N}(r, f(z+c_\nu)) \leq \alpha \overline{N}(r+C, f(z)) + S(r,f), \tag{1.23}$$

where $C := \max\{|c_1|, |c_2|, \ldots, |c_n|\}$, then either the order $\rho(f) = +\infty$, or

$$Q(z, f(z)) \equiv (f(z) + h(z))^q, \tag{1.24}$$

where $h(z)$ is a small meromorphic function relative to $f(z)$.

If the left hand side of (1.19) in Theorem 1.7 is replaced by the left hand side of (1.14) in Corollary 1.3, then (1.23) implies (1.22). Since we have

$$\overline{N}\left(r, \sum_{\lambda \in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right)\right) \leq \sum_{\nu=1}^{n} \sigma_\nu \overline{N}(r, f(z+c_\nu)) + S(r,f) \tag{1.25}$$

by the fundamental property of counting function. Therefore, we get the following result easily.

Corollary 1.8. *Suppose that $c_1, c_2, \ldots, c_n$ are distinct, nonzero complex numbers and that $f(z)$ is a transcendental meromorphic solution of*

$$\sum_{\lambda \in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{1.26}$$

where the coefficients $\alpha_\lambda(z) (\lambda \in I)$ are nonvanishing small functions relative to $f(z)$ and $P(z, f(z))$ and $Q(z, f(z))$ are relatively prime polynomials in $f(z)$ over the field of small functions relative to $f(z)$, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ is a finite index set, denote

$$\sigma_\nu = \max_{\lambda}\{l_{\lambda,\nu}\} \quad (\nu = 1, 2, \ldots, n), \ \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{1.27}$$

Moreover, we assume that $q = \deg_f Q > 0$,

$$\sigma = \max\{p, q\} := \max\left\{\deg_f P, \deg_f Q\right\}, \tag{1.28}$$

and that, without restricting generality, Q is a monic polynomial. If there exists $\alpha \in [0, \sigma)$ such that for all r sufficiently large,

$$\sum_{\nu=1}^{n} \sigma_\nu \overline{N}(r, f(z + c_\nu)) \leq \alpha \overline{N}(r + C, f(z)) + S(r, f), \tag{1.29}$$

where $C := \max\{|c_1|, |c_2|, \ldots, |c_n|\}$, then either the order $\rho(f) = +\infty$, or

$$Q(z, f(z)) \equiv (f(z) + h(z))^q, \tag{1.30}$$

where $h(z)$ is a small meromorphic function relative to $f(z)$.

Finally, we give a result corresponding to Theorem G.

Theorem 1.9. *Let $c_1, c_2, \ldots, c_n$ be distinct, nonzero complex numbers and suppose that f is a transcendental meromorphic solution of*

$$\frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{\nu=1}^{n} f(z + c_\nu)^{l_{\lambda,\nu}} \right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{\nu=1}^{n} f(z + c_\nu)^{m_{\mu,\nu}} \right)} = f(p(z)), \tag{1.31}$$

where $p(z)$ is a polynomial of degree $k \geq 2$, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ are two finite index sets. Denote

$$\sigma_\nu = \max_{\lambda,\mu} \{l_{\lambda,\nu}, m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \ \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{1.32}$$

Moreover, we assume that the coefficients $\alpha_\lambda(z)(\lambda \in I)$ and $\beta_\mu(z)(\mu \in J)$ are small functions relative to f and that $\sigma \geq k$. Then

$$T(r, f) = O\left((\log r)^{\alpha + \varepsilon} \right), \tag{1.33}$$

where $\alpha = \log \sigma / \log k$.

