Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 389109, 21 pages
doi:10.1155/2010/389109

Research Article

Oscillation of Second-Order Mixed-Nonlinear Delay Dynamic Equations

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Received 19 January 2010; Accepted 20 March 2010

Academic Editor: Josef Diblik

New oscillation criteria are established for second-order mixed-nonlinear delay dynamic equations on time scales by utilizing an interval averaging technique. No restriction is imposed on the coefficient functions and the forcing term to be nonnegative.

1. Introduction

In this paper we are concerned with oscillatory behavior of the second-order nonlinear delay dynamic equation of the form

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p_0(t)x(\tau_0(t)) + \sum_{i=1}^{n} p_i(t)|x(\tau_i(t))|^{\alpha_i-1}x(\tau_i(t)) = e(t), \quad t \geq t_0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0, \quad (n > m \geq 1); \tag{1.2}$$

the functions $r, p_i, e: \mathbb{T} \to \mathbb{R}$ are right-dense continuous with $r > 0$ nondecreasing; the delay functions $\tau_i : \mathbb{T} \to \mathbb{T}$ are nondecreasing right-dense continuous and satisfy $\tau_i(t) \leq t$ for $t \in \mathbb{T}$ with $\tau_i(t) \to \infty$ as $t \to \infty$.

We assume that the time scale $\mathbb{T}$ is unbounded above, that is, $\sup \mathbb{T} = \infty$ and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. It is also assumed that the reader is already familiar with the time scale calculus. A comprehensive treatment of calculus on time scales can be found in [1–3].

By a solution of (1.1) we mean a nontrivial real valued function $x : \mathbb{T} \to \mathbb{R}$ such that $x \in C^1_{rd}[T,\infty)_{\mathbb{T}}$ and $rx^{\Delta} \in C^1_{rd}[T,\infty)_{\mathbb{T}}$ for all $T \in \mathbb{T}$ with $T \geq t_0$, and that x satisfies (1.1). A function x is called an oscillatory solution of (1.1) if x is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if and only if every solution x of (1.1) is oscillatory.

Notice that when $\mathbb{T} = \mathbb{R}$, (1.1) is reduced to the second-order nonlinear delay differential equation

$$\left(r(t)x'(t)\right)' + p_0(t)x(\tau_0(t)) + \sum_{i=1}^{n} p_i(t)|x(\tau_i(t))|^{\alpha_i-1}x(\tau_i(t)) = e(t), \quad t \geq t_0 \tag{1.3}$$

while when $\mathbb{T} = \mathbb{Z}$, it becomes a delay difference equation

$$\Delta(r(k)\Delta x(k)) + p_0(k)x(\tau_0(k)) + \sum_{i=1}^{n} p_i(k)|x(\tau_i(k))|^{\alpha_i-1}x(\tau_i(k)) = e(k), \quad k \geq k_0. \tag{1.4}$$

Another useful time scale is $\mathbb{T} = q^{\mathbb{N}} := \{q^m : m \in \mathbb{N} \text{ and } q > 1 \text{ is a real number}\}$, which leads to the quantum calculus. In this case, (1.1) is the q-difference equation

$$\Delta_q\left(r(t)\Delta_q x(t)\right) + p_0(t)x(\tau_0(t)) + \sum_{i=1}^{n} p_i(t)|x(\tau_i(t))|^{\alpha_i-1}x(\tau_i(t)) = e(t), \quad t \geq t_0, \tag{1.5}$$

where $\Delta_q f(t) = [f(\sigma(t)) - f(t)]/\mu(t)$, $\sigma(t) = qt$, and $\mu(t) = (q-1)t$.

Interval oscillation criteria are more natural in view of the Sturm comparison theory since it is stated on an interval rather than on infinite rays and hence it is necessary to establish more interval oscillation criteria for equations on arbitrary time scales as in $\mathbb{T} = \mathbb{R}$. As far as we know when $\mathbb{T} = \mathbb{R}$, an interval oscillation criterion for forced second-order linear differential equations was first established by El-Sayed [4]. In 2003, Sun [5] demonstrated nicely how the interval criteria method can be applied to delay differential equations of the form

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = e(t), \quad (\alpha \geq 1), \tag{1.6}$$

where the potential p and the forcing term e may oscillate. Some of these interval oscillation criteria were recently extended to second-order dynamic equations in [6–10]. Further results on oscillatory and nonoscillatory behavior of the second order nonlinear dynamic equations on time scales can be found in [11–23], and the references cited therein.

Therefore, motivated by Sun and Meng's paper [24], using similar techniques introduced in [17] by Kong and an arithmetic-geometric mean inequality, we give oscillation criteria for second-order nonlinear delay dynamic equations of the form (1.1). Examples are considered to illustrate the results.

2. Main Results

We need the following lemmas in proving our results. The first two lemmas can be found in [25, Lemma 1].

Lemma 2.1. *Let* $\{\alpha_i\}$, $i = 1, 2, \ldots, n$ *be the* n*-tuple satisfying* $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$. *Then, there exists an* n*-tuple* $\{\eta_1, \eta_2, \ldots, \eta_n\}$ *satisfying*

$$\sum_{i=1}^{n} \alpha_i \eta_i = 1, \quad \sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1. \tag{2.1}$$

Lemma 2.2. *Let* $\{\alpha_i\}$, $i = 1, 2, \ldots, n$ *be the* n*-tuple satisfying* $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$. *Then there exists an* n*-tuple* $\{\eta_1, \eta_2, \ldots, \eta_n\}$ *satisfying*

$$\sum_{i=1}^{n} \alpha_i \eta_i = 1, \quad \sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.2}$$

The next two lemmas are quite elementary via differential calculus; see [23, 25].

Lemma 2.3. *Let* u, A, *and* B *be nonnegative real numbers. Then*

$$Au^{\gamma} + B \geq \gamma (\gamma - 1)^{1/\gamma - 1} A^{1/\gamma} B^{1 - 1/\gamma} u, \quad \gamma > 1. \tag{2.3}$$

Lemma 2.4. *Let* u, A, *and* B *be nonnegative real numbers. Then*

$$Cu - Du^{\gamma} \geq (\gamma - 1) \gamma^{\gamma/(1-\gamma)} C^{\gamma/(\gamma-1)} D^{1/(1-\gamma)}, \quad 0 < \gamma < 1. \tag{2.4}$$

The last important lemma that we need is a special case of the one given in [6]. For completeness, we provide a proof.

Lemma 2.5. *Let* $\tau : \mathbb{T} \to \mathbb{T}$ *be a nondecreasing right-dense continuous function with* $\tau(t) \leq t$, *and* $a, b \in \mathbb{T}$ *with* $a < b$. *If* $x \in C^1_{rd}[\tau(a), b]_{\mathbb{T}}$ *is a positive function such that* $r(t)x^{\Delta}(t)$ *is nonincreasing on* $[\tau(a), b]_{\mathbb{T}}$ *with* $r > 0$ *nondecreasing, then*

$$\frac{x(\tau(t))}{x^{\sigma}(t)} \geq \frac{\tau(t) - \tau(a)}{\sigma(t) - \tau(a)}, \quad t \in [a, b]_{\mathbb{T}}. \tag{2.5}$$

Proof. By the Mean Value Theorem [2, Theorem 1.14]

$$x(t) - x(\tau(a)) \geq x^{\Delta}(\eta)(t - \tau(a)), \tag{2.6}$$

for some $\eta \in (\tau(a), t)_{\mathbb{T}}$, for any $t \in (\tau(a), b]_{\mathbb{T}}$. Since $r(t)x^{\Delta}(t)$ is nonincreasing and $r(t)$ is nondecreasing, we have

$$r(t)x^{\Delta}(t) \leq r(\eta)x^{\Delta}(\eta) \leq r(t)x^{\Delta}(\eta), \quad t > \eta \tag{2.7}$$

and so $x^{\Delta}(t) \le x^{\Delta}(\eta)$, $t \ge \eta$. Now

$$x(t) - x(\tau(a)) \ge x^{\Delta}(t)(t - \tau(a)), \quad t \in [\tau(a), b]_{\mathbb{T}}. \tag{2.8}$$

