Research Article

Neural Network Adaptive Control for Discrete-Time Nonlinear Nonnegative Dynamical Systems

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Nonnegative and compartmental dynamical system models are derived from mass and energy balance considerations that involve dynamic states whose values are nonnegative. These models are widespread in engineering and life sciences, and they typically involve the exchange of nonnegative quantities between subsystems or compartments, wherein each compartment is assumed to be kinetically homogeneous. In this paper, we develop a neuroadaptive control framework for adaptive set-point regulation of discrete-time nonlinear uncertain nonnegative and compartmental systems. The proposed framework is Lyapunov-based and guarantees ultimate boundedness of the error signals corresponding to the physical system states and the neural network weighting gains. In addition, the neuroadaptive controller guarantees that the physical system states remain in the nonnegative orthant of the state space for nonnegative initial conditions.

1. Introduction

Neural networks have provided an ideal framework for online identification and control of many complex uncertain engineering systems because of their great flexibility in approximating a large class of continuous maps and their adaptability due to their inherently parallel architecture. Even though neuroadaptive control has been applied to numerous engineering problems, neuroadaptive methods have not been widely considered for problems involving systems with nonnegative state and control constraints [1, 2]. Such systems are commonly referred to as *nonnegative dynamical systems* in the literature [3–8]. A subclass of

nonnegative dynamical systems are *compartmental systems* [8–18]. Compartmental systems involve dynamical models that are characterized by conservation laws (e.g., mass and energy) capturing the exchange of material between coupled macroscopic subsystems known as compartments. The range of applications of nonnegative systems and compartmental systems includes pharmacological systems, queuing systems, stochastic systems (whose state variables represent probabilities), ecological systems, economic systems, demographic systems, telecommunications systems, and transportation systems, to cite but a few examples. Due to the severe complexities, nonlinearities, and uncertainties inherent in these systems, neural networks provide an ideal framework for online adaptive control because of their parallel processing flexibility and adaptability.

In this paper, we extend the results of [2] to develop a neuroadaptive control framework for discrete-time nonlinear uncertain nonnegative and compartmental systems. The proposed framework is Lyapunov-based and guarantees ultimate boundedness of the error signals corresponding to the physical system states as well as the neural network weighting gains. The neuroadaptive controllers are constructed *without* requiring knowledge of the system dynamics while guaranteeing that the physical system states remain in the nonnegative orthant of the state space. The proposed neuro control architecture is modular in the sense that if a nominal linear design model is available, the neuroadaptive controller can be augmented to the nominal design to account for system nonlinearities and system uncertainty. Furthermore, since in certain applications of nonnegative and compartmental systems (e.g., pharmacological systems for active drug administration) control (source) inputs as well as the system states need to be nonnegative, we also develop neuroadaptive controllers that guarantee the control signal as well as the physical system states remain nonnegative for nonnegative initial conditions.

The contents of the paper are as follows. In Section 2, we provide mathematical preliminaries on nonnegative dynamical systems that are necessary for developing the main results of this paper. In Section 3, we develop *new* Lyapunov-like theorems for partial boundedness and partial ultimate boundedness for nonlinear dynamical systems necessary for obtaining less conservative ultimate bounds for neuroadaptive controllers as compared to ultimate bounds derived using classical boundedness and ultimate boundedness notions. In Section 4, we present our main neuroadaptive control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. In Section 5, we extend the results of Section 4 to the case where control inputs are constrained to be nonnegative. Finally, in Section 6 we draw some conclusions.

2. Mathematical preliminaries

In this section we introduce notation, several definitions, and some key results concerning linear and nonlinear discrete-time nonnegative dynamical systems [19] that are necessary for developing the main results of this paper. Specifically, for $x \in \mathbb{R}^n$ we write $x \geq\geq 0$ (resp., $x >> 0$) to indicate that every component of x is nonnegative (resp., positive). In this case, we say that x is *nonnegative* or *positive*, respectively. Likewise, $A \in \mathbb{R}^{n \times m}$ is *nonnegative* or *positive* if every entry of A is nonnegative or positive, respectively, which is written as $A \geq\geq 0$ or $A >> 0$, respectively. In this paper it is important to distinguish between a square nonnegative (resp., positive) matrix and a nonnegative-definite (resp., positive-definite) matrix. Let $\overline{\mathbb{R}}_+^n$ and $\mathbb{R}_+^n$ denote the nonnegative and positive orthants of $\mathbb{R}^n$, that is, if $x \in \mathbb{R}^n$, then $x \in \overline{\mathbb{R}}_+^n$ and