2. Main Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 (see [10]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j}, \tag{2.1}$$

such that the meromorphic coefficients $a_i(z), b_j(z)$ *satisfy*

$$T(r, a_i) = S(r, f), \quad i = 0, 1, \ldots, p,$$
$$T(r, b_j) = S(r, f), \quad j = 0, 1, \ldots, q, \tag{2.2}$$

one has

$$T(r, R(z, f)) = \max\{p, q\} \cdot T(r, f) + S(r, f). \tag{2.3}$$

Lemma 2.2 (see [11]). *Let* $f_1, f_2, \ldots, f_n$ *be distinct meromorphic functions and*

$$F(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{\lambda \in I} f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}}{\sum_{\mu \in J} f_1^{m_{\mu,1}} f_2^{m_{\mu,2}} \cdots f_n^{m_{\mu,n}}}. \tag{2.4}$$

Then

$$m(r, F) \le \sum_{\nu=1}^{n} \sigma_\nu m(r, f_\nu) + N(r, Q) - N\left(r, \frac{1}{Q}\right) + O(1),$$
$$T(r, F) \le \sum_{\nu=1}^{n} \sigma_\nu T(r, f_\nu) + O(1), \tag{2.5}$$

where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ *and* $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \cup \{0\}, \nu = 1, 2, \ldots, n\}$ *are two finite index sets, and* $\sigma_\nu = \max_{\lambda,\mu}\{l_{\lambda,\nu}, m_{\mu,\nu}\}$ $(\nu = 1, 2, \ldots, n)$.

Remark 2.3. If we suppose that $\alpha_\lambda(z) = o(T(r, f_\nu)(\lambda \in I))$ and $\beta_\mu(z) = o(T(r, f_\nu)(\mu \in J))$ hold for all $\nu \in \{1, 2, \ldots, n\}$, and denote $T(r, \alpha_\lambda) = S(r, f)(\lambda \in I)$ and $T(r, \beta_\mu) = S(r, f)(\mu \in J)$, then we have the following estimation by the proof of Lemma 2.2

$$T\left(r, \frac{\sum_{\lambda \in I} \alpha_\lambda(z) f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}}{\sum_{\mu \in J} \beta_\mu(z) f_1^{m_{\mu,1}} f_2^{m_{\mu,2}} \cdots f_n^{m_{\mu,n}}}\right) \le \sum_{\nu=1}^{n} \sigma_\nu T(r, f_\nu) + S(r, f). \tag{2.6}$$

Lemma 2.4 (see [6]). *Let* $f(z)$ *be a meromorphic function with order* $\rho = \rho(f)$, $\rho < +\infty$, *and let* c *be a fixed nonzero complex number, then for each* $\varepsilon > 0$, *one has*

$$T(r, f(z+c)) = T(r, f) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r). \tag{2.7}$$

Lemma 2.5 (see [12]). *Let $f(z)$ be a meromorphic function and let ϕ be given by*

$$\phi = f^n + a_{n-1}f^{n-1} + \cdots + a_0, \tag{2.8}$$

where $a_i(i = 0, 1, \ldots, n-1)$ are small meromorphic functions relative to $f(z)$. Then either

$$\phi = \left(f + \frac{a_{n-1}}{n}\right)^n, \tag{2.9}$$

or

$$T(r, f) \le \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}(r, f) + S(r, f). \tag{2.10}$$

Lemma 2.6 (see [9, 13]). *Let $f(z)$ be a nonconstant meromorphic function and let $P(z, f)$, $Q(z, f)$ be two polynomials in $f(z)$ with meromorphic coefficients small relative to $f(z)$. If $P(z, f)$ and $Q(z, f)$ have no common factors of positive degree in $f(z)$ over the field of small functions relative to $f(z)$, then*

$$\overline{N}\left(r, \frac{1}{Q(z, f)}\right) \le \overline{N}\left(r, \frac{P(z, f)}{Q(z, f)}\right) + S(r, f). \tag{2.11}$$

Lemma 2.7 (see [14]). *Let f be a transcendental meromorphic function, and $p(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0, a_k \ne 0$ be a nonconstant polynomial of degree k. Given $0 < \delta < |a_k|$, denote $\lambda = |a_k| + \delta$ and $\mu = |a_k| - \delta$. Then given $\varepsilon > 0$ and $a \in \mathbb{C} \cup \{\infty\}$, one has*

$$\begin{gathered}
kn\left(\mu r^k, a, f\right) \le n(r, a, f(p(z))) \le kn\left(\lambda r^k, a, f\right), \\
N\left(\mu r^k, a, f\right) + O(\log r) \le N(r, a, f(p(z))) \le N\left(\lambda r^k, a, f\right) + O(\log r), \\
(1-\varepsilon)T\left(\mu r^k, f\right) \le T(r, f(p(z))) \le (1+\varepsilon)T\left(\lambda r^k, f\right)
\end{gathered} \tag{2.12}$$

for all r large enough.

