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Research Article

Structure of Eigenvalues of Multi-Point Boundary Value Problems

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The structure of eigenvalues of $-y''+q(x)y = \lambda y$, $y(0) = 0$, and $y(1) = \sum_{k=1}^{m} \alpha_k y(\eta_k)$, will be studied, where $q \in L^1([0,1],\mathbb{R})$, $\alpha = (\alpha_k) \in \mathbb{R}^m$, and $0 < \eta_1 < \cdots < \eta_m < 1$. Due to the nonsymmetry of the problem, this equation may admit complex eigenvalues. In this paper, a complete structure of all complex eigenvalues of this equation will be obtained. In particular, it is proved that this equation has always a sequence of real eigenvalues tending to $+\infty$. Moreover, there exists some constant $A_q > 0$ depending on q, such that when α satisfies $\|\alpha\| \le A_q$, all eigenvalues of this equation are necessarily real.

1. Introduction

In the recent years, multi-point boundary value problems of ordinary differential equations have received much attention. Some remarkable results have been obtained, especially for the existence and multiplicity of (positive) solutions for nonlinear second-order ordinary differential equations [1–10]. However, as noted in [5, 6], although it is important in many nonlinear problems, the corresponding eigenvalue theory for linear problems is incomplete. The main reason is that the linear operators are no longer symmetric with respect to multi-point boundary conditions.

In this paper, we will establish some fundamental results for eigenvalue theory of multi-point boundary value problems. Precisely, for a real potential $q \in L^1_{\mathbb{R}} := L^1([0,1],\mathbb{R})$, we consider the eigenvalue problem

$$-y'' + q(x)y = \lambda y, \quad x \in [0,1], \tag{1.1}$$

associated with the $(m+2)$-point boundary condition

$$y(0) = 0, \qquad y(1) - \sum_{k=1}^{m} \alpha_k y(\eta_k) = 0. \tag{1.2}$$

Here $m \in \mathbb{N}$ and the boundary data are $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and

$$\eta = (\eta_1, \dots, \eta_m) \in \Delta^m := \{(\eta_1, \dots, \eta_m) \in \mathbb{R}^m : 0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1\}. \tag{1.3}$$

As usual, $\lambda \in \mathbb{C}$ is called an *eigenvalue* of (1.1) and (1.2) if (1.1) has a nonzero complex solution $y = y(x)$ satisfying conditions of (1.2). The set of all eigenvalues of problem (1.1) and (1.2) is denoted by $\Sigma_{\alpha,\eta}^q \subset \mathbb{C}$, called the *spectrum*.

When $\alpha = (\alpha_k) = 0$, boundary condition (1.2) is reduced to the Dirichlet boundary condition

$$y(0) = y(1) = 0. \tag{1.4}$$

Problem (1.1)–(1.4) is symmetric and has only real eigenvalues [11, 12]. However, in case $\alpha \neq 0$, problem (1.1) and (1.2) is not symmetric, thus $\Sigma_{\alpha,\eta}^q$ may contain nonreal eigenvalues. A simple example is given by Example 2.1.

When $q(x) \equiv 0$, (1.1) is

$$-y'' = \lambda y, \quad x \in [0,1]. \tag{1.5}$$

Eigenvalues of problem (1.5)–(1.2) can be analyzed using elementary method, because all solutions of (1.5) can be found explicitly. However, as far as the authors know, even for this simple eigenvalue problem, the spectrum theory is incomplete in the literature. In [5, 6], Ma and O'Regan have constructed all *real* eigenvalues of problem (1.5)–(1.2) when all η_k are *rational*, and $\alpha = (\alpha_k)$ satisfies certain nondegeneracy condition. In [8, 9], Rynne has obtained all real eigenvalues for general $\eta \in \Delta^m$. See [13] for further extension.

The main topic of this paper is the structure of $\Sigma_{\alpha,\eta}^q$. Much attention will be paid to the real eigenvalues due to important applications in nonlinear problems.

Theorem 1.1. *Given $q \in L^1_{\mathbb{R}}$ and $(\alpha, \eta) \in \mathbb{R}^m \times \Delta^m$, then $\Sigma_{\alpha,\eta}^q$ is composed of a sequence $\{\lambda_n = \lambda_{n,\alpha,\eta}(q)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ which satisfies*

$$\operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 \le \cdots \le \operatorname{Re}\lambda_n \le \cdots, \qquad \lim_{n\to+\infty} \operatorname{Re}\lambda_n = +\infty. \tag{1.6}$$

Theorem 1.2. *Given $q \in L^1_{\mathbb{R}}$ and $(\alpha, \eta) \in \mathbb{R}^m \times \Delta^m$, then $\Sigma_{\alpha,\eta}^q \cap \mathbb{R} = \{\overline{\lambda}_n = \overline{\lambda}_{n,\alpha,\eta}(q)\}_{n\in\mathbb{N}}$, where*

$$\overline{\lambda}_1 < \overline{\lambda}_2 < \cdots < \overline{\lambda}_n < \cdots, \qquad \lim_{n\to+\infty} \overline{\lambda}_n = +\infty. \tag{1.7}$$

For $\alpha \in \mathbb{R}^m$, the norm is $\|\alpha\| = \|\alpha\|_{l^1} := \sum_{k=1}^m |\alpha_k|$. For $q \in L^1_{\mathbb{R}}$, the L^1 norm is denoted by $\|q\| := \|q\|_{L^1[0,1]}$. With some restrictions on α, we are able to prove that $\Sigma_{\alpha,\eta}^q$ contains only real eigenvalues.

Theorem 1.3. *If $\alpha \in \mathbb{R}^m$ satisfies $\|\alpha\| \leq 1/2$, then the spectrum $\Sigma^q_{\alpha,\eta}$ contains at most finitely many nonreal eigenvalues.*

Theorem 1.4. *Given $q \in L^1_{\mathbb{R}}$, there exists some constant $A(\|q\|) > 0$, depending on the norm $\|q\|$ only, such that if $\alpha \in \mathbb{R}^m$ satisfies $\|\alpha\| \leq A(\|q\|)$, then one has $\Sigma^q_{\alpha,\eta} \subset \mathbb{R}$.*

To sketch our proofs, let us denote

$$\begin{aligned}(M_q) &= \text{Problem} \quad (1.1) \quad (1.2),\\ (M_0) &= \text{Problem} \quad (1.4) \quad (1.2),\\ (D_0) &= \text{Problem} \quad (1.4) \quad (1.3).\end{aligned} \tag{1.8}$$

Basically, eigenvalues of (M_q) are zeros of some entire functions. See (2.24) and (3.3). In order to study the distributions of eigenvalues, we will consider (M_q) as a perturbation of (M_0) or of (D_0). To obtain the existence of infinitely many real eigenvalues as in Theorem 1.2, some properties of almost periodic functions [14, 15] will be used. See Lemmas 2.3 and 3.2. In order to pass the results of $\Sigma^0_{\alpha,\eta}$ to general potentials q, many techniques like implicit function theorem and the Rouché theorem will be exploited. Moreover, some basic estimates in [11] for fundamental solutions of (1.1) play an important role, especially in the proofs of Theorems 1.3 and 1.4. Due to the non-symmetry of problem (M_q), the proofs are complicated than that in [11] where the Dirichlet problem is considered.

The paper is organized as follows. In Section 2, we will give some detailed analysis on problem (M_0). In Section 3, after developing some basic estimates, we will prove Theorems 1.1 and 1.2. In Section 4, we will develop some techniques to exclude nonreal eigenvalues and complete the proofs of Theorems 1.3 and 1.4. Some open problem on the spectrum of (M_q) will be mentioned.

2. Structure of Eigenvalues of the Zero Potential

In order to motivate our consideration for $\Sigma^q_{\alpha,\eta}$ with non-zero potentials q, in this section we consider the spectrum $\Sigma^0_{\alpha,\eta}$ with the zero potential.

2.1. An Example of Nonreal Eigenvalues

Let $m = 1$. Boundary condition (1.2) is the following three-point boundary condition:

$$y(0) = 0, \qquad y(1) - \alpha y(\eta) = 0, \tag{2.1}$$

where $\alpha \in \mathbb{R}$ and $\eta \in \Delta^1 = (0,1)$. We consider the eigenvalue problems (1.5)–(2.1).