$x \in \mathbb{R}_+^n$ are equivalent, respectively, to $x \geq\geq 0$ and $x >> 0$. Finally, we write $(\cdot)^{\mathrm{T}}$ to denote transpose, $\mathrm{tr}(\cdot)$ for the trace operator, $\lambda_{\min}(\cdot)$ (resp., $\lambda_{\max}(\cdot)$) to denote the minimum (resp., maximum) eigenvalue of a Hermitian matrix, $\|\cdot\|$ for a vector norm, and $\overline{\mathbb{Z}}_+$ for the set of all nonnegative integers. The following definition introduces the notion of a nonnegative (resp., positive) function.

Definition 2.1. A real function $u : \overline{\mathbb{Z}}_+ \to \mathbb{R}^m$ is a *nonnegative* (resp., *positive*) *function* if $u(k) \geq\geq 0$ (resp., $u(k) >> 0$), $k \in \overline{\mathbb{Z}}_+$.

The following theorems give necessary and sufficient conditions for asymptotic stability of the discrete-time linear nonnegative dynamical system

$$x(k+1) = Ax(k), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+, \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$ is nonnegative and $x_0 \in \overline{\mathbb{R}}_+^n$, using *linear* and *quadratic* Lyapunov functions, respectively.

Theorem 2.2 (see [19]). *Consider the linear dynamical system $\mathcal{G}$ given by (2.1) where $A \in \mathbb{R}^{n \times n}$ is nonnegative. Then $\mathcal{G}$ is asymptotically stable if and only if there exist vectors $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy*

$$p = A^T p + r. \tag{2.2}$$

Theorem 2.3 (see [6, 19]). *Consider the linear dynamical system $\mathcal{G}$ given by (2.1) where $A \in \mathbb{R}^{n \times n}$ is nonnegative. Then $\mathcal{G}$ is asymptotically stable if and only if there exist a positive diagonal matrix $P \in \mathbb{R}^{n \times n}$ and an $n \times n$ positive-definite matrix R such that*

$$P = A^T P A + R. \tag{2.3}$$

Next, consider the controlled discrete-time linear dynamical system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+, \tag{2.4}$$

where

$$B = \begin{bmatrix} \widehat{B} \\ 0_{(n-m) \times m} \end{bmatrix}, \tag{2.5}$$

$A \in \mathbb{R}^{n \times n}$ is nonnegative and $\widehat{B} \in \mathbb{R}^{m \times m}$ is nonnegative such that rank $\widehat{B} = m$. The following theorem shows that discrete-time linear stabilizable nonnegative systems possess asymptotically stable zero dynamics with $\widehat{x} \triangleq [x_1, \dots, x_m]$ viewed as the output. For the statement of this result, let $\mathrm{spec}(A)$ denote the spectrum of A, let $\overline{\mathbb{C}}_1 \triangleq \{s \in \mathbb{C} : |s| \geq 1\}$, and let $A \in \mathbb{R}^{n \times n}$ in (2.4) be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.6}$$

where $A_{11} \in \mathbb{R}^{m \times m}$, $A_{12} \in \mathbb{R}^{m \times (n-m)}$, $A_{21} \in \mathbb{R}^{(n-m) \times m}$, and $A_{22} \in \mathbb{R}^{(n-m) \times (n-m)}$ are nonnegative matrices.

Theorem 2.4. *Consider the discrete-time linear dynamical system $\mathcal{G}$ given by (2.4), where $A \in \mathbb{R}^{n \times n}$ is nonnegative and partitioned as in (2.6), and $B \in \mathbb{R}^{n \times m}$ is nonnegative and is partitioned as in (2.5) with rank $\widehat{B} = m$. Then there exists a gain matrix $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is nonnegative and asymptotically stable if and only if A_{22} is asymptotically stable.*