Lemma 2.8 (see [15]). *Let $\phi : [r_0, +\infty) \to (0, +\infty)$ be positive and bounded in every finite interval, and suppose that $\phi(\mu r^m) \le A\phi(r) + B$ holds for all r large enough, where $\mu > 0, m > 1, A > 1$ and B are real constants. Then*

$$\phi(r) = O((\log r)^{\alpha}), \tag{2.13}$$

where $\alpha = \log A / \log m$.

3. Proof of Theorems

Proof of Theorem 1.2. We assume that $f(z)$ is a meromorphic solution of finite order of (1.12). It follows from Lemmas 2.1, 2.2, and 2.4 that for each $\varepsilon > 0$,

$$
\begin{aligned}
\max\{p,q\}T(r,f) &= T(r,R(z,f)) + S(r,f) \\
&= T\left(r, \frac{\sum_{\lambda\in I}\alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J}\beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{m_{\mu,\nu}}\right)}\right) + S(r,f) \\
&\le \sum_{\nu=1}^{n}\sigma_\nu T(r,f(z+c_\nu)) + S(r,f) + O(1) \\
&= \sum_{\nu=1}^{n}\sigma_\nu T(r,f(z)) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r) + S(r,f) \\
&= \left(\sum_{\nu=1}^{n}\sigma_\nu\right) T(r,f(z)) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r) + S(r,f) \\
&= \sigma T(r,f(z)) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r) + S(r,f).
\end{aligned}
\tag{3.1}
$$

This yields the asserted result. □

Proof of Theorem 1.7. Suppose $f(z)$ is a transcendental meromorphic solution of (1.19) and the second alternative of the conclusion is not true. Then according to Lemmas 2.5 and 2.6, we get

$$
\begin{aligned}
T(r,f) &\le \overline{N}\left(r, \frac{1}{Q(z,f(z))}\right) + \overline{N}(r,f) + S(r,f) \\
&\le \overline{N}\left(r, \frac{P(z,f(z))}{Q(z,f(z))}\right) + \overline{N}(r,f) + S(r,f) \\
&= \overline{N}\left(r, \frac{\sum_{\lambda\in I}\alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J}\beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{m_{\mu,\nu}}\right)}\right) + \overline{N}(r,f) + S(r,f) \\
&\le \alpha\overline{N}(r+C,f(z)) + \overline{N}(r,f) + S(r,f).
\end{aligned}
\tag{3.2}
$$

Thus, we have

$$
T(r,f) - \overline{N}(r,f) \le \alpha\overline{N}(r+C,f(z)) + S(r,f), \tag{3.3}
$$

Now assuming the order $\rho(f) < +\infty$, then we have $S(r, f(z+c_\nu)) = S(r, f)$ and

$$T(r, f(z+c_\nu)) - \overline{N}(r, f(z+c_\nu)) \le \alpha \overline{N}(r+C, f(z+c_\nu)) + S(r, f). \tag{3.4}$$

for all $\nu = 1, 2, \ldots, n$. By using Lemmas 2.1 and 2.2, we conclude that

$$\begin{aligned}
\sigma T(r, f) &= T\left(r, \frac{\sum_{\lambda\in I} \alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J} \beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{m_{\mu,\nu}}\right)}\right) + S(r, f) \\
&\le \sum_{\nu=1}^{n} \sigma_\nu T(r, f(z+c_\nu)) + S(r, f) \\
&= \sum_{\nu=1}^{n} \sigma_\nu \Big(T(r, f(z+c_\nu)) - \overline{N}(r, f(z+c_\nu))\Big) + \sum_{\nu=1}^{n} \sigma_\nu \overline{N}(r, f(z+c_\nu)) + S(r, f) \\
&\le \sum_{\nu=1}^{n} \sigma_\nu \alpha \overline{N}(r+C, f(z+c_\nu)) + \alpha \overline{N}(r+C, f(z)) + S(r, f) \\
&\le \sum_{\nu=1}^{n} \sigma_\nu \alpha \overline{N}(r+2C, f(z)) + \alpha \overline{N}(r+C, f(z)) + S(r, f) \\
&\le \left(\sum_{\nu=1}^{n} \sigma_\nu\right) \alpha \overline{N}(r+2C, f(z)) + \alpha \overline{N}(r+2C, f(z)) + S(r, f) \\
&= (\sigma+1)\alpha \overline{N}(r+2C, f(z)) + S(r, f).
\end{aligned} \tag{3.5}$$