Define

$$\mu(s) := x(s) - (s - \tau(a))x^{\Delta}(s), \quad s \in [\tau(t), \sigma(t)]_{\mathbb{T}},\ t \in [a, b)_{\mathbb{T}}. \tag{2.9}$$

It follows from (2.8) that $\mu(s) \ge x(\tau(a)) > 0$ for $s \in [\tau(t), \sigma(t)]_{\mathbb{T}}$ and $t \in [a, b)_{\mathbb{T}}$. Thus, we have

$$0 < \int_{\tau(t)}^{\sigma(t)} \frac{\mu(s)}{x(s)x^{\sigma}(s)} \Delta s = \int_{\tau(t)}^{\sigma(t)} \left(\frac{s - \tau(a)}{x(s)}\right)^{\Delta} \Delta s = \frac{\sigma(t) - \tau(a)}{x^{\sigma}(t)} - \frac{\tau(t) - \tau(a)}{x(\tau(t))}, \tag{2.10}$$

which completes the proof. □

In what follows we say that a function $H(t, s) : \mathbb{T}^2 \to \mathbb{R}$ belongs to $\mathscr{H}_{\mathbb{T}}$ if and only if H is right-dense continuous function on $\{(t, s) \in \mathbb{T}^2 : t \ge s \ge t_0\}$ having continuous Δ-partial derivatives on $\{(t, s) \in \mathbb{T}^2 : t > s \ge t_0\}$, with $H(t, t) = 0$ for all t and $H(t, s) \ne 0$ for all $t \ne s$. Note that in case $\mathscr{H}_{\mathbb{R}}$, the Δ-partial derivatives become the usual partial derivatives of $H(t, s)$. The partial derivatives for the cases $\mathscr{H}_{\mathbb{Z}}$ and $\mathscr{H}_{\mathbb{N}}$ will be explicitly given later.

Denoting the Δ-partial derivatives $H^{\Delta_t}(t, s)$ and $H^{\Delta_s}(t, s)$ of $H(t, s)$ with respect to t and s by $H_1(t, s)$ and $H_2(t, s)$, respectively, the theorems below extend the results obtained in [5] to nonlinear delay dynamic equation on arbitrary time scales and coincide with them when $H^2(t, s)$ is replaced by $H(t, s)$. Indeed, if we set $H(t, s) = \sqrt{U(t, s)}$, then it follows that

$$H_1(t, s) = \frac{U_1(t, s)}{\sqrt{U(\sigma(t), s)} + \sqrt{U(t, s)}}, \qquad H_2(t, s) = \frac{U_2(t, s)}{\sqrt{U(t, \sigma(s))} + \sqrt{U(t, s)}}. \tag{2.11}$$

When $\mathbb{T} = \mathbb{R}$, they become

$$\frac{\partial H(t, s)}{\partial t} = \frac{\partial U(t, s)/\partial t}{2\sqrt{U(t, s)}}, \qquad \frac{\partial H(t, s)}{\partial s} = \frac{\partial U(t, s)/\partial s}{2\sqrt{U(t, s)}} \tag{2.12}$$

as in [5]. However, we prefer using $H^2(t, s)$ instead of $U(t, s)$ for simplicity.

Theorem 2.6. *Suppose that for any given (arbitrarily large) $T \in \mathbb{T}$ there exist subintervals $[a_1, b_1]_{\mathbb{T}}$ and $[a_2, b_2]_{\mathbb{T}}$ of $[T, \infty)_{\mathbb{T}}$, where $a_1 < b_1$ and $a_2 < b_2$ such that*

$$\begin{gathered} p_i(t) \ge 0 \quad \text{for} \quad t \in [\overline{a}_1, b_1]_{\mathbb{T}} \cup [\overline{a}_2, b_2]_{\mathbb{T}},\ (i = 0, 1, 2, \dots, n), \\ (-1)^l e(t) > 0 \quad \text{for } t \in [\overline{a}_l, b_l]_{\mathbb{T}},\ (l = 1, 2), \end{gathered} \tag{2.13}$$

where

$$\overline{a}_l = \min\{\tau_j(a_l) : j = 0, 1, 2, \dots, n\} \tag{2.14}$$

hold. Let $\{\eta_1, \eta_2, \dots \eta_n\}$ *be an* n*-tuple satisfying* (2.1) *of Lemma 2.1. If there exist a function* $H \in \mathscr{H}_{\mathbb{T}}$ *and numbers* $c_\nu \in (a_\nu, b_\nu)_{\mathbb{T}}$ *such that*

$$\begin{aligned} &\frac{1}{H^2(c_\nu, a_\nu)} \int_{a_\nu}^{c_\nu} \left[Q(t)H^2(\sigma(t), a_\nu) - r(t)H_1^2(t, a_\nu)\right]\Delta t \\ &\quad + \frac{1}{H^2(b_\nu, c_\nu)} \int_{c_\nu}^{b_\nu} \left[Q(t)H^2(b_\nu, \sigma(t)) - r(t)H_2^2(b_\nu, t)\right]\Delta t > 0 \end{aligned} \tag{2.15}$$

for $\nu = 1, 2$*, where*

$$\begin{gathered} Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(a_\nu)}{\sigma(t) - \tau_0(a_\nu)} + k_0|e(t)|^{\eta_0}\prod_{i=1}^{n}(p_i(t))^{\eta_i}\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{\sigma(t) - \tau_i(a_\nu)}\right)^{\alpha_i\eta_i}, \\ k_0 = \prod_{i=0}^{n}\eta_i^{-\eta_i}, \qquad \eta_0 = 1 - \sum_{i=1}^{n}\eta_i, \end{gathered} \tag{2.16}$$

then (1.1) *is oscillatory.*

Proof. Suppose on the contrary that x is a nonoscillatory solution of (1.1). First assume that $x(t)$ and $x(\tau_j(t))$ $(j = 0, 1, 2 \dots, n)$ are positive for all $t \geq t_1$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Choose a_1 sufficiently large so that $\tau_j(\tau_j(a_1)) \geq t_1$. Let $t \in [a_1, b_1]_{\mathbb{T}}$.

Define

$$w(t) = -r(t)\frac{x^\Delta(t)}{x(t)}, \quad t \geq t_1. \tag{2.17}$$

Using the delta quotient rule, we have

$$w^\Delta(t) = -\frac{\left(r(t)x^\Delta(t)\right)^\Delta x(t) - r(t)\left(x^\Delta(t)\right)^2}{x(t)x^\sigma(t)} = -\frac{\left(r(t)x^\Delta(t)\right)^\Delta}{x^\sigma(t)} + \frac{r(t)\left(x^\Delta(t)\right)^2}{x(t)x^\sigma(t)}. \tag{2.18}$$

Notice that

$$x(t)x^\sigma(t) = x(t)\left[x(t) + \mu(t)x^\Delta(t)\right] = x^2(t)\left[1 - \mu(t)\frac{w(t)}{r(t)}\right] = \frac{x^2(t)}{r(t)}\left[r(t) - \mu(t)w(t)\right] \tag{2.19}$$

which implies

$$r(t) - \mu(t)w(t) = r(t)\frac{x^\sigma(t)}{x(t)} > 0. \tag{2.20}$$

Hence, we obtain

$$w^\Delta(t) = -\frac{\left(r(t)x^\Delta(t)\right)^\Delta}{x^\sigma(t)} + \frac{w^2(t)}{r(t) - \mu(t)w(t)}. \tag{2.21}$$

Substituting (2.21) into (1.1) yields

$$w^{\Delta}(t)=\frac{p_0(t)x(\tau_0(t))}{x^{\sigma}(t)}+\frac{w^2(t)}{r(t)-\mu(t)w(t)}+\sum_{i=1}^{n}p_i(t)|x(\tau_i(t))|^{\alpha_i-1}\frac{x(\tau_i(t))}{x^{\sigma}(t)}-\frac{e(t)}{x^{\sigma}(t)}. \tag{2.22}$$