Proof. First, let K be partitioned as $K = [K_1, K_2]$, where $K_1 \in \mathbb{R}^{m \times m}$ and $K_2 \in \mathbb{R}^{m \times (n-m)}$, and note that

$$(A + BK)^{\mathrm{T}} = \begin{bmatrix} \left(A_{11} + \widehat{B}K_1\right)^{\mathrm{T}} & A_{21}^{\mathrm{T}} \\ \left(A_{12} + \widehat{B}K_2\right)^{\mathrm{T}} & A_{22}^{\mathrm{T}} \end{bmatrix}. \tag{2.7}$$

Assume that $A + BK$ is nonnegative and asymptotically stable, and suppose that, *ad absurdum*, A_{22} is not asymptotically stable. Then, it follows from Theorem 2.2 that there does not exist a positive vector $p_2 \in \mathbb{R}_+^{n-m}$ such that $(A_{22}^{\mathrm{T}} - I)p_2 << 0$. Next, since $A_{12} + \widehat{B}K_2$ is nonnegative it follows that $\left(A_{12} + \widehat{B}K_2\right)^{\mathrm{T}} p_1 \geq\geq 0$ for any positive vector $p_1 \in \mathbb{R}_+^m$. Thus, there does not exist a positive vector $p \triangleq [p_1^{\mathrm{T}}, p_2^{\mathrm{T}}]^{\mathrm{T}}$ such that $[(A + BK)^{\mathrm{T}} - I]p << 0$, and hence, it follows from Theorem 2.2 that $A + BK$ is not asymptotically stable leading to a contradiction. Hence, A_{22} is asymptotically stable. Conversely, suppose that A_{22} is asymptotically stable. Then taking $K_1 = \widehat{B}^{-1}(A_{\mathrm{s}} - A_{11})$ and $K_2 = -\widehat{B}^{-1}A_{12}$, where A_{s} is nonnegative and asymptotically stable, it follows that $\mathrm{spec}(A + BK) \cap \overline{\mathbb{C}}_1 = [\mathrm{spec}(A_{\mathrm{s}}) \cup \mathrm{spec}(A_{22})] \cap \overline{\mathbb{C}}_1 = \varnothing$, and hence, $A + BK$ is nonnegative and asymptotically stable. $\square$

Next, consider the discrete-time nonlinear dynamical system

$$x(k + 1) = f(x(k)), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+, \tag{2.8}$$

where $x(k) \in \mathcal{D}$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ with $0 \in \mathcal{D}$, and $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$. Recall that the point $x_{\mathrm{e}} \in \mathcal{D}$ is an *equilibrium point* of (2.8) if $x_{\mathrm{e}} = f(x_{\mathrm{e}})$. Furthermore, a subset $\mathcal{D}_{\mathrm{c}} \subseteq \mathcal{D}$ is an *invariant set* with respect to (2.8) if $\mathcal{D}_{\mathrm{c}}$ contains the orbits of all its points. The following definition introduces the notion of nonnegative vector fields [19].

Definition 2.5. Let $f = [f_1, \ldots, f_n]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then f is *nonnegative with respect to* $\widehat{x} \triangleq [x_1, \ldots, x_m]^{\mathrm{T}}$, $m \leq n$, if $f_i(x) \geq 0$ for all $i = 1, \ldots, m$, and $x \in \overline{\mathbb{R}}_+^n$. f is *nonnegative* if $f_i(x) \geq 0$ for all $i = 1, \ldots, n$, and $x \in \overline{\mathbb{R}}_+^n$.

Note that if $f(x) = Ax$, where $A \in \mathbb{R}^{n \times n}$, then f is nonnegative if and only if A is nonnegative [19].

Proposition 2.6 (see [19]). *Suppose $\overline{\mathbb{R}}_+^n \subset \mathcal{D}$. Then $\overline{\mathbb{R}}_+^n$ is an invariant set with respect to (2.8) if and only if $f : \mathcal{D} \to \mathbb{R}^n$ is nonnegative.*

In this paper, we consider controlled discrete-time nonlinear dynamical systems of the form

$$x(k + 1) = f(x(k)) + G(x(k))u(k), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+, \tag{2.9}$$

where $x(k) \in \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, $u(k) \in \mathbb{R}^m$, $k \in \overline{\mathbb{Z}}_+$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies $f(0) = 0$, and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is continuous.

The following definition and proposition are needed for the main results of the paper.

Definition 2.7. The discrete-time nonlinear dynamical system given by (2.9) is *nonnegative* if for every $x(0) \in \overline{\mathbb{R}}_+^n$ and $u(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, the solution $x(k)$, $k \in \overline{\mathbb{Z}}_+$, to (2.9) is nonnegative.