Let $\lambda \in \mathbb{C}$. Complex solutions $y(x)$ of (1.5) satisfying $y(0) = 0$ are $y(x) = cS_\lambda(x)$, $c \in \mathbb{C}$, where

$$S_\lambda(x) := \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\lambda^k x^{2k+1}, \quad x \in [0,1]. \tag{2.2}$$

Notice that $S_\lambda(x)$ is an entire function of $\lambda \in \mathbb{C}$. Define

$$T_0(\lambda) := S_\lambda(1) - \alpha S_\lambda(\eta) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - \frac{\alpha \sin \eta\sqrt{\lambda}}{\sqrt{\lambda}}, \quad \lambda \in \mathbb{C}. \tag{2.3}$$

Obviously, $T_0(\lambda)$ depends on the boundary data (α, η) as well. Then $\lambda \in \Sigma^0_{\alpha,\eta}$ if and only if λ satisfies

$$T_0(\lambda) = 0. \tag{2.4}$$

Example 2.1. Let $\eta = 1/3$. By (2.3) and (2.4), $\lambda = \omega^2 \in \Sigma^0_{\alpha,1/3}$ if and only if ω satisfies

$$T_0(\lambda) = -4\frac{\sin(\omega/3)}{\omega}\left(\sin^2\frac{\omega}{3} - \frac{(3-\alpha)}{4}\right) = 0. \tag{2.5}$$

That is, either

$$\frac{\sin(\omega/3)}{\omega} = 0 \tag{2.6}$$

or

$$\sin^2\frac{\omega}{3} = \frac{3-\alpha}{4} \tag{2.7}$$

Equation (2.6) shows that $\Sigma^0_{\alpha,1/3}$ always contains positive eigenvalues $(3n\pi)^2$, $n \in \mathbb{N}$.

Equation (2.7) has real solutions ω if and only if $\alpha \in [-1, 3]$. In this case, $\Sigma^0_{\alpha,1/3}$ consists of non-negative eigenvalues. More precisely,

$$\begin{gathered} \alpha \in [-1, 3) \Longrightarrow \Sigma^0_{\alpha,1/3} \subset (0, \infty), \\ \alpha = 3 \Longrightarrow \Sigma^0_{\alpha,1/3} \subset [0, \infty), \quad 0 \in \Sigma^0_{\alpha,1/3}. \end{gathered} \tag{2.8}$$

Equation (2.7) has nonreal solutions ω if and only if $\alpha \in (-\infty, -1) \cup (3, \infty)$. In this case, we have

$$\alpha \in (-\infty, -1) \cup (3, \infty) \Longrightarrow \Sigma^0_{\alpha,1/3} \setminus \mathbb{R} \neq \emptyset. \tag{2.9}$$

For example, one has

$$\alpha \in (-\infty, -1) \Longrightarrow \lambda = \left(\frac{3\pi}{2} + 3i \log \frac{\sqrt{-1-\alpha} + \sqrt{3-\alpha}}{2}\right)^2 \in \Sigma^0_{\alpha,1/3}, \tag{2.10}$$

$$\alpha \in (3, +\infty) \Longrightarrow \lambda = \left(3\pi + 3i \log \frac{\sqrt{\alpha - 3} + \sqrt{\alpha + 1}}{2}\right)^2 \in \Sigma^0_{\alpha,1/3}. \tag{2.11}$$

Notice that all eigenvalues obtained from (2.7) can be constructed explicitly as (2.10) and (2.11). For example, $\Sigma^0_{\alpha,1/3}$ contains negative eigenvalues if and only if $\alpha \in (3, \infty)$. Moreover, in this case, one has the unique negative eigenvalue given by

$$\lambda = -9\left(\log \frac{\sqrt{\alpha - 3} + \sqrt{\alpha + 1}}{2}\right)^2. \tag{2.12}$$

For more details, see [5, 6, 8].

Results (2.10) and (2.11) show that to guarantee that $\Sigma^0_{\alpha,\eta}$ contains only real eigenvalues, some restrictions on parameters (α, η) are necessary.

2.2. Real Eigenvalues with General Parameters

In the following we consider general $\alpha \in \mathbb{R}^m$, based on properties of almost periodic functions [14, 15].

Definition 2.2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. One calls that f is *almost periodic*, if for any $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that for any $a \in \mathbb{R}$, there exists $b = b_{a,\varepsilon} \in [a, a + l_\varepsilon]$ such that

$$\|f(\cdot + b) - f(\cdot)\|_{L^\infty} := \sup_{u\in\mathbb{R}} |f(u + b) - f(u)| < \varepsilon. \tag{2.13}$$

Any almost periodic function f admits a well-defined *mean value*

$$\overline{f} := \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(u)\mathrm{d}u \in \mathbb{R}. \tag{2.14}$$

To study (M_0) and (M_q), let us prove some properties on almost periodic functions.

Lemma 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic function.*

(i) *For any $A \in \mathbb{R}$, one has*

$$\inf_{u\in[A,\infty)} f(u) = \inf_{u\in\mathbb{R}} f(u), \tag{2.15}$$

$$\sup_{u\in[A,\infty)} f(u) = \sup_{u\in\mathbb{R}} f(u). \tag{2.16}$$

(ii) *Assume that f is non-zero and $\overline{f} = 0$. Then $f(u)$ is oscillatory as $u \to +\infty$, that is,*

$$\forall A \in \mathbb{R} \, \exists u_1, \quad u_2 > A \quad s.t. \quad f(u_1)f(u_2) < 0. \tag{2.17}$$

In particular, $f(u)$ has a sequence of positive zeros tending to $+\infty$.

Proof. (i) Let us only prove (2.15) because (2.16) is similar. For any $\varepsilon > 0$, choose $a_0 \in \mathbb{R}$ such that

$$f_0 \le f(a_0) < f_0 + \varepsilon, \qquad f_0 := \inf_{u\in\mathbb{R}} f(u) \in \mathbb{R}. \tag{2.18}$$

By (2.13), there exists $b_0 \in [a_0, a_0 + l_\varepsilon]$ such that $\|f(\cdot + b_0) - f(\cdot)\|_{L^\infty} < \varepsilon$. For any $A \in \mathbb{R}$, let us take

$$a = \max(a_0, A) + l_\varepsilon. \tag{2.19}$$

By (2.13) again, there exists $b \in [a, a + l_\varepsilon]$ such that $\|f(\cdot + b) - f(\cdot)\|_{L^\infty} < \varepsilon$. Hence

$$\left\| f(\cdot + b) - f(\cdot + b_0) \right\|_{L^\infty} < 2\varepsilon. \tag{2.20}$$

In particular,

$$\left| f((a_0 - b_0) + b) - f((a_0 - b_0) + b_0) \right| \le \left\| f(\cdot + b) - f(\cdot + b_0) \right\|_{L^\infty} < 2\varepsilon. \tag{2.21}$$

By the choice of a, one has $u_0 := a_0 - b_0 + b \ge a - l_\varepsilon \ge A$. Hence

$$f_0 \le \inf_{u\in[A,\infty)} f(u) \le f(u_0) < f(a_0) + 2\varepsilon < f_0 + 3\varepsilon. \tag{2.22}$$

This proves (2.15).

(ii) If $f \neq 0$ and f has mean value 0, it is easy to see that

$$\inf_{u\in\mathbb{R}} f(u) < 0 < \sup_{u\in\mathbb{R}} f(u). \tag{2.23}$$

Now result (2.17) can be deduced simply from (2.15) and (2.16). □

Like (2.3) and (2.4), all eigenvalues $\lambda \in \mathbb{C}$ of problem (M_0) are determined by the following equation:

$$M_0(\lambda) = 0, \tag{2.24}$$

where

$$M_0(\lambda) := S_\lambda(1) - \sum_{k=1}^{m} \alpha_k S_\lambda(\eta_k) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - \sum_{k=1}^{m} \frac{\alpha_k \sin \eta_k \sqrt{\lambda}}{\sqrt{\lambda}}, \quad \lambda \in \mathbb{C}. \tag{2.25}$$

Notice that $M_0(\lambda)$ is an entire function of $\lambda \in \mathbb{C}$. Hence (2.24) has only isolated zeros in $\mathbb{C}$. For $u, v \in \mathbb{R}$, we have the following elementary equalities:

$$\sin(u + iv) = \sin u \cosh v + i \cos u \sinh v, \qquad |\sin(u + iv)|^2 = \sin^2 u + \sinh^2 v. \tag{2.26}$$

For real eigenvalues of problem (M_0), we have the following result.