By assumption, we can choose $a_1, b_1 \geq t_1$ such that $p_i(t) \geq 0$ $(i = 1, 2, 3\ldots, n)$ and $e(t) \leq 0$ for all $t \in [\overline{a}_1, b_1]_{\mathbb{T}}$, where $\overline{a}_1$ is defined as in (2.14). Clearly, the conditions of Lemma 2.5 are satisfied when, τ replaced with τ_j for each fixed $(j = 0, 1, 2, \ldots, n)$. Therefore, from (2.5), we have

$$\frac{x(\tau_j(t))}{x^{\sigma}(t)} \geq \frac{\tau_j(t)-\tau_j(a_1)}{\sigma(t)-\tau_j(a_1)}, \quad t\in[a_1,b_1]_{\mathbb{T}} \tag{2.23}$$

and taking into account (2.22) yields

$$w^{\Delta}(t)\geq p_0(t)\frac{\tau_0(t)-\tau_0(a_1)}{\sigma(t)-\tau_0(a_1)}+\frac{w^2(t)}{r(t)-\mu(t)w(t)}+\sum_{i=1}^{n}p_k(t)\left(\frac{\tau_i(t)-\tau_i(a_1)}{\sigma(t)-\tau_i(a_1)}\right)^{\alpha_i}(x^{\sigma}(t))^{\alpha_i-1}+\frac{|e(t)|}{x^{\sigma}(t)}. \tag{2.24}$$

Denote

$$Q_0^*(t):=p_0(t)\frac{\tau_0(t)-\tau_0(a_1)}{\sigma(t)-\tau_0(a_1)}, \qquad Q_i^*(t):=p_i(t)\left(\frac{\tau_i(t)-\tau_i(a_1)}{\sigma(t)-\tau_i(a_1)}\right)^{\alpha_i}. \tag{2.25}$$

From (2.24), we have

$$w^{\Delta}(t)\geq Q_0^*(t)+\frac{w^2(t)}{r(t)-\mu(t)w(t)}+\sum_{i=1}^{n}Q_i^*(t)(x^{\sigma}(t))^{\alpha_i-1}+\frac{|e(t)|}{x^{\sigma}(t)}. \tag{2.26}$$

Now recall the well-known arithmetic-geometric mean inequality, see [26],

$$\sum_{i=0}^{n}u_i\eta_i\geq\prod_{i=0}^{n}u_i^{\eta i}, \tag{2.27}$$

where $\eta_0 = 1-\sum_{i=1}^{n}\eta_i$ and $\eta_i > 0$, $i = 1, 2, \ldots, n$. Setting

$$u_0\eta_0:=\frac{|e(t)|}{x^{\sigma}(t)}, \qquad u_i\eta_i:=Q_i^*(t)(x^{\sigma}(t))^{\alpha_i-1} \tag{2.28}$$

in (2.26) yields

$$w^{\Delta}(t)\geq Q_0^*(t)+\frac{w^2(t)}{r(t)-\mu(t)w(t)}+\sum_{i=1}^{n}u_i\eta_i+u_0\eta_0=Q_0^*(t)+\frac{w^2(t)}{r(t)-\mu(t)w(t)}+\sum_{i=0}^{n}u_i\eta_i. \tag{2.29}$$

From (2.29) and taking into account (2.27), we get

$$w^{\Delta}(t) \geq Q_0^*(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)} + \prod_{i=0}^{n} u_i^{\eta_i} \tag{2.30}$$

and hence,

$$\begin{aligned} w^{\Delta}(t) &\geq Q_0^*(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)} + \eta_0^{-\eta_0} \frac{|e(t)|^{\eta_0}}{(x^{\sigma}(t))^{\eta_0}} \prod_{i=1}^{n} \eta_i^{-\eta_i} (Q_i^*(t))^{\eta_i} \left((x^{\sigma}(t))^{\alpha_i - 1} \right)^{\eta_i} \\ &= Q_0^*(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)} + \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} (Q_i^*(t))^{\eta_i} (x^{\sigma}(t))^{-\eta_0 + \sum_{j=1}^{n} (\alpha_j \eta_j - \eta_j)} \\ &= Q_0^*(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)} + \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} (Q_i^*(t))^{\eta_i} \end{aligned} \tag{2.31}$$

which yields

$$\begin{aligned} w^{\Delta}(t) &\geq Q_0^*(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)} + \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} (p_i(t))^{\eta_i} \left(\frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)} \right)^{\alpha_i \eta_i} \\ &= Q(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)}, \end{aligned} \tag{2.32}$$

where

$$Q(t) = Q_0^*(t) + \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} (p_i(t))^{\eta_i} \left(\frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)} \right)^{\alpha_i \eta_i}. \tag{2.33}$$

Multiplying both sides of (2.32) by $H^2(\sigma(t), a_1)$ and integrating both sides of the resulting inequality from a_1 to c_1, $a_1 < c_1 < b_1$ yield

$$\int_{a_1}^{c_1} w^{\Delta}(t) H^2(\sigma(t), a_1) \Delta t \geq \int_{a_1}^{c_1} Q(t) H^2(\sigma(t), a_1) \Delta t + \int_{a_1}^{c_1} \frac{w^2(t) H^2(\sigma(t), a_1)}{r(t) - \mu(t)w(t)} \Delta t. \tag{2.34}$$

Fix s and note that

$$\begin{aligned} \left(w(t) H^2(t,s) \right)^{\Delta_t} &= H^2(\sigma(t), s) w^{\Delta}(t) + \left(H^2(t,s) \right)^{\Delta_t} w(t) \\ &= H^2(\sigma(t), s) w^{\Delta}(t) + H_1(t,s) H(\sigma(t), s) w(t) + H(t,s) H_1(t,s) w(t), \end{aligned} \tag{2.35}$$

from which we obtain

$$H^2(\sigma(t), s) w^{\Delta}(t) = \left(w(t) H^2(t,s) \right)^{\Delta_t} - H_1(t,s) H(\sigma(t), s) w(t) - H(t,s) H_1(t,s) w(t). \tag{2.36}$$

Therefore,

$$\int_{a_1}^{c_1} w^{\Delta}(t)H^2(\sigma(t),a_1)\Delta t = \int_{a_1}^{c_1}\left(w(t)H^2(t,a_1)\right)^{\Delta_t}\Delta t - \int_{a_1}^{c_1}[H_1(t,a_1)H(\sigma(t),a_1)w(t) + H(t,a_1)H_1(t,a_1)w(t)]\Delta t. \tag{2.37}$$

Notice that

$$\int_{a_1}^{c_1}\left(w(t)H^2(t,a_1)\right)^{\Delta_t}\Delta t = w(c_1)H^2(c_1,a_1) - w(a_1)H^2(a_1,a_1) = w(c_1)H^2(c_1,a_1) \tag{2.38}$$

since $H(a_1,a_1) = 0$ and hence, we obtain from (2.34) that

$$w(c_1)H^2(c_1,a_1) \geq \int_{a_1}^{c_1} Q(t)H^2(\sigma(t),a_1)\Delta t + \int_{a_1}^{c_1}\frac{w^2(t)}{r(t)-\mu(t)w(t)}H^2(\sigma(t),a_1)\Delta t + \int_{a_1}^{c_1}[H_1(t,a_1)H(\sigma(t),a_1)w(t) + H(t,a_1)H_1(t,a_1)w(t)]\Delta t. \tag{2.39}$$

On the other hand,

$$\begin{aligned}\frac{w^2(t)H^2(\sigma(t),s)}{r(t)-\mu(t)w(t)} &+ w(t)H(\sigma(t),s)H_1(t,s) + H(t,s)H_1(t,s)w(t)\\ &= \left[\frac{w(t)H(\sigma(t),s)}{\sqrt{r(t)-\mu(t)w(t)}} + \sqrt{r(t)-\mu(t)w(t)}H_1(t,s)\right]^2\\ &\quad - \left(r(t)-\mu(t)w(t)\right)H_1^2(t,s) - w(t)H(\sigma(t),s)H_1(t,s) + H(t,s)H_1(t,s)w(t).\end{aligned} \tag{2.40}$$

Taking into account that $H(\sigma(t),s) = H(t,s) + \mu(t)H_1(t,s)$, we have

$$\frac{w^2(t)H^2(\sigma(t),a_1)}{r(t)-\mu(t)w(t)} + w(t)H(\sigma(t),a_1)H_1(t,a_1) + H(t,a_1)H_1(t,a_1)w(t) \geq -r(t)H_1^2(t,a_1). \tag{2.41}$$