Proposition 2.8 (see [19]). *The discrete-time nonlinear dynamical system given by (2.9) is nonnegative if $f(x) \geq\geq 0$ and $G(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$.*

It follows from Proposition 2.8 that a nonnegative input signal $G(x(k))u(k)$, $k \in \overline{\mathbb{Z}}_+$, is sufficient to guarantee the nonnegativity of the state of (2.9).

Next, we present a time-varying extension to Proposition 2.8 needed for the main theorems of this paper. Specifically, we consider the time-varying system

$$x(k+1) = f(k, x(k)) + G(x(k))u(k), \quad x(k_0) = x_0, \ k \geq k_0, \tag{2.10}$$

where $f : \overline{\mathbb{Z}}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in k and x on $\overline{\mathbb{Z}}_+ \times \mathbb{R}^n$ and $f(k,0) = 0$, $k \in \overline{\mathbb{Z}}_+$, and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is continuous. For the following result, the definition of nonnegativity holds with (2.9) replaced by (2.10).

Proposition 2.9. *Consider the time-varying discrete-time dynamical system (2.10) where $f(k, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n$ for all $k \in \overline{\mathbb{Z}}_+$ and $f(\cdot, x) : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$ is continuous on $\overline{\mathbb{Z}}_+$ for all $x \in \mathbb{R}^n$. If for every $k \in \overline{\mathbb{Z}}_+$, $f(k, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is nonnegative, then the solution $x(k)$, $k \geq k_0$, to (2.10) is nonnegative.*

Proof. The result is a direct consequence of Proposition 2.8 by equivalently representing the time-varying discrete-time system (2.10) as an autonomous discrete-time nonlinear system by appending another state to represent time. Specifically, defining $y(k - k_0) \triangleq x(k)$ and $y_{n+1}(k - k_0) \triangleq k$, it follows that the solution $x(k)$, $k \geq k_0$, to (2.10) can be equivalently characterized by the solution $y(\kappa)$, $\kappa \geq 0$, where $\kappa \triangleq k - k_0$, to the discrete-time nonlinear autonomous system

$$y(\kappa+1) = f(y_{n+1}(\kappa), y(\kappa)) + G(y(\kappa))\widehat{u}(\kappa), \quad y(0) = y_0, \ \kappa \geq 0, \tag{2.11}$$

$$y_{n+1}(\kappa+1) = y_{n+1}(\kappa) + 1, \quad y_{n+1}(0) = k_0, \tag{2.12}$$

where $\widehat{u}(\kappa) \triangleq u(\kappa + k_0)$. Now, since $y_i(\kappa) \geq 0$, $\kappa \geq 0$, for $i = 1, \dots, n+1$, and $G(x(\kappa))\widehat{u}(\kappa) \geq\geq 0$, the result is a direct consequence of Proposition 2.8. $\qquad\square$

3. Partial boundedness and partial ultimate boundedness

In this section, we present Lyapunov-like theorems for *partial boundedness* and *partial ultimate boundedness* of discrete-time nonlinear dynamical systems. These notions allow us to develop less conservative ultimate bounds for neuroadaptive controllers as compared to ultimate bounds derived using classical boundedness and ultimate boundedness notions. Specifically, consider the discrete-time nonlinear autonomous interconnected dynamical system

$$x_1(k+1) = f_1(x_1(k), x_2(k)), \quad x_1(0) = x_{10}, \ k \in \overline{\mathbb{Z}}_+, \tag{3.1}$$

$$x_2(k+1) = f_2(x_1(k), x_2(k)), \quad x_2(0) = x_{20}, \tag{3.2}$$

where $x_1 \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^{n_1}$ is an open set such that $0 \in \mathcal{D}$, $x_2 \in \mathbb{R}^{n_2}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is continuous in x_1, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is continuous. Note that under the above assumptions the solution $(x_1(k), x_2(k))$ to (3.1) and (3.2) exists and is unique over $\overline{\mathbb{Z}}_+$.