Lemma 2.4. *Given* $(\alpha, \eta) \in \mathbb{R}^m \times \Delta^m$, *then* $\Sigma^0_{\alpha,\eta} \cap \mathbb{R} = \{\overline{\lambda}_n = \overline{\lambda}_{n,\alpha,\eta}\}_{n\in\mathbb{N}}$, *where*

$$\overline{\lambda}_1 < \cdots < \overline{\lambda}_n < \cdots, \qquad \lim_{n\to\infty} \overline{\lambda}_n = +\infty. \tag{2.27}$$

Proof. Let us first consider possible positive eigenvalues $\lambda = u^2$ of (M_0), where $u > 0$. By the first equality of (2.26), equation (2.24) is the same as

$$F(u) = F_{\alpha,\eta}(u) := \sin u - \sum_{k=1}^{m} \alpha_k \sin \eta_k u = 0. \tag{2.28}$$

The function $F_{\alpha,\eta}(u)$ is a non-zero, almost periodic function and has mean value 0. In fact, $F_{\alpha,\eta}(u)$ is quasiperiodic. By Lemma 2.3(ii), $F_{\alpha,\eta}$ has infinitely many positive zeros tending to $+\infty$. See Figure 1. Hence $\Sigma^0_{\alpha,\eta}$ contains a sequence of positive eigenvalues tending to $+\infty$.

Next we consider possible negative eigenvalues $\lambda = -u^2$ of (M_0), where $u > 0$. In this case, (2.24) is the same as

$$\overline{F}(u) = \overline{F}_{\alpha,\eta}(u) := \sinh u - \sum_{k=1}^{m} \alpha_k \sinh \eta_k u = 0. \tag{2.29}$$

See the first equality of (2.26). Notice that $\overline{F}(u)$ is analytic in u. As $\eta_k \in (0,1)$, one has

$$\lim_{u\to+\infty} \frac{\overline{F}(u)}{\sinh u} = 1. \tag{2.30}$$

Thus (2.29) has at most finitely many positive solutions. Hence $\Sigma^0_{\alpha,\eta}$ contains at most finitely many negative eigenvalues.

As both (2.28) and (2.29) have only isolated solutions, the above two cases show that all real eigenvalues of (M_0) can be listed as in (2.27). □

The quasi-periodic function $F_{\alpha,\eta}(u)$ is as in Figure 1.

2.3. Nonexistence of Nonreal Eigenvalues

To study real eigenvalues of problem (M_0), the authors of [5, 6, 8] have imposed some restrictions on $\alpha = (\alpha_k) \in \mathbb{R}^m$. The typical conditions are

$$\alpha_k > 0, \quad \forall k \in \{1, 2, \ldots, m\}, \quad \|\alpha\| < 1. \tag{2.31}$$

With some restrictions on $\alpha = (\alpha_k)$, we will prove that $\Sigma^0_{\alpha,\eta}$ consists of only real eigenvalues.

Lemma 2.5. *Suppose that* $\alpha = (\alpha_k) \in \mathbb{R}^m$ *satisfies*

$$\|\alpha\|_{l^2} := \left(\sum_{k=1}^{m} \alpha_k^2\right)^{1/2} \le \frac{1}{\sqrt{m}}. \tag{2.32}$$

Then $\Sigma^0_{\alpha,\eta}$ *contains only real eigenvalues. Moreover, one has* $\Sigma^0_{\alpha,\eta} \subset ((\pi/2)^2, +\infty)$.

Proof. When $\alpha = 0$, problem (1.5)–(1.2) is the Dirichlet problem and $\Sigma^0_{\alpha,\eta} = \{(n\pi)^2 : n \in \mathbb{N}\}$. In the following, assume that $\alpha \ne 0$.

Suppose that $\lambda = \omega^2 \in \Sigma^0_{\alpha,\eta}$, where $\omega = u + iv$, $u, v \in \mathbb{R}$. We assert that $v = 0$ under assumption (2.32). Otherwise, assume that $v \ne 0$. By (2.26), equation (2.24) is the following system for $(u, v) \in \mathbb{R}^2$:

$$\sinh v \cos u = \sum_{k=1}^{m} \alpha_k \sinh \eta_k v \cos \eta_k u, \qquad \cosh v \sin u = \sum_{k=1}^{m} \alpha_k \cosh \eta_k v \sin \eta_k u. \tag{2.33}$$

It follows from the Hölder inequality that

$$\begin{aligned}
1 &= \cos^2 u + \sin^2 u \\
&= \left(\sum_{k=1}^{m} \frac{\alpha_k \sinh \eta_k v}{\sinh v} \cos \eta_k u\right)^2 + \left(\sum_{k=1}^{m} \frac{\alpha_k \cosh \eta_k v}{\cosh v} \sin \eta_k u\right)^2 \\
&\le \left(\sum_{k=1}^{m} \alpha_k^2 \frac{\sinh^2 \eta_k v}{\sinh^2 v}\right)\left(\sum_{k=1}^{m} \cos^2 \eta_k u\right) + \left(\sum_{k=1}^{m} \alpha_k^2 \frac{\cosh^2 \eta_k v}{\cosh^2 v}\right)\left(\sum_{k=1}^{m} \sin^2 \eta_k u\right) \\
&< \|\alpha\|_{l^2}^2 \cdot \sum_{k=1}^{m} \cos^2 \eta_k u + \|\alpha\|_{l^2}^2 \cdot \sum_{k=1}^{m} \sin^2 \eta_k u \\
&= m \|\alpha\|_{l^2}^2,
\end{aligned} \tag{2.34}$$

which is impossible under assumption (2.32). Thus $v = 0$ and therefore $\lambda = u^2 \ge 0$.

Next, by (2.2), (2.25), and the Hölder inequality, we have

$$M_0(0) = 1 - \sum_{k=1}^{m} \alpha_k \eta_k \ge 1 - \|\alpha\|_{l^2} \|\eta\|_{l^2} > 0, \tag{2.35}$$

because $\|\alpha\|_{l^2} \le 1/\sqrt{m}$ and $\|\eta\|_{l^2} < \sqrt{m}$. By (2.24), $0 \notin \Sigma^0_{\alpha,\eta}$. Hence we have $\sum^0_{\alpha,\eta} \subset (0, \infty)$.

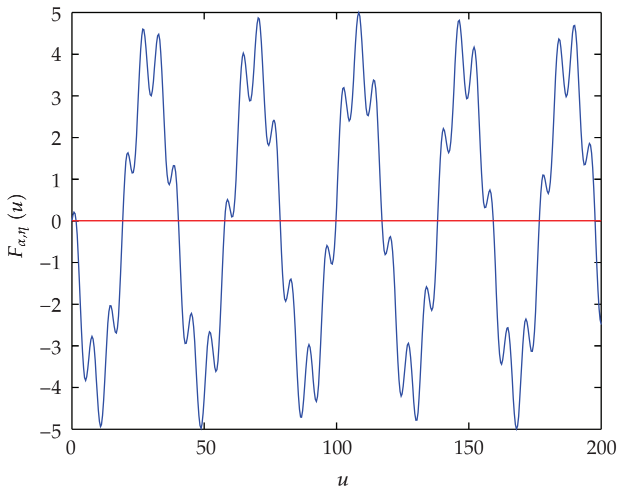

Figure 1: Function $F_{\alpha,\eta}(u)$ and its positive zeros, where $m = 1$, $\alpha = 4$ and $\eta = 1/(2\pi)$.

Finally, by the Hölder inequality, assumption (2.32) implies that $\|\alpha\| \leq \sqrt{m} \cdot \|\alpha\|_{l^2} \leq 1$. For any $u \in (0, \pi/2]$, the function $F_{\alpha,\eta}(u)$ of (2.28) satisfies

$$\begin{aligned} F_{\alpha,\eta}(u) &\geq \sin u - \sum_{k=1}^{m} |\alpha_k| \sin \eta_k u \\ &> \sin u - \left(\sum_{k=1}^{m} |\alpha_k| \right) \sin u \\ &\geq 0. \end{aligned} \tag{2.36}$$

Hence (2.28) shows that $\Sigma^0_{\alpha,\eta} \subset ((\pi/2)^2, \infty)$. □

Remark 2.6. Condition (2.32) is sharp. For example, let $m = 1$ and $\eta = 1/3$. Example 2.1 shows that $\Sigma^0_{\alpha,\eta}$ contains nonreal eigenvalues if $\alpha < -1$. Similarly, by letting $m = 1$ and $\eta = 1/5$, one can verify that $\Sigma^0_{\alpha,\eta}$ contains nonreal eigenvalues when $\alpha > 1$.