Using this inequality in (2.39), we have

$$w(c_1)H^2(c_1,a_1) \geq \int_{a_1}^{c_1}\left[Q(t)H^2(\sigma(t),a_1) - r(t)H_1^2(t,a_1)\right]\Delta t. \tag{2.42}$$

Similarly, by following the above calculation step by step, that is, multiplying both sides of (2.32) this time by $H^2(b_1, \sigma(s))$ after taking into account that

$$H^2(t,\sigma(s))w^{\Delta}(s) = \left(w(s)H^2(t,s)\right)^{\Delta_s} - H_2(t,s)H(t,\sigma(s))w(s) - H(t,s)H_2(t,s)w(s), \quad (2.43)$$

one can easily obtain

$$-w(c_1)H^2(b_1,c_1) \geq \int_{c_1}^{b_1} \left[Q(s)H^2(b_1,\sigma(s)) - r(s)H_2^2(b_1,s)\right]\Delta s. \quad (2.44)$$

Adding up (2.42) and (2.44), we obtain

$$\begin{aligned} 0 \geq \frac{1}{H^2(c_1,a_1)} &\int_{a_1}^{c_1} \left[Q(t)H^2(\sigma(t),a_1) - r(t)H_1^2(t,a_1)\right]\Delta t \\ + \frac{1}{H^2(b_1,c_1)} &\int_{c_1}^{b_1} \left[Q(t)H^2(b_1,\sigma(t)) - r(s)H_2^2(b_1,t)\right]\Delta t. \end{aligned} \quad (2.45)$$

This contradiction completes the proof when $x(t)$ is eventually positive. The proof when $x(t)$ is eventually negative is analogous by repeating the above arguments on the interval $[\overline{a}_2, b_2]_{\mathbb{T}}$ instead of $[\overline{a}_1, b_1]_{\mathbb{T}}$. □

Corollary 2.7. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exist subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, \infty)$ such that*

$$\begin{gathered} p_i(t) \geq 0 \quad \text{for } t \in [\overline{a}_1, b_1] \cup [\overline{a}_2, b_2], \ (i = 0,1,2,\ldots,n), \\ (-1)^l e(t) \geq 0 \quad \text{for } t \in [\overline{a}_l, b_l], \ (l = 1,2), \end{gathered} \quad (2.46)$$

where $\overline{a}_l = \min\{\tau_j(a_l) : j = 0,1,2,\ldots,n\}$ holds. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.1) *of Lemma 2.1. If there exist a function $H \in \mathscr{H}_{\mathbb{R}}$ and numbers $c_\nu \in (a_\nu, b_\nu)$ such that*

$$\begin{aligned} \frac{1}{H^2(c_\nu,a_\nu)} &\int_{a_\nu}^{c_\nu} \left[Q(t)H^2(t,a_\nu) - r(t)H_1^2(t,a_\nu)\right]dt \\ + \frac{1}{H^2(b_\nu,c_\nu)} &\int_{c_\nu}^{b_\nu} \left[Q(t)H^2(b_\nu,t) - r(t)H_2^2(b_\nu,t)\right]dt > 0 \end{aligned} \quad (2.47)$$

for $\nu = 1,2$, where

$$\begin{gathered} Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(a_\nu)}{t - \tau_0(a_\nu)} + k_0|e(t)|^{\eta_0}\prod_{i=1}^{n}(p_i(t))^{\eta_i}\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{t - \tau_i(a_\nu)}\right)^{\alpha_i\eta_i}, \\ k_0 = \prod_{i=0}^{n}\eta_i^{-\eta_i}, \qquad \eta_0 = 1 - \sum_{i=1}^{n}\eta_i, \end{gathered} \quad (2.48)$$

then (1.3) *is oscillatory.*

Corollary 2.8. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exist $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ with $T \leq a_1 < b_1$ and $T \leq a_2 < b_2$ such that for each $i = 0, 1, 2, \ldots, n$,*

$$\begin{gathered} p_i(t) \geq 0 \quad \text{for } t \in \{\overline{a}_1, \overline{a}_1 + 1, \overline{a}_1 + 2, \ldots, b_1\} \cup \{\overline{a}_2, \overline{a}_2 + 1, \overline{a}_2 + 2, \ldots, b_2\}, \\ (-1)^l e(t) \geq 0 \quad \text{for } t \in \{\overline{a}_l, \overline{a}_l + 1, \overline{a}_l + 2, \ldots, b_l\} \ (l = 1, 2), \end{gathered} \tag{2.49}$$

where $\overline{a}_l = \min\{\tau_j(a_l) : j = 0, 1, 2, \ldots, n\}$ holds. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.1) *of Lemma 2.1. If there exist a function $H \in \mathscr{H}_{\mathbb{Z}}$ and numbers $c_\nu \in \{a_\nu + 1, a_\nu + 2, \ldots, b_\nu - 1\}$ such that*

$$\begin{aligned} &\frac{1}{H^2(c_\nu, a_\nu)} \sum_{t=a_\nu}^{c_\nu - 1} \left[Q(t) H^2(t+1, a_\nu) - r(t) H_1^2(t, a_\nu) \right] \\ &\quad + \frac{1}{H^2(b_\nu, c_\nu)} \sum_{t=c_\nu}^{b_\nu - 1} \left[Q(t) H^2(b_\nu, t+1) - r(t) H_2^2(b_\nu, t) \right] > 0 \end{aligned} \tag{2.50}$$

for $\nu = 1, 2$, where

$$\begin{gathered} H_1(t, a_\nu) := H(t+1, a_\nu) - H(t, a_\nu), \qquad H_2(b_\nu, t) := H(b_\nu, t+1) - H(b_\nu, t), \\ Q(t) = p_0(t) \frac{\tau_0(t) - \tau_0(a_\nu)}{t + 1 - \tau_0(a_\nu)} + k_0 |e(t)|^{\eta_0} \prod_{i=1}^{n} (p_i(t))^{\eta_i} \left(\frac{\tau_i(t) - \tau_i(a_\nu)}{t + 1 - \tau_i(a_\nu)} \right)^{\alpha_i \eta_i}, \\ k_0 = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \qquad \eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \end{gathered} \tag{2.51}$$

then (1.4) *is oscillatory.*

Corollary 2.9. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exist $a_1, b_1, a_2, b_2 \in \mathbb{N}$ with $T \leq a_1 < b_1$ and $T \leq a_2 < b_2$ such that for each $i = 0, 1, 2, \ldots, n$,*

$$\begin{gathered} p_i(t) \geq 0 \quad \text{for } t \in \left\{ q^{\overline{a}_1}, q^{\overline{a}_1 + 1}, \ldots, q^{b_1} \right\} \cup \left\{ q^{\overline{a}_2}, q^{\overline{a}_2 + 1}, \ldots, q^{b_2} \right\}, \\ (-1)^l e(t) \geq 0 \quad \text{for } t \in \left\{ q^{\overline{a}_l}, q^{\overline{a}_l + 1}, \ldots, q^{b_l} \right\}, \ (l = 1, 2) \end{gathered} \tag{2.52}$$

where $q^{\overline{a}_l} = \min\{\tau_j(q^{a_l}) : j = 0, 1, 2, \ldots, n\}$ holds. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.1) *of Lemma 2.1. If there exist a function $H \in \mathscr{H}_q$ and numbers $q^{c_\nu} \in \{q^{a_\nu + 1}, q^{a_\nu + 2}, \ldots, q^{b_\nu - 1}\}$ such that*

$$\begin{aligned} &\frac{1}{H^2(q^{c_\nu}, q^{a_\nu})} \sum_{m=a_\nu}^{c_\nu - 1} q^m \left[Q(q^m) H^2\left(q^{m+1}, q^{a_\nu}\right) - r(q^m) H_1^2(q^m, q^{a_\nu}) \right] \\ &\quad + \frac{1}{H^2(q^{b_\nu}, q^{c_\nu})} \sum_{m=c_\nu}^{b_\nu - 1} q^m \left[Q(q^m) H^2\left(q^{b_\nu}, q^{m+1}\right) - r(q^m) H_2^2\left(q^{b_\nu}, q^m\right) \right] > 0 \end{aligned} \tag{2.53}$$

for $\nu = 1, 2$, *where*

$$H_1(q^m, q^{a_\nu}) := \frac{H(q^{m+1}, q^{a_\nu}) - H(q^m, q^{a_\nu})}{(q-1)q^m}, \qquad H_2\left(q^{b_\nu}, q^m\right) := \frac{H(q^{b_\nu}, q^{m+1}) - H(q^{b_\nu}, q^m)}{(q-1)q^m},$$

$$Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(q^{a_\nu})}{qt - \tau_0(q^{a_\nu})} + k_0|e(t)|^{\eta_0}\prod_{i=1}^{n}(p_i(t))^{\eta_i}\left(\frac{\tau_i(t) - \tau_i(q^{a_\nu})}{qt - \tau_i(q^{a_\nu})}\right)^{\alpha_i\eta_i},$$

$$k_0 = \prod_{i=0}^{n}\eta_i^{-\eta_i}, \qquad \eta_0 = 1 - \sum_{i=1}^{n}\eta_i, \tag{2.54}$$

then (1.5) *is oscillatory.*

Notice that Theorem 2.6 does not apply if there is no forcing term, that is, $e(t) \equiv 0$. In this case we have the following theorem.