Definition 3.1 (see [20]). (i) The discrete-time nonlinear dynamical system (3.1) and (3.2) is *bounded with respect to x_1 uniformly in x_{20}* if there exists $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(k)\| < \varepsilon$ for all $k \in \overline{\mathbb{Z}}_+$. The discrete-time nonlinear dynamical system (3.1) and (3.2) is *globally bounded with respect to x_1 uniformly in x_{20}* if, for every $\delta \in (0, \infty)$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(k)\| < \varepsilon$ for all $k \in \overline{\mathbb{Z}}_+$.

(ii) The discrete-time nonlinear dynamical system (3.1) and (3.2) is *ultimately bounded with respect to x_1 uniformly in x_{20} with ultimate bound ε* if there exists $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists $K = K(\delta, \varepsilon) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(k)\| < \varepsilon$, $k \geq K$. The discrete-time nonlinear dynamical system (3.1) and (3.2) is *globally ultimately bounded with respect to x_1 uniformly in x_{20} with ultimate bound ε* if, for every $\delta \in (0, \infty)$, there exists $K = K(\delta, \varepsilon) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(k)\| < \varepsilon$, $k \geq K$.

Note that if a discrete-time nonlinear dynamical system is (globally) bounded with respect to x_1 uniformly in x_{20}, then there exists $\varepsilon > 0$, such that it is (globally) ultimately bounded with respect to x_1 uniformly in x_{20} with an ultimate bound ε. Conversely, if a discrete-time nonlinear dynamical system is (globally) ultimately bounded with respect to x_1 uniformly in x_{20} with an ultimate bound ε, then it is (globally) bounded with respect to x_1 uniformly in x_{20}. The following results present Lyapunov-like theorems for boundedness and ultimate boundedness for discrete-time nonlinear systems. For these results define $\Delta V(x_1, x_2) \triangleq V(f(x_1, x_2)) - V(x_1, x_2)$, where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$ and $V : \mathfrak{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is a given continuous function. Furthermore, let $\mathcal{B}_\delta(x)$, $x \in \mathbb{R}^n$, $\delta > 0$, denote the open ball centered at x with radius δ and let $\overline{\mathcal{B}}_\delta(x)$ denote the closure of $\mathcal{B}_\delta(x)$, and recall the definitions of class-$\mathcal{K}$, class-$\mathcal{K}_\infty$, and class-$\mathcal{KL}$ functions [20].

Theorem 3.2. *Consider the discrete-time nonlinear dynamical system (3.1) and (3.2). Assume that there exist a continuous function $V : \mathfrak{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class-$\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that*

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad x_1 \in \mathfrak{D}, \; x_2 \in \mathbb{R}^{n_2}, \tag{3.3}$$

$$\Delta V(x_1, x_2) \leq 0, \quad x_1 \in \mathfrak{D}, \; \|x_1\| > \mu, \; x_2 \in \mathbb{R}^{n_2}, \tag{3.4}$$

where $\mu > 0$ is such that $\mathcal{B}_{\alpha^{-1}(\beta(\mu))}(0) \subset \mathfrak{D}$. Furthermore, assume that $\sup_{(x_1, x_2) \in \overline{\mathcal{B}}_\mu(0) \times \mathbb{R}^{n_2}} V(f(x_1, x_2))$ exists. Then the discrete-time nonlinear dynamical system (3.1) and (3.2) is bounded with respect to x_1 uniformly in x_{20}. Furthermore, for every $\delta \in (0, \gamma)$, $x_{10} \in \overline{\mathcal{B}}_\delta(0)$ implies that $\|x_1(k)\| \leq \varepsilon$, $k \in \overline{\mathbb{Z}}_+$, where

$$\varepsilon = \varepsilon(\delta) \triangleq \alpha^{-1}(\max\{\eta, \beta(\delta)\}), \tag{3.5}$$

$\eta \geq \max\{\beta(\mu), \sup_{(x_1, x_2) \in \overline{\mathcal{B}}_\mu(0) \times \mathbb{R}^{n_2}} V(f(x_1, x_2))\} = \max\{\beta(\mu), \sup_{(x_1, x_2) \in \overline{\mathcal{B}}_\mu(0) \times \mathbb{R}^{n_2}} (V(x_1, x_2) + \Delta V(x_1, x_2))\}$, and $\gamma \triangleq \sup\{r > 0 : \mathcal{B}_{\alpha^{-1}(\beta(r))}(0) \subset \mathfrak{D}\}$. If, in addition, $\mathfrak{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ is a class-$\mathcal{K}_\infty$ function, then the discrete-time nonlinear dynamical system (3.1) and (3.2) is globally bounded with respect to x_1 uniformly in x_{20} and for every $x_{10} \in \mathbb{R}^{n_1}$, $\|x_1(k)\| \leq \varepsilon$, $k \in \overline{\mathbb{Z}}_+$, where ε is given by (3.5) with $\delta = \|x_{10}\|$.