3. Structure of Eigenvalues of Non-Zero Potentials

Given $q \in L^1_{\mathbb{R}}$ and complex parameter $\lambda \in \mathbb{C}$, the fundamental solutions of (1.1) are denoted by $y_k(x, \lambda, q)$, $k = 1, 2$. That is, they are solutions of (1.1) satisfying the initial values

$$y_1(0, \lambda, q) = y_2'(0, \lambda, q) = 1, \qquad y_1'(0, \lambda, q) = y_2(0, \lambda, q) = 0. \tag{3.1}$$

Notice that $y_k(x, \lambda, q)$ are entire functions of $\lambda \in \mathbb{C}$. See [11]. To study (M_q), let us introduce

$$M_q(\lambda) := y_2(1, \lambda, q) - \sum_{k=1}^{m} \alpha_k y_2(\eta_k, \lambda, q), \quad \lambda \in \mathbb{C}, \tag{3.2}$$

which is an entire function of $\lambda \in \mathbb{C}$. See (2.25) for the case $q = 0$. Notice that $M_q(\lambda)$ is real for $\lambda \in \mathbb{R}$. Then $\lambda \in \Sigma^q_{\alpha,\eta}$ if and only if

$$M_q(\lambda) = 0. \tag{3.3}$$

3.1. Basic Estimates

Lemma 3.1. *Given $\beta \in (0,1)$, one has*

$$\lim_{\substack{v\in\mathbb{R}\\ |v|\to+\infty}} \frac{|\sin(u+iv)|}{\exp|v|} = \frac{1}{2}, \qquad \lim_{\substack{v\in\mathbb{R}\\ |v|\to+\infty}} \frac{|\sin\beta(u+iv)|}{\exp|v|} = 0 \tag{3.4}$$

uniformly in $u \in \mathbb{R}$.

Proof. Suppose that $u,\ v \in \mathbb{R}$. We have from (2.26)

$$\begin{aligned} |\sin(u+iv)| &= \sqrt{\sin^2 u + \sinh^2 v} \in [\sinh|v|, \cosh v], \\ |\sin\beta(u+iv)| &= \sqrt{\sin^2\beta u + \sinh^2\beta v} \le \cosh\beta v \le \exp(\beta|v|). \end{aligned} \tag{3.5}$$

The uniform limits in (3.4) are evident. □

For the function $F(u) = F_{\alpha,\eta}(u)$ of (2.28), one has the following result on its amplitude.

Lemma 3.2. *Given $(\alpha,\eta) \in \mathbb{R}^m \times \Delta^m$, there exist a constant $c_{\alpha,\eta} > 0$ and a sequence $\{a_n\}_{n\in\mathbb{N}}$ of increasing positive numbers such that $a_n \uparrow +\infty$ and*

$$(-1)^n F(a_n) \ge c_{\alpha,\eta} \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Proof. Recall that $F(u)$ is quasi-periodic and has the mean value 0. Denote that

$$c_{\alpha,\eta} := \frac{1}{2}\max\left(\sup_{u\in\mathbb{R}} F(u), -\inf_{u\in\mathbb{R}} F(u)\right). \tag{3.7}$$

Then $c_{\alpha,\eta} > 0$. The construction for $\{a_n\}_{n\in\mathbb{N}}$ is as follows. By (2.15), one has some $a_1 \in (0,\infty)$ such that $F(a_1) \le -c_{\alpha,\eta}$. By letting $A = a_1 + 1$ in (2.16), we have some $a_2 \in [a_1 + 1, \infty)$ such that $F_2(a_2) \ge c_{\alpha,\eta}$. Then, by letting $A = a_2 + 1$ in (2.15), we have some $a_3 \in [a_2 + 1, \infty)$ such that $F_2(a_3) \le -c_{\alpha,\eta}$. Inductively, we can use (2.15) and (2.16) to find a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \ge a_1 + n - 1$, and (3.6) is satisfied for all $n \in \mathbb{N}$. □

Lemma 3.3 (basic estimates, [11, page 13, Theorem 3]). *Let $q \in L^1_{\mathbb{R}}$ and $\lambda \in \mathbb{C}$. There hold the following estimates for all $x \in [0,1]$*

$$\left|y_1(x,\lambda,q) - C_\lambda(x)\right| \le \frac{1}{\left|\sqrt{\lambda}\right|}\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x + \|q\|_{L^1[0,x]}\right), \tag{3.8}$$

$$\left|y_2(x,\lambda,q) - S_\lambda(x)\right| \le \frac{1}{|\lambda|}\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x + \|q\|_{L^1[0,x]}\right), \tag{3.9}$$

$$\left|y_1'(x,\lambda,q) - C_\lambda'(x)\right| \le \|q\|\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x + \|q\|_{L^1[0,x]}\right), \tag{3.10}$$

$$\left|y_2'(x,\lambda,q) - S_\lambda'(x)\right| \le \frac{\|q\|}{\left|\sqrt{\lambda}\right|}\exp\left(\left|\operatorname{Im}\sqrt{\lambda}\right|x + \|q\|_{L^1[0,x]}\right). \tag{3.11}$$

Remark 3.4. For their purpose, the authors of [11] have proved (3.8)–(3.11) for complex potentials $q \in L^2_{\mathbb{C}} := L^2([0,1],\mathbb{C})$. For example, in (3.8)–(3.11), the terms $\|q\|$ and $\|q\|_{L^1[0,x]}$ are replaced by $\|q\|_{L^2[0,1]}$ and $\|q\|_{L^2[0,1]} \cdot \sqrt{x}$, respectively in [11]. Inspecting their proofs, especially the proof of [11, pages 7–9, Theorem 1], one can find that estimates (3.8)–(3.11) are also true for L^1 potentials q. Moreover, these estimates can be established even for linear measure differential equations with general measures [16]. By the Hölder inequality, one has

$$\|q\| = \|q\|_{L^1[0,1]} \le \|q\|_{L^2[0,1]}, \qquad \|q\|_{L^1[0,x]} \le \|q\|_{L^2[0,1]} \cdot \sqrt{x}. \tag{3.12}$$

This is why the authors of [11] have used these terms in (3.8)–(3.11).

Lemma 3.5. *There holds the following estimate for $M_q(\lambda)$:*

$$\left|M_q(\lambda) - M_0(\lambda)\right| \le \frac{B\exp(|Im\,\omega|)}{|\omega|^2}, \quad \omega := \sqrt{\lambda} \in \mathbb{C}, \tag{3.13}$$

where

$$B = B(\alpha, q) := (1 + \|\alpha\|)\exp\left(\|q\|\right) \in [1,\infty). \tag{3.14}$$

Proof. Define $\varphi(x) := y_2(x,\lambda,q) - S_\lambda(x)$. From (3.9), we have

$$\left|\varphi(x)\right| \le \exp\left(\|q\|\right)|\omega|^{-2}\exp(|\operatorname{Im}\omega|) \quad \forall x \in [0,1]. \tag{3.15}$$

By (2.25) and (3.2), we have

$$\begin{aligned} \left|M_q(\lambda) - M_0(\lambda)\right| &= \left|\varphi(1) - \sum_{k=1}^{m} \alpha_k \varphi(\eta_k)\right| \\ &\le \left|\varphi(1)\right| + \sum_{k=1}^{m} |\alpha_k| \left|\varphi(\eta_k)\right| \\ &\le (1 + \|\alpha\|) \exp(\|q\|) |\omega|^{-2} \exp(|\operatorname{Im} \omega|). \end{aligned} \tag{3.16}$$

This gives (3.13). □

Lemma 3.6. *One has $M_q(\lambda) \not\equiv 0$ on $\mathbb{R}$. Consequently, there exists $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 \notin \Sigma^q_{\alpha,\eta}$.*

Proof. Otherwise, we have $M_q(\lambda) \equiv 0$ on $\mathbb{R}$. Notice that

$$M_0\left(u^2\right) \equiv \frac{F(u)}{u}, \quad u > 0. \tag{3.17}$$

Let $\lambda = a_n^2$ in (3.13), where $\{a_n\}_{n\in\mathbb{N}}$ is as in Lemma 3.2. We have

$$\left|\frac{F(a_n)}{a_n}\right| = \left|M_0\left(a_n^2\right)\right| \le \frac{B}{a_n^2} \quad \forall n \in \mathbb{N}. \tag{3.18}$$

Hence $\lim_{n\to+\infty} |F(a_n)| \le B/a_n \to 0$, a contradiction with (3.6). □

3.2. Eigenvalues with General Parameters

The most general results on spectrum $\Sigma^q_{\alpha,\eta}$ of (M_q) are stated as in Theorem 1.1.