Theorem 2.10. *Suppose that for any given (arbitrarily large)* $T \in \mathbb{T}$ *there exists a subinterval* $[a, b]_{\mathbb{T}}$ *of* $[T, \infty)_{\mathbb{T}}$, *where* $a < b$ *such that*

$$p_i(t) \geq 0 \quad \text{for } t \in [\overline{a}, b]_{\mathbb{T}}, \quad (i = 0, 1, 2, \ldots, n), \tag{2.55}$$

where $\overline{a} = \min\{\tau_j(a) : j = 0, 1, 2, \ldots, n\}$ *holds. Let* $\{\eta_1, \eta_2, \ldots, \eta_n\}$ *be an* n*-tuple satisfying* (2.2) *in Lemma 2.2. If there exist a function* $H \in \mathscr{H}_{\mathbb{T}}$ *and a number* $c \in (a, b)_{\mathbb{T}}$ *such that*

$$\begin{aligned} &\frac{1}{H^2(c, a)}\int_a^c \left[Q(t)H^2(\sigma(t), a) - r(t)H_1^2(t, a)\right]\Delta t \\ &\quad + \frac{1}{H^2(b, c)}\int_c^b \left[Q(t)H^2(b, \sigma(t)) - r(s)H_2^2(b, t)^2\right]\Delta t > 0, \end{aligned} \tag{2.56}$$

where

$$Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(a)}{\sigma(t) - \tau_0(a)} + k_0\prod_{i=1}^{n}(p_i(t))^{\eta_i}\left(\frac{\tau_i(t) - \tau_i(a)}{\sigma(t) - \tau_i(a)}\right)^{\alpha_i\eta_i}, \quad k_0 = \prod_{i=1}^{n}\eta_i^{-\eta_i}, \tag{2.57}$$

then (1.1) *with* $e(t) \equiv 0$ *is oscillatory.*

Proof. We will just highlight the proof since it is the same as the proof of Theorem 2.6. We should remark here that taking $e(t) \equiv 0$ and $\eta_0 = 0$ in proof of Theorem 2.6, we arrive at

$$w^{\Delta}(t) \geq Q_0^*(t) + \frac{w^2(t)}{r(t) - \mu(t)w(t)} + \sum_{i=1}^{n}u_i\eta_i. \tag{2.58}$$

The arithmetic-geometric mean inequality we now need is

$$\sum_{i=1}^{n} u_i \eta_i \geq \prod_{i=1}^{n} u_i^{\eta_i}, \tag{2.59}$$

where $1 = \sum_{i=1}^{n} \eta_i$ and $\eta_i > 0$, $i = 1, 2, \ldots, n$ are as in Lemma 2.2. □

Corollary 2.11. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exists a subinterval $[a, b]$ of $[T, \infty)$, where $T \leq a < b$ with $a, b \in \mathbb{R}$ such that*

$$p_i(t) \geq 0 \quad \text{for } t \in [\overline{a}, b], \ (i = 0, 1, 2, \ldots, n), \tag{2.60}$$

where $\overline{a} = \min\{\tau_j(a) : j = 0, 1, 2, \ldots, n\}$ holds. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) in Lemma 2.2. If there exist a function $H \in \mathscr{H}_{\mathbb{R}}$ and a number $c \in (a, b)$ such that

$$\begin{aligned} &\frac{1}{H^2(c, a)} \int_a^c \left[Q(t) H^2(t, a) - r(t) H_1^2(t, a) \right] dt \\ &\quad + \frac{1}{H^2(b, c)} \int_c^b \left[Q(s) H^2(b, t) - r(t) H_2^2(b, t) \right] dt > 0, \end{aligned} \tag{2.61}$$

where

$$Q(t) = p_0(t) \frac{\tau_0(t) - \tau_0(a)}{t - \tau_0(a)} + k_0 \prod_{i=1}^{n} (p_i(t))^{\eta_i} \left(\frac{\tau_i(t) - \tau_i(a)}{t - \tau_i(a)} \right)^{\alpha_i \eta_i}, \quad k_0 = \prod_{i=1}^{n} \eta_i^{-\eta_i}, \tag{2.62}$$

then (1.3) with $e(t) \equiv 0$ is oscillatory.

Corollary 2.12. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exists $a, b \in \mathbb{Z}$ with $T \leq a < b$ such that*

$$p_i(t) \geq 0 \quad \text{for } t \in \{\overline{a}, \overline{a} + 1, \ldots, b\}, \quad (i = 0, 1, 2, \ldots, n), \tag{2.63}$$

where $\overline{a} = \min\{\tau_j(a) : j = 0, 1, 2, \ldots, n\}$ holds. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) in Lemma 2.2. If there exist a function $H \in \mathscr{H}_{\mathbb{Z}}$ and a number $c \in \{a + 1, a + 2, \ldots, b - 1\}$ such that

$$\begin{aligned} &\frac{1}{H^2(c, a)} \sum_{t=a}^{c-1} \left[Q(t) H^2(t + 1, a) - r(t) H_1^2(t, a) \right] \\ &\quad + \frac{1}{H^2(b, c)} \sum_{t=c}^{b-1} \left[Q(t) H^2(b, t + 1) - r(t) H_2^2(b, t) \right] > 0, \end{aligned} \tag{2.64}$$

where

$$H_1(t,a) := H(t+1,a) - H(t,a), \qquad H_2(b,t) := H(b,t+1) - H(b,t),$$
$$Q(t) = p_0(t)\frac{\tau_0(t)-\tau_0(a)}{t+1-\tau_0(a)} + k_0\prod_{i=1}^{n}(p_i(t))^{\eta_i}\left(\frac{\tau_i(t)-\tau_i(a)}{t+1-\tau_i(a)}\right)^{\alpha_i\eta_i}, \quad k_0 = \prod_{i=1}^{n}\eta_i^{-\eta_i}, \tag{2.65}$$

then (1.4) *with* $e(t) \equiv 0$ *is oscillatory.*

Corollary 2.13. *Suppose that for any given (arbitrarily large)* $T \geq t_0$ *there exist* $a,b \in \mathbb{N}$ *with* $T \leq a < b$ *such that*

$$p_i(t) \geq 0 \quad \text{for} \quad t \in \left\{q^{\overline{a}}, q^{\overline{a}+1}, \dots, q^b\right\}, \ (i = 0,1,2,\dots,n) \tag{2.66}$$

where $q^{\overline{a}} = \min\{\tau_j(q^a) : j = 0,1,2,\dots,n\}$ *holds. Let* $\{\eta_1,\eta_2,\dots,\eta_n\}$ *be an* n*-tuple satisfying* (2.2) *in Lemma 2.2. If there exist a function* $H \in \mathscr{H}_{q^{\mathbb{N}}}$ *and a number* $q^c \in \{q^a, q^{a+1},\dots,q^b\}$ *such that*