It follows from this that

$$T(r, f) - \overline{N}(r, f) \le \frac{\sigma+1}{\sigma} \alpha \overline{N}(r+2C, f) - \overline{N}(r, f) + S(r, f). \tag{3.6}$$

We prove the following inequality by induction:

$$T(r, f) - \overline{N}(r, f) \le \frac{\sigma+m}{\sigma} \alpha \overline{N}(r+2mC, f) - m\overline{N}(r, f) + S(r, f). \tag{3.7}$$

The case $m = 1$ has been proved. We assume that above inequality holds when $m = k$. Next, we prove that inequality (3.7) holds for $m = k + 1$. We have

$$
\begin{aligned}
\sigma T(r, f) &\le \sum_{\nu=1}^{n} \sigma_\nu \Big(T(r, f(z + c_\nu)) - \overline{N}(r, f(z + c_\nu)) \Big) + \alpha \overline{N}(r + C, f(z)) + S(r, f) \\
&\le \sum_{\nu=1}^{n} \sigma_\nu \left(\frac{\sigma + k}{\sigma} \alpha \overline{N}(r + 2kC, f(z + c_\nu)) - k\overline{N}(r, f(z + c_\nu)) \right) \\
&\quad + \alpha \overline{N}(r + C, f(z)) + S(r, f) \\
&\le \sum_{\nu=1}^{n} \sigma_\nu \left(\frac{\sigma + k}{\sigma} \alpha \overline{N}(r + 2kC + C, f(z)) - k\overline{N}(r - C, f(z)) \right) \\
&\quad + \alpha \overline{N}(r + C, f(z)) + S(r, f) \\
&\le \left(\sum_{\nu=1}^{n} \sigma_\nu \right) \left(\frac{\sigma + k}{\sigma} \alpha \overline{N}(r + 2kC + C, f(z)) - k\overline{N}(r - C, f(z)) \right) \\
&\quad + \alpha \overline{N}(r + 2kC + C, f(z)) + S(r, f) \\
&= (\sigma + k + 1)\alpha \overline{N}(r + 2kC + C, f(z)) - \sigma k \overline{N}(r - C, f(z)) + S(r, f).
\end{aligned}
\tag{3.8}
$$

Noting that $T(r, f(z)) \le T(r + C, f(z))$, thus we have

$$
\begin{aligned}
\sigma T(r, f(z)) &\le \sigma T(r + C, f(z)) \\
&\le (\sigma + k + 1)\alpha \overline{N}(r + 2kC + 2C, f(z)) - \sigma k \overline{N}(r, f(z)) + S(r, f)
\end{aligned}
\tag{3.9}
$$

and so

$$
T(r, f(z)) \le \frac{\sigma + k + 1}{\sigma} \alpha \overline{N}(r + 2(k + 1)C, f(z)) - k\overline{N}(r, f(z)) + S(r, f). \tag{3.10}
$$

This implies that

$$
T(r, f(z)) - \overline{N}(r, f(z)) \le \frac{\sigma + k + 1}{\sigma} \alpha \overline{N}(r + 2(k + 1)C, f(z)) - (k + 1)\overline{N}(r, f(z)) + S(r, f). \tag{3.11}
$$

It follows from (3.7) that

$$
\overline{N}(r, f(z)) \le \frac{\sigma + m}{\sigma m} \alpha \overline{N}(r + 2mC, f) + S(r, f). \tag{3.12}
$$