Proof of Theorem 1.1. We argue as in general spectrum theory [12]. By Lemma 3.6, there exists $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 \notin \Sigma^q_{\alpha,\eta}$. That is, the following equation:

$$-y'' + q(x)y - \lambda_0 y = 0 \tag{3.19}$$

has only the trivial solution $y = 0$ satisfying boundary condition (1.2). Let $G_0(x,u)$ be the Green function associated with problem (3.19)-(1.2). Then $\lambda \in \Sigma^q_{\alpha,\eta}$ if and only if $\lambda \ne \lambda_0$ and

$$-y'' + \left(q(x) - \lambda_0\right)y = (\lambda - \lambda_0)y \tag{3.20}$$

has nontrivial solution $y(x)$ satisfying (1.2). In other words, $\lambda \in \Sigma^q_{\alpha,\eta}$ if and only if the following equation:

$$y = (\lambda - \lambda_0) L_q y \tag{3.21}$$

has non-trivial solution y, where

$$L_q y(x) := \int_0^1 G_0(x,u)\big(q(u)-\lambda_0\big)y(u)\mathrm{d}u. \tag{3.22}$$

Since L_q is a compact linear operator, one sees that this happens when and only when $1/(\lambda - \lambda_0) \in \sigma(L_q) \subset \mathbb{C}$, where $\sigma(L_q)$ is the spectrum of L_q. Hence $\Sigma^q_{\alpha,\eta}$ consists of a sequence of eigenvalues which can accumulate only at infinity of $\mathbb{C}$.

For $\lambda \in \mathbb{C}$, denote that

$$\lambda = \omega^2, \qquad \omega = \sqrt{\lambda} = u + iv, \quad u, v \in \mathbb{R}. \tag{3.23}$$

Suppose that $\lambda \in \Sigma^q_{\alpha,\eta}$ and $\lambda \neq 0$. Then $M_q(\lambda) = 0$ and (3.13) implies that

$$\begin{aligned} \frac{B\exp|v|}{|\omega|^2} &\geq \left|M_q(\lambda) - M_0(\lambda)\right| = |M_0(\lambda)| \\ &= \left|\frac{\sin\omega - \sum_{k=1}^m \alpha_k \sin\eta_k\omega}{\omega}\right| \\ &\geq \frac{|\sin(u+iv)| - \sum_{k=1}^m |\alpha_k|\left|\sin\eta_k(u+iv)\right|}{|\omega|}. \end{aligned} \tag{3.24}$$

We conclude that all non-zero eigenvalues $\lambda \in \Sigma^q_{\alpha,\eta}$ satisfy

$$|\omega| \cdot \frac{|\sin(u+iv)| - \sum_{k=1}^m |\alpha_k|\left|\sin\eta_k(u+iv)\right|}{\exp|v|} \leq B. \tag{3.25}$$

Let us derive some consequences from estimate (3.25) for $\lambda \in \Sigma^q_{\alpha,\eta}$.

(i) Since $|\omega| \geq |v|$, it follows from the uniform limits in (3.4) that

$$\lim_{|v|=|\mathrm{Im}\ \omega|\to+\infty} |\omega| \cdot \frac{|\sin(u+iv)| - \sum_{k=1}^m |\alpha_k|\left|\sin\eta_k(u+iv)\right|}{\exp|v|} = +\infty. \tag{3.26}$$

Thus there exists some $h = h_{\alpha,\eta,q} > 0$ such that

$$\lambda \in \Sigma^q_{\alpha,\eta} \Longrightarrow \omega = \sqrt{\lambda} \in H_h := \{\omega \in \mathbb{C} : |\mathrm{Im}\ \omega| < h\}. \tag{3.27}$$

The horizontal strip H_h of (3.27) in the ω-plane is transformed by (3.23) to the following half-plane P_r in the λ-plane:

$$\Sigma^q_{\alpha,\eta} \subset P_r := \{\lambda \in \mathbb{C} : \mathrm{Re}\ \lambda > r\}, \quad \text{where } r := -h^2. \tag{3.28}$$

(ii) Let $\hat{r} > -h^2$. We assert that

$$\Sigma_{\alpha,\eta}^q \cap \{\lambda \in \mathbb{C} : \text{Re } \lambda \le \hat{r}\} = \Sigma_{\alpha,\eta}^q \cap \left\{\lambda \in \mathbb{C} : -h^2 < Re\,\lambda \le \hat{r}\right\} \tag{3.29}$$

contains at most finitely many eigenvalues. Otherwise, suppose that

$$\Sigma_{\alpha,\eta}^q \cap \left\{\lambda \in \mathbb{C} : -h^2 < Re\,\lambda \le \hat{r}\right\} \tag{3.30}$$

contains infinitely many λ_n, $n \in \mathbb{N}$. Since (3.3) has only isolated solutions, we have necessarily $|\text{Im } \lambda_n| \to +\infty$. By denoting $\sqrt{\lambda_n} = u_n + iv_n$, one has

$$-h^2 < u_n^2 - v_n^2 \le \hat{r}, \qquad 2|u_n||v_n| \longrightarrow +\infty. \tag{3.31}$$

In particular, $|v_n| \to +\infty$. Now estimate (3.25) reads as

$$\frac{|\sin(u_n + iv_n)|}{\exp|v_n|} \le \frac{\sum_{k=1}^m |\alpha_k| \left|\sin \eta_k(u_n + iv_n)\right|}{\exp|v_n|} + o(1) \quad \text{as } n \longrightarrow \infty. \tag{3.32}$$

This is impossible because we have the uniform limits (3.4).

Combining (i) and (ii), we know that $\Sigma_{\alpha,\eta}^q$ can be listed as in (1.6). □

Though problem (M_q) is not symmetric, $\Sigma_{\alpha,\eta}^q$ always contains infinitely many real eigenvalues, as stated in Theorem 1.2.

Proof of Theorem 1.2. We need to only consider positive eigenvalues of (M_q). Let $\lambda = a_n^2$ in (3.13), where $\{a_n\}_{n\in\mathbb{N}}$ is as in Lemma 3.2. By using (3.17), we have

$$\left|M_q\left(a_n^2\right) - \frac{F(a_n)}{a_n}\right| \le \frac{B}{a_n^2} \quad \forall n \in \mathbb{N}. \tag{3.33}$$

Since $a_n \uparrow +\infty$, w.l.o.g., we can assume that $a_n \ge 2B/c_{\alpha,\eta}$ for all $n \in \mathbb{N}$. Thus

$$\left|a_n M_q\left(a_n^2\right) - F(a_n)\right| \le \frac{B}{a_n} \le \frac{c_{\alpha,\eta}}{2} \quad \forall n \in \mathbb{N}. \tag{3.34}$$

By using (3.6), we conclude that

$$(-1)^n M_q\left(a_n^2\right) > 0 \quad \forall n \in \mathbb{N}. \tag{3.35}$$

Hence (3.3) has at least one positive solution $\overline{\lambda}_n$ in each interval (a_n^2, a_{n+1}^2), $n \in \mathbb{N}$. Combining with Theorem 1.1, $\Sigma_{\alpha,\eta}^q \cap \mathbb{R}$ consists of a sequence of real eigenvalues tending to $+\infty$. Hence $\Sigma_{\alpha,\eta}^q \cap \mathbb{R}$ can be listed as in (1.7). □

4. Nonexistence of Nonreal Eigenvalues for Small α

We will apply the Rouché theorem to give further results on $\Sigma^q_{\alpha,\eta}$ when $\|\alpha\|$ is small, following the approach in [11] for the Dirichlet problem (1.1)-(1.4), which corresponds to (M_q) with $\alpha = 0$. Let us recall the Rouché theorem.