$$\frac{1}{H^2(q^c,q^a)}\sum_{m=a}^{c-1} q^m\left[Q(q^m)H^2\left(q^{m+1},q^a\right) - r(q^m)(H_1(q^m,q^a))^2\right]$$
$$+ \frac{1}{H^2(q^b,q^c)}\sum_{m=c}^{b-1} q^m\left[Q(q^m)H^2\left(q^b,q^{m+1}\right) - r(q^m)\left(H_2(q^b,q^m)\right)^2\right] > 0, \tag{2.67}$$

where

$$H_1(q^m,q^a) := \frac{H(q^{m+1},q^a) - H(q^m,q^a)}{(q-1)q^m}, \qquad H_2\left(q^b,q^m\right) := \frac{H(q^b,q^{m+1}) - H(q^b,q^m)}{(q-1)q^m},$$
$$Q(t) = p_0(t)\frac{\tau_0(t)-\tau_0(q^a)}{qt-\tau_0(q^a)} + k_0\prod_{i=1}^{n}(p_i(t))^{\eta_i}\left(\frac{\tau_i(t)-\tau_i(q^a)}{qt-\tau_i(q^a)}\right)^{\alpha_i\eta_i}, \quad k_0 = \prod_{i=1}^{n}\eta_i^{-\eta_i}, \tag{2.68}$$

then (1.5) *with* $e(t) \equiv 0$ *is oscillatory.*

It is obvious that Theorem 2.6 is not applicable if the functions $p_i(t)$ are nonpositive for $i = m+1, m+2,\dots,n$. In this case the theorem below is valid.

Theorem 2.14. *Suppose that for any given (arbitrarily large)* $T \in \mathbb{T}$ *there exist subintervals* $[a_1,b_1]_{\mathbb{T}}$ *and* $[a_2,b_2]_{\mathbb{T}}$ *of* $[T,\infty)_{\mathbb{T}}$*, where* $a_1 < b_1$ *and* $a_2 < b_2$ *such that*

$$p_i(t) \geq 0 \quad \text{for } t \in [\overline{a}_1,b_1]_{\mathbb{T}} \cup [\overline{a}_2,b_2]_{\mathbb{T}}, \ (i = 0,1,2,\dots,n),$$
$$(-1)^l e(t) > 0 \quad \text{for } t \in [\overline{a}_l,b_l]_{\mathbb{T}}, \ (l = 1,2), \tag{2.69}$$

where $\overline{a}_l = \min\{\tau_j(a_l) : j = 0, 1, 2, \ldots, n\}$ *holds. If there exist a function* $H \in \mathscr{H}_{\mathbb{T}}$, *positive numbers* λ_i *and* ν_i *satisfying*

$$\sum_{i=1}^{m}\lambda_i + \sum_{i=m+1}^{n}\nu_i = 1, \tag{2.70}$$

and numbers $c_\nu \in (a_\nu, b_\nu)_{\mathbb{T}}$ *such that*

$$\begin{aligned} &\frac{1}{H^2(c_\nu, a_\nu)}\int_{a_\nu}^{c_\nu}\left[Q(t)H^2(\sigma(t), a_\nu) - r(t)H_1^2(t, a_\nu)\right]\Delta t \\ &\quad + \frac{1}{H^2(b_\nu, c_\nu)}\int_{c_\nu}^{b_\nu}\left[Q(t)H^2(b_\nu, \sigma(t)) - r(t)H_2^2(b_\nu, t)\right]\Delta t > 0 \end{aligned} \tag{2.71}$$

for $\nu = 1, 2$, *where*

$$\begin{aligned} Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(a_\nu)}{\sigma(t) - \tau_0(a_\nu)} &+ \sum_{i=1}^{m}\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)}p_i^{1/\alpha_i}(t)\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{\sigma(t) - \tau_i(a_\nu)}\right) \\ &- \sum_{i=m+1}^{n}\beta_i(\nu_i|e(t)|)^{1-(1/\alpha_i)}\widetilde{p}_i^{\,1/\alpha_i}(t)\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{\sigma(t) - \tau_i(a_\nu)}\right), \end{aligned} \tag{2.72}$$

with

$$\mu_i = \alpha_i(\alpha_i - 1)^{(1/\alpha_i)-1}, \qquad \beta_i = \alpha_i(1 - \alpha_i)^{(1/\alpha_i)-1}, \qquad \widetilde{p}_i = \max\{-p_i(t), 0\}, \tag{2.73}$$

then (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution. Without losss of generality, we may assume that $x(t)$ and $x(\tau_i(t))$ $(i = 0, 1, 2, \ldots, n)$ are eventually positive on $[a_1, b_1]_{\mathbb{T}}$ when a_1 is sufficiently large. If $x(t)$ is eventually negative, one may repeat the same proof step by step on the interval $[a_2, b_2]_{\mathbb{T}}$.

Rewriting (1.1) for $t \in [a_1, b_1]_{\mathbb{T}}$ as

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p_0(t)x(\tau_0(t)) + \sum_{i=1}^{m}\left[p_i(t)x^{\alpha_i}(\tau_i(t)) + \lambda_i|e(t)|\right] + \sum_{i=m+1}^{n}\left[p_i(t)x^{\alpha_i}(\tau_i(t)) + \nu_i|e(t)|\right] = 0 \tag{2.74}$$

and applying Lemma 2.3 to each term in the first sum, we obtain

$$\begin{aligned} &\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p_0(t)x(\tau_0(t)) + \sum_{i=1}^{m}\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)}p_i^{1/\alpha_i}(t)x(\tau_i(t)) \\ &\quad + \sum_{i=m+1}^{n}\left[p_i(t)x^{\alpha_i}(\tau_i(t)) + \nu_i|e(t)|\right] \le 0, \end{aligned} \tag{2.75}$$

where $\mu_i = \alpha_i(\alpha_i - 1)^{(1/\alpha_i)-1}$ for $i = 1, 2, \dots, m$. Setting

$$w(t) = -r(t)\frac{x^{\Delta}(t)}{x(t)} \tag{2.76}$$

yields

$$w^{\Delta}(t) = -\frac{\left(r(t)x^{\Delta}(t)\right)^{\Delta}}{x^{\sigma}(t)} + \frac{w^2(t)}{r(t) - \mu(t)w(t)}. \tag{2.77}$$

Substituting the above last equality into (2.75), we have

$$\begin{aligned} w^{\Delta}(t) \geq{}& p_0(t)\frac{x(\tau_0(t))}{x^{\sigma}(t)} + \sum_{i=1}^{m}\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)}p_i^{1/\alpha_i}(t)\frac{x(\tau_i(t))}{x^{\sigma}(t)} \\ &+ \frac{1}{x^{\sigma}(t)}\sum_{i=m+1}^{n}\left[p_i(t)x^{\alpha_i}(\tau_i(t)) + \nu_i|e(t)|\right] + \frac{w^2(t)}{r(t) - \mu(t)w(t)}. \end{aligned} \tag{2.78}$$

It follows from (2.5) that

$$\frac{x(\tau_0(t))}{x^{\sigma}(t)} \geq \frac{\tau_0(t) - \tau_0(a_1)}{\sigma(t) - \tau_0(a_1)}, \tag{2.79}$$

$$\frac{x(\tau_i(t))}{x^{\sigma}(t)} \geq \frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)}, \tag{2.80}$$

$$\frac{x^{\alpha_i}(\tau_i(t))}{x^{\sigma}(t)} \geq x^{\alpha_i - 1}(\tau_i(t))\frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)}. \tag{2.81}$$