Let m be large enough such that

$$
\frac{1}{\gamma} := \frac{\sigma + m}{\sigma m} \alpha = \left(\frac{1}{m} + \frac{1}{\sigma} \right) \alpha < 1. \tag{3.13}
$$

Since

$$\overline{N}(r, f(z)) \leq \frac{1}{\gamma}\overline{N}(r + 2mC, f(z)) + S(r, f), \tag{3.14}$$

we have for any $s \in \mathbb{N}$,

$$\overline{N}(r, f(z)) \leq \frac{1}{\gamma^s}\overline{N}(r + 2smC, f) + S(r + (s-1)mC, f) \tag{3.15}$$

thus for each $\varepsilon > 0$,

$$\begin{aligned} \gamma^s\overline{N}(r, f(z)) &\leq \overline{N}(r + 2smC, f) + S(r + (s-1)mC, f) \\ &\leq (1+\varepsilon)T(r + 2smC, f(z)), \end{aligned} \tag{3.16}$$

for $r + 2smC$ large enough holds. We now fix $r = r_0$, and let $r_0 + 2smC = t$, thus

$$\begin{gathered} \gamma^{(t-r_0)/2mC}\overline{N}(r_0, f(z)) \leq (1+\varepsilon)T(t, f), \\ \frac{\log T(t, f)}{\log t} + \frac{\log(1+\varepsilon)}{\log t} \geq \frac{t\log\gamma}{2mC\log t} - \frac{r_0\log\gamma}{2mC\log t} + \frac{\log\overline{N}(r_0, f)}{\log t}. \end{gathered} \tag{3.17}$$

Finally, let $t \to \infty$, and we conclude that the order $\rho(f) = \infty$. Therefore, we get a contradiction and the assertion follows. □

Proof of Theorem 1.9. We assume $f(z)$ is a transcendental meromorphic solution of (1.31). Denoting again $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. According to the last assertion of Lemmas 2.7 and 2.2, we get that

$$\begin{aligned} (1-\varepsilon)T\left(\mu r^k, f\right) &\leq T(r, f(p(z))) \\ &= T\left(r, \frac{\sum_{\lambda\in I}\alpha_\lambda(z)\left(\prod_{\nu=1}^n f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J}\beta_\mu(z)\left(\prod_{\nu=1}^n f(z+c_\nu)^{m_{\mu,\nu}}\right)}\right) \\ &\leq \sum_{\nu=1}^n \sigma_\nu T(r, f(z+c_\nu)) + S(r, f) \\ &\leq \sum_{\nu=1}^n \sigma_\nu T(r + C, f(z)) + S(r, f) \\ &= \left(\sum_{\nu=1}^n \sigma_\nu\right)T(r + C, f(z)) + S(r, f) \\ &= \sigma T(r + C, f(z)) + S(r, f). \end{aligned} \tag{3.18}$$

Since $T(r+C,f) \leq T(\beta r,f)$ holds for r large enough for $\beta > 1$, we may assume r to be large enough to satisfy

$$(1-\varepsilon)T\left(\mu r^k, f\right) \leq \sigma(1+\varepsilon)T(\beta r, f) \tag{3.19}$$

outside a possible exceptional set of finite linear measure. By the standard idea of removing the exceptional set (see [4, page 5]), we know that whenever $\gamma > 1$,

$$(1-\varepsilon)T\left(\mu r^k, f\right) \leq \sigma(1+\varepsilon)T(\gamma\beta r, f) \tag{3.20}$$

holds for all r large enough. Denote $t = \gamma\beta r$, thus inequality (3.20) may be written in the form

$$T\left(\frac{\mu}{(\gamma\beta)^k}t^k, f\right) \leq \frac{\sigma(1+\varepsilon)}{1-\varepsilon}T(t,f). \tag{3.21}$$

By Lemma 2.8, we have

$$T(r,f) = O\left((\log r)^s\right), \tag{3.22}$$

where

$$s = \frac{\log(\sigma(1+\varepsilon)/(1-\varepsilon))}{\log k} = \frac{\log\sigma}{\log k} + o(1). \tag{3.23}$$

Denoting now $\alpha = \log\sigma/\log k$, thus we obtain the required form. Theorem 1.9 is proved. □

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