Lemma 4.1 (Rouché theorem). *Suppose that $f(z), g(z)$ are entire functions of $z \in \mathbb{C}$. If $|g(z)| < |f(z)|$ on a Jordan curve C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C, counted multiplicities.*

For later use, let us introduce the following elementary function:

$$\overline{G}(\omega) := \frac{\sqrt{\left(\sin^2 u + \sinh^2 v\right)}}{\cosh v} = \sqrt{1 - \left(\frac{\cos u}{\cosh v}\right)^2}, \quad \omega = u + iv \in \mathbb{C}. \tag{4.1}$$

Then $\overline{G}(\omega) \equiv \overline{G}(\omega + \pi)$. Obviously, $\overline{G}(\omega) = 0$ if and only if $\omega = n\pi$, $n \in \mathbb{Z}$. Define

$$G(r) := \min_{\omega \in C_r} G(\omega) \in [0,1], \quad r \in [0,\infty), \tag{4.2}$$

where C_r is the circle in the ω-plane

$$C_r := \{\omega = u + iv \in \mathbb{C} : |\omega| = r\}. \tag{4.3}$$

Then $G(n\pi) = 0$, $n \in \mathbb{Z}^+ := \{0, 1, \ldots, n, \ldots\}$, and $0 < G(r) < 1$ for all $r \in [0,\infty) \setminus \pi\mathbb{Z}^+$. Let $\mathbf{r}_0$ be the unique solution of the following equation:

$$\tanh r = \sin r, \quad r \in (0,\pi). \tag{4.4}$$

Numerically, $\mathbf{r}_0 \doteq 1.8751 \doteq 0.5968\pi$. The following facts can be verified by elementary arguments.

Lemma 4.2. *One has*

$$G(r) = \begin{cases} \tanh r & \text{for } r \in [0, \mathbf{r}_0], \\ \sin r & \text{for } r \in [\mathbf{r}_0, \pi], \end{cases}$$
$$\lim_{n \to +\infty} G\left(\left(n + \frac{1}{2}\right)\pi\right) = 1. \tag{4.5}$$

For the graph of $G(r)$, see Figure 2.

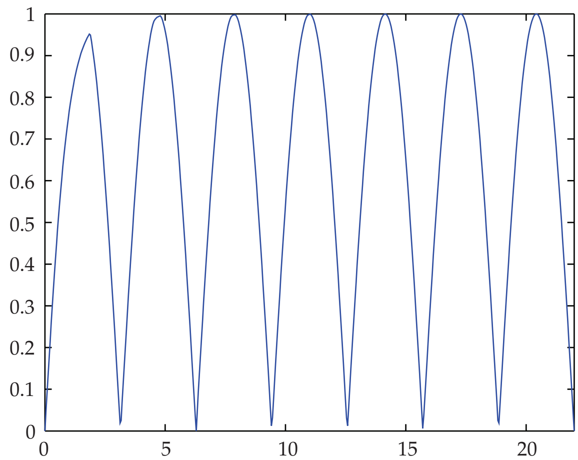

Figure 2: The function $G(r)$, where $r \in [0, 7\pi]$.

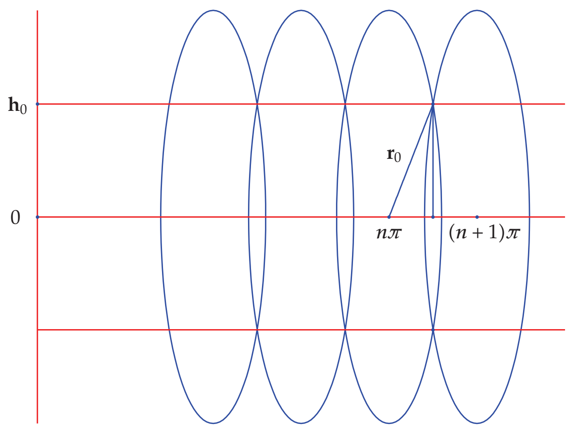

Figure 3: Finding zeros of $M_q(\omega^2)$ in the ω-plane.

4.1. Large Eigenvalues

In the following we apply the Rouché theorem to study the spectrum $\Sigma^q_{\alpha,\eta}$, that is, the zeros of the function $M_q(\lambda)$ in the λ-plane. To this end, we consider problem (M_q) as a perturbation of the Dirichlet problem (D_0), whose eigenvalues are zeros of the function

$$D_0(\lambda) := \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}, \quad \lambda \in \mathbb{C}. \tag{4.6}$$

Let $\lambda = \omega^2$. Equation (3.3) is the same as

$$D_0\left(\omega^2\right) + \left(M_q\left(\omega^2\right) - D_0\left(\omega^2\right)\right) = 0, \quad \omega \in \mathbb{C}, \tag{4.7}$$

which is considered as a perturbation of the following equation:

$$D_0\left(\omega^2\right) = \frac{\sin\omega}{\omega} = 0, \quad \omega \in \mathbb{C}. \tag{4.8}$$

Due to the form of (4.7) and (4.8), one needs to only consider solutions in the right half-plane $\mathbb{C}_+$ of ω. Notice that all solutions of (4.8) are $n\pi$, $n \in \mathbb{N}$, which are simple zeros of $D_0(\omega^2)$. For any $\alpha \neq 0$, we do not know whether all zeros of (4.7) are real. In order to overcome this, the proof is complicated than that in [11].

Let us derive another consequence from estimate (3.25) with some restriction on $\alpha = (\alpha_k)$. Suppose that $\alpha \in \mathbb{R}^m$ satisfies $\|\alpha\| < 1$. Define the positive function

$$W(h) = W(h;\alpha,q) \stackrel{\text{def}}{=} \frac{B\exp h}{\sinh h - \|\alpha\|\cosh h}, \quad h \in (\arctan h\ \|\alpha\|, \infty), \tag{4.9}$$

where $B = B(\alpha, q)$ is as in (3.14). Then $W(h)$ is decreasing in $h \in (\operatorname{arctanh}\|\alpha\|, \infty)$.

Lemma 4.3. *Suppose that*

$$\|\alpha\| < 1, \qquad h > \operatorname{arctanh}\|\alpha\|. \tag{4.10}$$

Then for any $\lambda = \omega^2 \in \Sigma^q_{\alpha,\eta}$, where $\eta \in \Delta^m$, one has

$$\text{either } |\operatorname{Im}\omega| < h, \quad \text{or} \quad |\omega| \le W(h). \tag{4.11}$$

Proof. We keep the notations in (3.23) Let $\lambda = \omega^2 \in \Sigma^q_{\alpha,\eta}$. If $|v| = |\operatorname{Im}\omega| \ge h$, it follows from (2.26) and (3.25) that

$$\begin{aligned}\frac{B\exp|v|}{|\omega|} &\ge |\sin(u+iv)| - \sum_{k=1}^m |\alpha_k| \left|\sin\ \eta_k(u+iv)\right| \\ &= \sqrt{\sin^2 u + \sinh^2 v} - \sum_{k=1}^m |\alpha_k| \sqrt{\sin^2\eta_k u + \sinh^2\eta_k v} \\ &\ge \sinh|v| - \sum_{k=1}^m |\alpha_k| \sqrt{1+\sinh^2 v} \\ &= \sinh|v| - \|\alpha\|\cosh v.\end{aligned} \tag{4.12}$$

Using the function $W(\cdot)$ in (4.9), we obtain $|\omega| \le W(|v|) \le W(h)$. This proves (4.11). □

Consider the following circles of the ω-plane:

$$C_{n,r} := \{\omega \in \mathbb{C} : |\omega - n\pi| = r\} = n\pi + C_r, \quad n \in \mathbb{Z},\ r \in (0,\infty) \setminus \pi\mathbb{N}. \tag{4.13}$$

Lemma 4.4. *Let* (n, r) *be as in* (4.13). *one has*

$$\left|\frac{M_q(\lambda) - D_0(\lambda)}{D_0(\lambda)}\right| \le \frac{1}{G(r)}\left(\|\alpha\| + \frac{2B}{|\omega|}\right) \quad \forall \omega = \sqrt{\lambda} \in C_{n,r}. \tag{4.14}$$

Proof. Let $\omega = n\pi + u + iv \in C_{n,r}$, where $u + iv \in C_r$. Then $\sin\omega \neq 0$. By (2.26) we have

$$\begin{aligned}\left|\frac{\sin\eta_k\omega}{\sin\omega}\right| &= \frac{\sqrt{\sin^2\eta_k(n\pi+u)+\sinh^2\eta_k v}}{\sqrt{\sin^2 u+\sinh^2 v}} \\ &\le \frac{\sqrt{1+\sinh^2 v}}{\sqrt{\sin^2 u+\sinh^2 v}} \\ &= \frac{1}{\overline{G}(u+iv)} \le \frac{1}{G(r)}.\end{aligned} \tag{4.15}$$

See (4.1) and (4.2). Notice that

$$\left|\frac{M_q(\lambda) - D_0(\lambda)}{D_0(\lambda)}\right| \le \frac{|M_0(\lambda) - D_0(\lambda)|}{|D_0(\lambda)|} + \frac{\left|M_q(\lambda) - M_0(\lambda)\right|}{|D_0(\lambda)|} =: T_1 + T_2. \tag{4.16}$$