Proof. See [20, page 786]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 3.3. *Consider the discrete-time nonlinear dynamical system* (3.1) *and* (3.2). *Assume there exist a continuous function* $V : \mathfrak{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ *and class-$\mathcal{K}$ functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *such that* (3.3) *holds. Furthermore, assume that there exists a continuous function* $W : \mathfrak{D} \to \mathbb{R}$ *such that* $W(x_1) > 0$, $\|x_1\| > \mu$, *and*

$$\Delta V(x_1, x_2) \leq -W(x_1), \quad x_1 \in \mathfrak{D}, \|x_1\| > \mu, x_2 \in \mathbb{R}^{n_2}, \tag{3.6}$$

where $\mu > 0$ *is such that* $\mathcal{B}_{\alpha^{-1}(\beta(\mu))}(0) \subset \mathfrak{D}$. *Finally, assume* $\sup_{(x_1,x_2)\in\overline{\mathcal{B}}_\mu(0)\times\mathbb{R}^{n_2}} V(f(x_1, x_2))$ *exists. Then the nonlinear dynamical system* (3.1), (3.2) *is ultimately bounded with respect to* x_1 *uniformly in* x_{20} *with ultimate bound* $\varepsilon \triangleq \alpha^{-1}(\eta)$, *where* $\eta > \max\{\beta(\mu), \sup_{(x_1,x_2)\in\overline{\mathcal{B}}_\mu(0)\times\mathbb{R}^{n_2}} V(f(x_1, x_2))\} = \max\{\beta(\mu), \sup_{(x_1,x_2)\in\overline{\mathcal{B}}_\mu(0)\times\mathbb{R}^{n_2}} (V(x_1, x_2) + \Delta V(x_1, x_2))\}$. *Furthermore,* $\limsup_{k\to\infty} \|x_1(k)\| \leq \alpha^{-1}(\eta)$. *If, in addition,* $\mathfrak{D} = \mathbb{R}^n$ *and* $\alpha(\cdot)$ *is a class-$\mathcal{K}_\infty$ function, then the nonlinear dynamical system* (3.1) *and* (3.2) *is globally ultimately bounded with respect to* x_1 *uniformly in* x_{20} *with ultimate bound* ε.

Proof. See [20, page 787]. □

The following result on ultimate boundedness of interconnected systems is needed for the main theorems in this paper. For this result, recall the definition of input-to-state stability given in [21].

Proposition 3.4. *Consider the discrete-time nonlinear interconnected dynamical system* (3.1) *and* (3.2). *If* (3.2) *is input-to-state stable with* x_1 *viewed as the input and* (3.1) *and* (3.2) *are ultimately bounded with respect to* x_1 *uniformly in* x_{20}, *then the solution* $(x_1(k), x_2(k))$, $k \in \overline{\mathbb{Z}}_+$, *of the interconnected dynamical system* (3.1)-(3.2), *is ultimately bounded.*

Proof. Since system (3.1)-(3.2) is ultimately bounded with respect to x_1 (uniformly in x_{20}), there exist positive constants ε and $K = K(\delta, \varepsilon)$ such that $\|x_1(k)\| < \varepsilon, k \geq K$. Furthermore, since (3.2) is input-to-state stable with x_1 viewed as the input, it follows that $x_2(K)$ is finite, and hence, there exist a class-$\mathcal{KL}$ function $\eta(\cdot, \cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that

$$
\begin{aligned}
\|x_2(k)\| &\leq \eta(\|x_2(K)\|, k - K) + \gamma\left(\max_{K\leq i\leq k}\|x_1(i)\|\right) \\
&< \eta(\|x_2(K)\|, k - K) + \gamma(\varepsilon) \\
&\leq \eta(\|x_2(K)\|, 0) + \gamma(\varepsilon), \quad k \geq K,
\end{aligned}
\tag{3.7}
$$

which proves that the solution $(x_1(k), x_2(k))$, $k \in \overline{\mathbb{Z}}_+$ to (3.1) and (3.2) is ultimately bounded. □