Notice that the second sum in (2.78) can be written as

$$\begin{aligned} \frac{1}{x^{\sigma}(t)}\sum_{i=m+1}^{n}\left[p_i(t)x^{\alpha_i}(\tau_i(t)) + \nu_i|e(t)|\right] &= \sum_{i=m+1}^{n}\left[p_i(t)\frac{x^{\alpha_i}(\tau_i(t))}{x^{\sigma}(t)} + \frac{\nu_i|e(t)|}{x^{\sigma}(t)}\right] \\ &= \sum_{i=m+1}^{n}\left[\frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)}\right]\left[\nu_i|e(t)|\frac{1}{x(\tau_i(t))} - \widetilde{p}_i(t)\left(\frac{1}{x(\tau_i(t))}\right)^{1-\alpha_i}\right], \end{aligned} \tag{2.82}$$

and hence applying the Lemma 2.4 yields

$$\begin{aligned} &\sum_{i=m+1}^{n}\left[\frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)}\right]\left[\nu_i|e(t)|\frac{1}{x(\tau_i(t))} - \widetilde{p}_i(t)\left(\frac{1}{x(\tau_i(t))}\right)^{1-\alpha_i}\right] \\ &\quad \geq -\sum_{i=m+1}^{n}\left[\frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)}\right]\beta_i(\nu_i|e(t)|)^{1-(1/\alpha_i)}\widetilde{p}_i^{\,1/\alpha_i}(t), \end{aligned} \tag{2.83}$$

where $\beta_i = \alpha_i(1-\alpha_i)^{(1/\alpha_i)-1}$ and $\widetilde{p}_i = \max\{-p_i(t), 0\}$ for $i = m+1, m+2, \dots, n$. Using (2.79), (2.80), and (2.78) into (2.78), we obtain

$$
\begin{aligned}
w^{\Delta}(t) \geq{}& p_0(t)\frac{\tau_0(t)-\tau_0(a_1)}{\sigma(t)-\tau_0(a_1)} + \sum_{i=1}^{m}\left[\frac{\tau_i(t)-\tau_i(a_1)}{\sigma(t)-\tau_i(a_1)}\right]\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)}p_i^{1/\alpha_i}(t) \\
&- \sum_{i=m+1}^{n}\left[\frac{\tau_i(t)-\tau_i(a_1)}{\sigma(t)-\tau_i(a_1)}\right]\beta_i(\nu_i|e(t)|)^{1-(1/\alpha_i)}\widetilde{p}_i^{\,1/\alpha_i}(t) + \frac{w^2(t)}{r(t)-\mu(t)w(t)}.
\end{aligned}
\tag{2.84}
$$

Setting

$$
\begin{aligned}
Q(t) ={}& p_0(t)\frac{\tau_0(t)-\tau_0(a_1)}{\sigma(t)-\tau_0(a_1)} + \sum_{i=1}^{m}\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)}p_i^{1/\alpha_i}(t)\left(\frac{\tau_i(t)-\tau_i(a_1)}{\sigma(t)-\tau_i(a_1)}\right) \\
&- \sum_{i=m+1}^{n}\beta_i(\nu_i|e(t)|)^{1-(1/\alpha_i)}\widetilde{p}_i^{\,1/\alpha_i}(t)\left(\frac{\tau_i(t)-\tau_i(a_1)}{\sigma(t)-\tau_i(a_1)}\right),
\end{aligned}
\tag{2.85}
$$

we have

$$
w^{\Delta}(t) \geq Q(t) + \frac{w^2(t)}{r(t)-\mu(t)w(t)}. \tag{2.86}
$$

The rest of the proof is the same as that of Theorem 2.6 and hence it is omitted. □

Corollary 2.15. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exist subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, \infty)$, where $T \leq a_1 < b_1$ and $T \leq a_2 < b_2$ such that*

$$
\begin{gathered}
p_i(t) \geq 0 \quad \text{for } t \in [\overline{a}_1, b_1] \cup [\overline{a}_2, b_2],\ (i = 0, 1, 2, \dots, n), \\
(-1)^l e(t) > 0 \quad \text{for } t \in [\overline{a}_l, b_l],\ (l = 1, 2),
\end{gathered}
\tag{2.87}
$$

where $\overline{a}_l = \min\{\tau_j(a_l) : j = 0, 1, 2, \dots, n\}$ holds. If there exist a function $H \in \mathscr{H}_{\mathbb{R}}$, positive numbers λ_i and ν_i satisfying

$$
\sum_{i=1}^{m}\lambda_i + \sum_{i=m+1}^{n}\nu_i = 1, \tag{2.88}
$$

and numbers $c_\nu \in (a_\nu, b_\nu)$ such that

$$
\begin{aligned}
&\frac{1}{H^2(c_\nu, a_\nu)}\int_{a_\nu}^{c_\nu}\left[Q(t)H^2(t, a_\nu) - r(t)H_1^2(t, a_\nu)\right]dt \\
&\quad + \frac{1}{H^2(b_\nu, c_\nu)}\int_{c_\nu}^{b_\nu}\left[Q(t)H^2(b_\nu, t) - r(t)H_2^2(b_\nu, t)\right]dt > 0
\end{aligned}
\tag{2.89}
$$

for $\nu = 1, 2$, *where*

$$Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(a_\nu)}{t - \tau_0(a_\nu)} + \sum_{i=1}^{m}\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)} p_i^{1/\alpha_i}(t)\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{t - \tau_i(a_\nu)}\right) - \sum_{i=m+1}^{n}\beta_i(\nu_i|e(t)|)^{1-(1/\alpha_i)} \widetilde{p}_i^{\,1/\alpha_i}(t)\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{t - \tau_i(a_\nu)}\right) \tag{2.90}$$

with

$$\mu_i = \alpha_i(\alpha_i - 1)^{(1/\alpha_i)-1}, \qquad \beta_i = \alpha_i(1 - \alpha_i)^{(1/\alpha_i)-1}, \quad \widetilde{p}_i = \max\{-p_i(t), 0\}, \tag{2.91}$$

then (1.3) *is oscillatory.*

Corollary 2.16. *Suppose that for any given (arbitrarily large)* $T \geq t_0$ *there exist* $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ *with* $T \leq a_1 < b_1$ *and* $T \leq a_2 < b_2$ *such that for each* $i = 0, 1, 2, \ldots, n$,

$$p_i(t) \geq 0 \quad \text{for } t \in \{\overline{a}_1, \overline{a}_1 + 1, \ldots, b_1\} \cup \{\overline{a}_2, \overline{a}_2 + 1, \ldots, b_2\}$$
$$(-1)^l e(t) > 0 \quad \text{for } t \in \{\overline{a}_l, \overline{a}_l + 1, \ldots, b_l\}, \; (l = 1, 2), \tag{2.92}$$

where $\overline{a}_l = \min\{\tau_j(a_l) : j = 0, 1, 2, \ldots, n\}$ *holds. If there exist a function* $H \in \mathscr{H}_{\mathbb{Z}}$, *positive numbers* λ_i *and* ν_i *satisfying*

$$\sum_{i=1}^{m}\lambda_i + \sum_{i=m+1}^{n}\nu_i = 1, \tag{2.93}$$

and numbers $c_\nu \in \{a_\nu + 1, a_\nu + 2, \ldots, b_\nu - 1\}$ *such that*

$$\frac{1}{H^2(c_\nu, a_\nu)}\sum_{t=a_\nu}^{c_\nu - 1}\left[Q(t)H^2(t+1, a_\nu) - r(t)H_1^2(t, a_\nu)\right] + \frac{1}{H^2(b_\nu, c_\nu)}\sum_{t=c_\nu}^{b_\nu - 1}\left[Q(t)H^2(b_\nu, t+1) - r(t)H_2^2(b_\nu, t)\right] > 0 \tag{2.94}$$

for $\nu = 1, 2$, *where*

$$H_1(t, a_\nu) := H(t+1, a_\nu) - H(t, a_\nu), \qquad H_2(b_\nu, t) := H(b_\nu, t+1) - H(b_\nu, t),$$

$$Q(t) = p_0(t)\frac{\tau_0(t) - \tau_0(a_\nu)}{t + 1 - \tau_0(a_\nu)} + \sum_{i=1}^{m}\mu_i(\lambda_i|e(t)|)^{1-(1/\alpha_i)} p_i^{1/\alpha_i}(t)\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{t + 1 - \tau_i(a_\nu)}\right) - \sum_{i=m+1}^{n}\beta_i(\nu_i|e(t)|)^{1-(1/\alpha_i)} \widetilde{p}_i^{\,1/\alpha_i}(t)\left(\frac{\tau_i(t) - \tau_i(a_\nu)}{t + 1 - \tau_i(a_\nu)}\right) \tag{2.95}$$

with

$$\mu_i = \alpha_i(\alpha_i - 1)^{(1/\alpha_i)-1}, \qquad \beta_i = \alpha_i(1-\alpha_i)^{(1/\alpha_i)-1}, \quad \widetilde{p}_i = \max\{-p_i(t), 0\}, \tag{2.96}$$

then (1.4) *is oscillatory.*

Corollary 2.17. *Suppose that for any given (arbitrarily large) $T \geq t_0$ there exist $a_1, b_1, a_2, b_2 \in \mathbb{N}$ with $T \leq a_1 < b_1$ and $T \leq a_2 < b_2$ such that for each $i = 0, 1, 2, \ldots, n$,*

$$\begin{gathered} p_i(t) \geq 0 \quad \text{for } t \in \left\{q^{\overline{a}_1}, q^{\overline{a}_1+1}, \ldots, q^{b_1}\right\} \cup \left\{q^{\overline{a}_2}, q^{\overline{a}_2+1}, \ldots, q^{b_2}\right\}, \\ (-1)^l e(t) > 0 \quad \text{for } t \in \left\{q^{\overline{a}_l}, q^{\overline{a}_l+1}, \ldots, q^{b_l}\right\}, \ (l = 1, 2), \end{gathered} \tag{2.97}$$

where $q^{\overline{a}_l} = \min\{\tau_j(q^{a_l}) : j = 0, 1, 2, \ldots, n\}$ holds. If there exist a function $H \in \mathscr{H}_q$, positive numbers λ_i and ν_i satisfying