By (2.25) and (4.15), we have

$$T_1 \le \sum_{k=1}^{m}|\alpha_k|\left|\frac{\sin\eta_k\omega}{\sin\omega}\right| \le \frac{\|\alpha\|}{G(r)}. \tag{4.17}$$

Since

$$\frac{\exp(|\mathrm{Im}\ \omega|)}{|\sin\omega|} = \frac{\exp|v|}{\cosh v}\frac{\cosh v}{\sqrt{\sin^2 u+\sinh^2 v}} = \frac{\exp|v|/\cosh v}{\widehat{G}(u+iv)} \le \frac{2}{G(r)}, \tag{4.18}$$

Compared with (4.1) and (4.2), it follows from (3.13) that

$$T_2 \le \frac{\exp(|\mathrm{Im}\ \ \omega|)}{|\sin\omega|}\frac{B}{|\omega|} \le \frac{2B}{G(r)|\omega|}. \tag{4.19}$$

Thus one has (4.14). □

Proof of Theorem 1.3. Let

$$\mathbf{h}_0 \overset{\text{def}}{=} \sqrt{\mathbf{r}_0^2 - \left(\frac{\pi}{2}\right)^2} \doteq 1.0240, \tag{4.20}$$

where the constant $\mathbf{r}_0 \in (\pi/2, \pi)$ is as in Lemma 4.2. One has $\sin \mathbf{r}_0 \doteq 0.9541$ and $\tanh \mathbf{h}_0 \doteq 0.7714$. Denote by $\mathbb{D}_{n,\mathbf{r}_0}$ the disc enclosed by the circle $C_{n,\mathbf{r}_0}$, that is,

$$\mathbb{D}_{n,r_0} := \{\omega \in \mathbb{C} : |\omega - n\pi| < \mathbf{r}_0\}, \quad n \in \mathbb{Z}. \tag{4.21}$$

Since $\mathbf{r}_0 > \pi/2$, $\mathbb{D}_{n,\mathbf{r}_0}$ intersects $\mathbb{D}_{n+1,\mathbf{r}_0}$. See Figure 3.

In the following, we always assume that $\alpha \in \mathbb{R}^m$ satisfies

$$\|\alpha\| \leq \frac{1}{2} < \min(\sin \mathbf{r}_0, \tanh \mathbf{h}_0). \tag{4.22}$$

Suggested by (3.14), (4.9), and (4.11), we denote

$$c_0 \stackrel{\text{def}}{=} \frac{(1+1/2)\exp \mathbf{h}_0}{\sinh \mathbf{h}_0 - (1/2)\cosh \mathbf{h}_0} = \frac{3\exp \mathbf{h}_0}{2\sinh \mathbf{h}_0 - \cosh \mathbf{h}_0} \doteq 9.7873. \tag{4.23}$$

Then, for all α as in (4.22), by (4.9) one has

$$W(\mathbf{h}_0;\alpha,q) = \frac{(1+\|\alpha\|)\exp \mathbf{h}_0}{\sinh \mathbf{h}_0 - \|\alpha\|\cosh \mathbf{h}_0}\exp(\|q\|) \leq c_0 \exp(\|q\|). \tag{4.24}$$

Suppose that

$$\lambda_0 = \omega_0^2 \in \Sigma_{\alpha,\eta}^q, \qquad |\omega_0| \geq 11\exp(\|q\|). \tag{4.25}$$

Let us show that λ_0 must be positive. In fact, (4.25) implies that $|\omega_0| \geq 11\exp(\|q\|) > W(\mathbf{h}_0;\alpha,q)$. By result (4.11), we have $|\operatorname{Im} \omega_0| < \mathbf{h}_0$. Hence ω_0 is a zero of $M_q(\omega^2)$ inside some disc $\mathbb{D}_{n,\mathbf{r}_0}$. See Figure 3. W.l.o.g., let us assume that $n > 0$. Then n satisfies

$$n\pi \geq |\omega_0| - |\omega_0 - n\pi| > 11\exp(\|q\|) - \mathbf{r}_0. \tag{4.26}$$

For any $\omega \in C_{n,\mathbf{r}_0}$, one has

$$|\omega| \geq n\pi - |\omega - n\pi| \geq n\pi - \mathbf{r}_0 > 11\exp(\|q\|) - 2\mathbf{r}_0. \tag{4.27}$$

It follows from (4.5) that $G(\mathbf{r}_0) = \sinh \mathbf{r}_0 = \sin \mathbf{r}_0$. By (4.14), we have the estimate

$$\begin{aligned}
\left| \frac{M_q(\omega^2) - D_0(\omega^2)}{D_0(\omega^2)} \right| &\leq \frac{1}{G(\mathbf{r}_0)} \left(\|\alpha\| + \frac{2(1 + \|\alpha\|) \exp(\|q\|)}{|\omega|} \right) \\
&< \frac{1}{\sinh \mathbf{r}_0} \left(\frac{1}{2} + \frac{3 \exp(\|q\|)}{11 \exp(\|q\|) - 2\mathbf{r}_0} \right) \\
&\leq \frac{1}{\sinh \mathbf{r}_0} \left(\frac{1}{2} + \frac{3}{11 - 2\mathbf{r}_0} \right) \doteq 0.9578 \\
&< 1.
\end{aligned} \tag{4.28}$$

Since (4.8) has the unique, simple zero $\omega = n\pi$ in $\mathbb{D}_{n,\mathbf{r}_0}$, by the Rouché theorem, we conclude from estimate (4.28) that (4.7) has the unique, simple zero $\omega = \omega_n$ in $\mathbb{D}_{n,\mathbf{r}_0}$. Furthermore, denote that

$$\varepsilon_{\pm} := \frac{M_q\left((n\pi \pm \mathbf{r}_0)^2\right) - D_0\left((n\pi \pm \mathbf{r}_0)^2\right)}{D_0\left((n\pi \pm \mathbf{r}_0)^2\right)} \in (-1, 1). \tag{4.29}$$

See (4.28). We have

$$M_q\left((n\pi \pm \mathbf{r}_0)^2\right) = (1 + \varepsilon_{\pm}) D_0\left((n\pi \pm \mathbf{r}_0)^2\right) = \frac{\pm(-1)^n (1 + \varepsilon_{\pm}) \sin \mathbf{r}_0}{n\pi \pm \mathbf{r}_0}. \tag{4.30}$$

Thus

$$M_q\left((n\pi - \mathbf{r}_0)^2\right) \cdot M_q\left((n\pi + \mathbf{r}_0)^2\right) = -\frac{(1 + \varepsilon_+)(1 + \varepsilon_-) \sin^2 \mathbf{r}_0}{(n\pi)^2 - \mathbf{r}_0^2} < 0. \tag{4.31}$$

Hence (4.7) has at least one real solution in the interval $(n\pi - \mathbf{r}_0, n\pi + \mathbf{r}_0) \subset \mathbb{D}_{n,\mathbf{r}_0}$. Due to the uniqueness, all eigenvalues λ_0 as in (4.25) must be positive.

Finally, it follows from Theorem 1.1 that $\Sigma_{\alpha,\eta}^q$ contains at most finitely many λ which do not satisfy (4.25). Thus the proof of Theorem 1.3 is completed. □

4.2. Small Eigenvalues

In order to prove Theorem 1.4, we need to show that all "small eigenvalues" are also real provided that $\|\alpha\|$ is small. The proof below is a modification of the proof of Theorem 1.3.