4. Neuroadaptive control for discrete-time nonlinear nonnegative uncertain systems

In this section, we consider the problem of characterizing neuroadaptive feedback control laws for discrete-time nonlinear nonnegative and compartmental uncertain dynamical systems to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the controlled discrete-time nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$x(k+1) = f_x(x(k), z(k)) + G(x(k), z(k))u(k), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+, \tag{4.1}$$

$$z(k+1) = f_z(x(k), z(k)), \quad z(0) = z_0, \tag{4.2}$$

where $x(k) \in \mathbb{R}^{n_x}$, $k \in \overline{\mathbb{Z}}_+$, and $z(k) \in \mathbb{R}^{n_z}$, $k \in \overline{\mathbb{Z}}_+$, are the state vectors, $u(k) \in \mathbb{R}^m$, $k \in \overline{\mathbb{Z}}_+$, is the control input, $f_x : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is nonnegative with respect to x but otherwise unknown and satisfies $f_x(0, z) = 0$, $z \in \mathbb{R}^{n_z}$, $f_z : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$ is nonnegative with respect to z but otherwise unknown and satisfies $f_z(x, 0) = 0$, $x \in \mathbb{R}^{n_x}$, and $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x \times m}$ is a known nonnegative input matrix function. Here, we assume that we have m control inputs so that the input matrix function is given by

$$G(x, z) = \begin{bmatrix} B_{\mathrm{u}} G_{\mathrm{n}}(x, z) \\ 0_{(n-m) \times m} \end{bmatrix}, \tag{4.3}$$

where $B_{\mathrm{u}} = \mathrm{diag}[b_1, \dots, b_m]$ is a positive diagonal matrix and $G_{\mathrm{n}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m \times m}$ is a nonnegative matrix function such that $\det G_{\mathrm{n}}(x, z) \neq 0$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. The control input $u(\cdot)$ in (4.1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(k) \in \mathbb{R}^m$, $k \in \overline{\mathbb{Z}}_+$. In this section, we do not place any restriction on the sign of the control signal and design a neuroadaptive controller that guarantees that the system states remain in the nonnegative orthant of the state space for nonnegative initial conditions and are ultimately bounded in the neighborhood of a desired equilibrium point.

In this paper, we assume that $f_x(\cdot, \cdot)$ and $f_z(\cdot, \cdot)$ are unknown functions with $f_x(\cdot, \cdot)$ given by

$$f_x(x, z) = Ax + \Delta f(x, z), \tag{4.4}$$

where $A \in \mathbb{R}^{n_x \times n_x}$ is a known nonnegative matrix and $\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is an unknown nonnegative function with respect to x and belongs to the uncertainty set $\mathcal{F}$ given by

$$\mathcal{F} = \left\{ \Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x} : \Delta f(x, z) = B\delta(x, z), \ (x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \right\}, \tag{4.5}$$

where $B \triangleq [B_{\mathrm{u}}, 0_{m \times (n-m)}]^{\mathrm{T}}$ and $\delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^m$ is an uncertain continuous function such that $\delta(x, z)$ is nonnegative with respect to x. Furthermore, we assume that for a given $x_{\mathrm{e}} \in \mathbb{R}_+^{n_x}$ there exist $z_{\mathrm{e}} \in \overline{\mathbb{R}}_+^{n_z}$ and $u_{\mathrm{e}} \in \overline{\mathbb{R}}_+^m$ such that

$$x_{\mathrm{e}} = Ax_{\mathrm{e}} + \Delta f(x_{\mathrm{e}}, z_{\mathrm{e}}) + G(x_{\mathrm{e}}, z_{\mathrm{e}}) u_{\mathrm{e}}, \tag{4.6}$$

$$z_{\mathrm{e}} = f_z(x_{\mathrm{e}}, z_{\mathrm{e}}). \tag{4.7}$$

In addition, we assume that (4.2) is input-to-state stable at $z(k) \equiv z_{\mathrm{e}}$ with $x(k) - x_{\mathrm{e}}$ viewed as the input, that is, there exist a class-$\mathcal{KL}$ function $\eta(\cdot, \cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\|z(k) - z_{\mathrm{e}}\| \leq \eta(\|z_0 - z_{\mathrm{e}}\|, k) + \gamma\left(\max_{0 \leq i \leq k} \|x(i) - x_{\mathrm{e}}\|\right), \quad k \geq 0, \tag{4.8}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. Unless otherwise stated, henceforth we use $\|\cdot\|$ to denote the Euclidean vector norm. Note that $(x_{\mathrm{e}}, z_{\mathrm{e}}) \in \mathbb{R}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ is an equilibrium point of (4.1) and (4.2) if and only if there exists $u_{\mathrm{e}} \in \overline{\mathbb{R}}_+^m$ such that (4.6) and (4.7) hold.