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \nu_i = 1, \tag{2.98}$$

and numbers $q^{c_\nu} \in \{q^{a_\nu+1}, q^{a_\nu+2}, \ldots, q^{b_\nu-1}\}$ such that

$$\begin{aligned} &\frac{1}{H^2(q^{c_\nu}, q^{a_\nu})} \sum_{m=a_\nu}^{c_\nu-1} q^m \left[Q(q^m) H^2\left(q^{m+1}, q^{a_\nu}\right) - r(t) H_1^2(q^m, q^{a_\nu})\right] \\ &\quad + \frac{1}{H^2(q^{b_\nu}, q^{c_\nu})} \sum_{m=c_\nu}^{b_\nu-1} q^m \left[Q(q^m) H^2\left(q^{b_\nu}, q^{m+1}\right) - r(t) H_2^2\left(q^{b_\nu}, q^m\right)\right] > 0 \end{aligned} \tag{2.99}$$

for $\nu = 1, 2$, where

$$\begin{gathered} H_1(q^m, q^{a_\nu}) := \frac{H(q^{m+1}, q^{a_\nu}) - H(q^m, q^{a_\nu})}{(q-1)q^m}, \qquad H_2\left(q^{b_\nu}, q^m\right) := \frac{H(q^{b_\nu}, q^{m+1}) - H(q^{b_\nu}, q^m)}{(q-1)q^m}, \\ Q(t) = p_0(t) \frac{\tau_0(t) - \tau_0(q^{a_\nu})}{qt - \tau_0(q^{a_\nu})} + \sum_{i=1}^{m} \mu_i (\lambda_i |e(t)|)^{1-(1/\alpha_i)} p_i^{1/\alpha_i}(t) \left(\frac{\tau_i(t) - \tau_i(q^{a_\nu})}{qt - \tau_i(q^{a_\nu})}\right) \\ - \sum_{i=m+1}^{n} \beta_i (\nu_i |e(t)|)^{1-(1/\alpha_i)} \widetilde{p}_i^{\,1/\alpha_i}(t) \left(\frac{\tau_i(t) - \tau_i(q^{a_\nu})}{qt - \tau_i(q^{a_\nu})}\right) \end{gathered} \tag{2.100}$$

with

$$\mu_i = \alpha_i(\alpha_i - 1)^{1/\alpha_i - 1}, \qquad \beta_i = \alpha_i(1-\alpha_i)^{1/\alpha_i - 1}, \quad \widetilde{p}_i = \max\{-p_i(t), 0\}, \tag{2.101}$$

then (1.5) *is oscillatory.*

3. Examples

In this section we give three examples when $n = 2$, and $\alpha_1 = 2$, $\alpha_2 = 1/2$ in (1.1). That is, we consider

$$x^{\Delta\Delta}(t) + p_0(t)x(\tau_0(t)) + p_1(t)|x(\tau_1(t))|x(\tau_1(t)) + p_2(t)|x(\tau_1(t))|^{-1/2}x(\tau_2(t)) = 0. \tag{3.1}$$

For simplicity we take $H(t,s) = t - s$, thus $H_1(t,s) = -H_2(t,s) = 1$. Note that $\eta_1 = 1/3$ and $\eta_2 = 2/3$ by Lemma 2.2.

Example 3.1. Let $A \geq 0$ and $B, C > 0$ be constants. Consider the differential equation

$$x''(t) + Ax(t-1) + B|x(t-2)|x(t-2) + C|x(t-1)|^{-1/2}x(t-1) = 0. \tag{3.2}$$

Let $a = j$, $b = j+2$, and $c = j+1$, $j \in \mathbb{N}$.

We calculate

$$Q(t) = A\left(\frac{t-j}{t-j+1}\right) + \frac{3}{\sqrt[3]{4}}(B)^{1/3}(C)^{2/3}\frac{(t-j)}{(t-j+2)^{2/3}(t-j+1)^{1/3}} \tag{3.3}$$

and see that (2.61) holds if

$$4A + 9\left(BC^2\right)^{1/3} > 27. \tag{3.4}$$

Since all conditions of Corollary 2.11 are satisfied, we conclude that (3.2) is oscillatory when (3.4) holds.

Example 3.2. Let $A \geq 0$ and $B, C > 0$ be constants. Define $p_0(t) = A$, $p_1(t) = B$, and $p_2(t) = C$ for $t = 10j + k$, $k = -3, -2, -1, 0, 1, 2, 3$, $j \geq 1$; otherwise, the functions are defined arbitrarily. Consider the difference equation

$$\Delta^2 x(t) + p_0(t)x(t-1) + p_1(t)|x(t-2)|x(t-2) + p_2(t)|x(t-1)|^{-1/2}x(t-1) = 0. \tag{3.5}$$

Let $a = 10j$, $b = 10j + 3$, and $c = 10j + 1$. We derive

$$Q(t) = A\frac{t-10j}{t-10j+2} + \frac{3}{\sqrt[3]{4}}\left(BC^2\right)^{1/3}\frac{t-10j}{(t-3j+3)^{2/3}(t-10j+4)^{1/3}} \tag{3.6}$$

and see that positivity in (2.64) satisfies if

$$A + \frac{9\left(BC^2\right)^{1/3}}{4\sqrt[3]{5}} > \frac{48}{5}. \tag{3.7}$$

Since all conditions of Corollary 2.12 are satisfied, we conclude that (3.5) is oscillatory if (3.7) holds.

Example 3.3. Let $A \geq 0$ and $B, C > 0$ be constants. Define $p_0(t) = A$, $p_1(t) = B$ and $p_2(t) = C$ for $t = 2^{10j+k}$, $k = -3, -2, -1, 0, 1, 2, 3$, $j \geq 1$; otherwise, the functions are defined arbitrarily. Consider the q-difference equation, $(q = 2)$,

$$\Delta_q^2 x(t) + p_0(t)x\left(\frac{t}{2}\right) + p_1(t)\left|x\left(\frac{t}{4}\right)\right| x\left(\frac{t}{4}\right) + p_2(t)\left|x\left(\frac{t}{8}\right)\right|^{-1/2} x\left(\frac{t}{8}\right) = 0. \tag{3.8}$$

Let $a = 10j$, $b = 10j + 3$, and $c = 10j + 1$. We have

$$Q(t) = A\frac{t - 2^{10j}}{4t - 2^{10j}} + \frac{3}{\sqrt[3]{4}}\left(BC^2\right)^{1/3} \frac{t - 2^{10j}}{\left(8t - 2^{10j}\right)^{2/3}\left(16t - 2^{10j}\right)^{1/3}}. \tag{3.9}$$

We see that (2.67) holds for all $A \geq 0$ and $B, C > 0$. Since all conditions of Corollary 2.12 are satisfied, we conclude that (3.8) is oscillatory if $A \geq 0$ and $B, C > 0$ are positive.

Acknowledgments

The paper is supported in part by the Scientific and Research Council of Turkey (TUBITAK) under Contract 108T688. The authors would like to thank the referees for their valuable comments and suggestions.

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