Proof of Theorem 1.4. By (4.4), we can fix some $\mathbf{n} = \mathbf{n}_{\|q\|} \in \mathbb{N}$ such that

$$\mathbf{n} \geq \frac{11 \exp(\|q\|)}{\pi} - \frac{1}{2}, \tag{4.32}$$

$$\frac{1}{G((\mathbf{n}+1/2)\pi)}\left(\frac{1}{2} + \frac{3\exp(\|q\|)}{(\mathbf{n}+1/2)\pi}\right) < 1. \tag{4.33}$$

Denote that

$$\mathbf{r} = \mathbf{r}_{\|q\|} := \left(\left(\mathbf{n} + \frac{1}{2}\right)\pi\right)^2, \qquad \mathbb{D} = \mathbb{D}_{\|q\|} := \{\lambda \in \mathbb{C} : |\lambda| < \mathbf{r}\}. \tag{4.34}$$

In the following we assume that $\alpha \in \mathbb{R}^m$ satisfies (4.22), that is, $\|\alpha\| \leq 1/2$. From the proof of Theorem 1.3, $\Sigma^q_{\alpha,\eta} \cap (\mathbb{C} \setminus \mathbb{D})$ consists of positive eigenvalues. See conditions (4.25) and (4.32). Moreover, for $\lambda \in \partial\mathbb{D}$, that is, $|\sqrt{\lambda}| = (\mathbf{n}+1/2)\pi$, we obtain from estimate (4.14) and condition (4.33) that

$$\begin{aligned}\left|\frac{M_q(\lambda) - D_0(\lambda)}{D_0(\lambda)}\right| &\leq \frac{1}{G((\mathbf{n}+1/2)\pi)}\left(\|\alpha\| + \frac{2(1+\|\alpha\|)\exp(\|q\|)}{(\mathbf{n}+1/2)\pi}\right) \\ &\leq \frac{1}{G((\mathbf{n}+1/2)\pi)}\left(\frac{1}{2} + \frac{3\exp(\|q\|)}{(\mathbf{n}+1/2)\pi}\right) < 1.\end{aligned} \tag{4.35}$$

Notice that equation

$$D_0(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0, \quad \lambda \in \mathbb{D}, \tag{4.36}$$

has (simple) solutions $\lambda = (n\pi)^2$, $n = 1, \ldots, \mathbf{n}$. By the Rouché theorem, we conclude that, if $\|\alpha\| \leq 1/2$, the following problem:

$$M_q(\lambda, \alpha) = 0, \quad \lambda \in \mathbb{D}, \tag{4.37}$$

has precisely $\mathbf{n}$ solutions, counted multiplicity. Here $M_q(\lambda)$ has been written as $M_q(\lambda,\alpha)$ to emphasize the dependence on α.

Suppose that $\alpha = 0$. Equation (4.37) corresponds to the Dirichlet eigenvalue problem (1.1)–(1.4), which has only real eigenvalues. Moreover, all solutions of (4.37) are simple in this case [11]. Hence solutions of problem (4.37) can be denoted by $\lambda = \mu_n$, $1 \leq n \leq \mathbf{n}$, where

$$-\mathbf{r} < \mu_1 < \cdots < \mu_{\mathbf{n}} < \mathbf{r}. \tag{4.38}$$

They are the first $\mathbf{n}$ eigenvalues of problem (1.1)–(1.4).

In the following, we apply the implicit function theorem to prove that solutions of (4.37) inside $\mathbb{D}$ are actually real when $\|\alpha\|$ is small. Notice that $M_q(\lambda,\alpha)$ is a smooth real-valued function of $(\lambda,\alpha)\in\mathbb{R}^2$. By [11, page 21, Theorem 6], the derivative of $M_q(\lambda,\alpha)$ w.r.t. λ is

$$\partial_\lambda M_q(\lambda,\alpha)=\int_0^1 y_2(t,\lambda,q)\left(y_1(1,\lambda,q)y_2(t,\lambda,q)-y_2(1,\lambda,q)y_1(t,\lambda,q)\right)\mathrm{d}t \\ -\sum_{k=1}^{m}\alpha_k\int_0^{\eta_k} y_2(t,\lambda,q)\left(y_1(\eta_k,\lambda,q)y_2(t,\lambda,q)-y_2(\eta_k,\lambda,q)y_1(t,\lambda,q)\right)\mathrm{d}t. \tag{4.39}$$

In particular,

$$\partial_\lambda M_q(\lambda,\alpha)|_{(\lambda,\alpha)=(\mu_n,0)}=a_n\int_0^1 y_2^2(t,\mu_n,q)\mathrm{d}t-b_n\int_0^1 y_1(t,\mu_n,q)y_2(t,\mu_n,q)\mathrm{d}t, \tag{4.40}$$

where

$$a_n:=y_1(1,\mu_n,q),\qquad b_n:=y_2(1,\mu_n,q). \tag{4.41}$$

Since μ_n is a Dirichlet eigenvalue of problem (1.1), we have $b_n=0$. Moreover, the Liouville theorem for (1.1) implies that

$$1=\det\begin{pmatrix} y_1(1,\mu_n,q) & y_2(1,\mu_n,q)\\ y_1'(1,\mu_n,q) & y_2'(1,\mu_n,q)\end{pmatrix}=\det\begin{pmatrix} y_1(1,\mu_n,q) & 0\\ y_1'(1,\mu_n,q) & y_2'(1,\mu_n,q)\end{pmatrix}. \tag{4.42}$$

In particular, $a_n\neq 0$. Hence

$$\partial_\lambda M_q(\lambda,\alpha)|_{(\lambda,\alpha)=(\mu_n,0)}=a_n\int_0^1 y_2^2(t,\mu_n,q)\mathrm{d}t\neq 0. \tag{4.43}$$

Now the implicit function theorem is applicable to (4.37). In conclusion, there exist some constant $A=A_{q,\eta}>0$ and a continuously differentiable real-valued functions $\lambda_n(\alpha)$ of α such that

$$\lambda_n(0)=\mu_n,\qquad M_q(\lambda_n(\alpha),\alpha)\equiv 0,\qquad \|\alpha\|\le A_{q,\eta},\quad 1\le n\le \mathbf{n}. \tag{4.44}$$

Due to (4.38)–(4.44) and the continuity of $\lambda_n(\alpha)$, one can assume that

$$-\mathbf{r}<\lambda_1(\alpha)<\cdots<\lambda_{\mathbf{n}}(\alpha)<\mathbf{r}\quad\forall\ \|\alpha\|\le A_{q,\eta}. \tag{4.45}$$

Thus $\{\lambda_n(\alpha)\}_{1\le n\le\mathbf{n}}$ are different eigenvalues of (M_q) located in the interval $(-\mathbf{r},\mathbf{r})$. Since (4.37) has precisely $\mathbf{n}$ solutions inside $\mathbb{D}$, we conclude that all solutions of (4.37) inside $\mathbb{D}$ are necessarily real. Now we have proved that $\Sigma_{\alpha,\eta}^q\subset\mathbb{R}$ for all $\|\alpha\|\le A_{q,\eta}$.

Notice that the constant $A_{q,\eta}$ in (4.44) is constructed from the implicit function theorem. Generally speaking, $A_{q,\eta}$ depends on $\eta \in (0,1)$ and all information of the potential $q \in L^1_{\mathbb{R}}$. However, during the application of the implicit function theorem to (4.37), the derivatives of $\partial_\alpha \lambda_n(\alpha)$ can be well controlled using estimates in [11], like (3.8)–(3.11). It is possible to choose some $A_{q,\eta}$ such that it depends on the norm $\|q\|$ only. We will not give the detailed construction. Note that this has been already observed for large eigenvalues. For example, **n** and **r** depend only on the norm $\|q\|$ of q. □

We end the paper with an open problem. Given $(\alpha,\eta) \in \mathbb{R}^m \times \Delta^m$, for any $q \in L^1_{\mathbb{R}}$, due to Theorem 1.2, problem (M_q) has always a sequence of real eigenvalues $\overline{\lambda}_n(q) = \overline{\lambda}_{n,\alpha,\eta}(q)$ which tends to $+\infty$. In applications of eigenvalues to nonlinear problems, the smallest (real) eigenvalues $\overline{\lambda}_1(q)$ are of great importance. The main reason is that solutions of problem (1.1)–(1.2) are oscillatory only when $\lambda > \overline{\lambda}_1(q)$. As for the smallest eigenvalue of the Dirichlet problem (1.1)–(1.4), denoted by $\lambda_1(q)$, one has the following variational characterization:

$$\lambda_1(q) = \inf_{y \in H_0^1(0,1),\ y \neq 0} \frac{\int_0^1 \left(y'^2 + q(x)y^2\right) \mathrm{d}x}{\int_0^1 y^2 \, \mathrm{d}x}. \tag{4.46}$$

An open problem is what is the characterization like (4.46) for the smallest eigenvalue $\overline{\lambda}_1(q)$ of (M_q). Once this is clear, some results on nonlinear problems in [5, 6, 8] can be extended by using eigenvalues of (M_q).

Finally, let us remark that the approaches in this paper also can be applied to other multi-point boundary conditions like

$$y'(0) = 0, \qquad y(1) - \sum_{k=1}^{m} \alpha_k y(\eta_k) = 0 \tag{4.47}$$

or to more general Stieltjes boundary conditions [17]. In this sense, eigenvalue theory can be established for these nonsymmetric problems.

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