Furthermore, we assume that, for a given $\varepsilon_i^* > 0$, the ith component of the vector function $\delta(x, z) - \delta(x_{\mathrm{e}}, z_{\mathrm{e}}) - G_{\mathrm{n}}(x_{\mathrm{e}}, z_{\mathrm{e}}) u_{\mathrm{e}}$ can be approximated over a compact set $\mathfrak{D}_{cx} \times \mathfrak{D}_{cz} \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$

by a linear in the parameters neural network up to a desired accuracy so that for $i = 1, \ldots, m$, there exists $\varepsilon_i(\cdot, \cdot)$ such that $|\varepsilon_i(x, z)| < \varepsilon_i^*$, $(x, z) \in \mathcal{D}_{cx} \times \mathcal{D}_{cz}$, and

$$\delta_i(x, z) - \delta_i(x_e, z_e) - [G_n(x_e, z_e)u_e]_i = W_i^T \sigma_i(x, z) + \varepsilon_i(x, z), \quad (x, z) \in \mathcal{D}_{cx} \times \mathcal{D}_{cz}, \quad (4.9)$$

where $W_i \in \mathbb{R}^{s_i}$, $i = 1, \ldots, m$, are optimal *unknown* (constant) weights that minimize the approximation error over $\mathcal{D}_{cx} \times \mathcal{D}_{cz}$, $\sigma_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{s_i}$, $i = 1, \ldots, m$, are a set of basis functions such that each component of $\sigma_i(\cdot, \cdot)$ takes values between 0 and 1, $\varepsilon_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}$, $i = 1, \ldots, m$, are the modeling errors, and $\|W_i\| \leq w_i^*$, where w_i^*, $i = 1, \ldots, m$, are bounds for the optimal weights W_i, $i = 1, \ldots, m$.

Since $f_x(\cdot, \cdot)$ is continuous, we can choose $\sigma_i(\cdot, \cdot)$, $i = 1, \ldots, m$, from a linear space $\mathcal{X}$ of continuous functions that forms an algebra and separates points in $\mathcal{D}_{cx} \times \mathcal{D}_{cz}$. In this case, it follows from the Stone-Weierstrass theorem [22, page 212] that $\mathcal{X}$ is a dense subset of the set of continuous functions on $\mathcal{D}_{cx} \times \mathcal{D}_{cz}$. Now, as is the case in the standard neuroadaptive control literature [23], we can construct the signal $u_{\text{ad}_i} = \widehat{W}_i^T \sigma_i(x, z)$ involving the estimates of the optimal weights as our adaptive control signal. However, even though $\widehat{W}_i^T \sigma_i(x, z)$, $i = 1, \ldots, m$, provides adaptive cancellation of the system uncertainty, it does not necessarily guarantee that the state trajectory of the closed-loop system remains in the nonnegative orthant of the state space for nonnegative initial conditions.

To ensure nonnegativity of the closed-loop plant states, the adaptive control signal is assumed to be of the form $\widehat{W}_i^T \widehat{\sigma}_i(x, z, \widehat{W}_i)$, $i = 1, \ldots, m$, where $\widehat{\sigma}_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s_i} \to \mathbb{R}^{s_i}$ is such that each component of $\widehat{\sigma}_i(\cdot, \cdot, \cdot)$ takes values between 0 and 1 and $\widehat{\sigma}_{i(j)}(x, z, \widehat{W}_i) = 0$, whenever $\widehat{W}_{i(j)} > 0$ for all $i = 1, \ldots, m$, $j = 1, \ldots, s_i$, where $\widehat{\sigma}_{i(j)}(\cdot, \cdot, \cdot)$ and $\widehat{W}_{i(j)}$ are the jth element of $\widehat{\sigma}_i(\cdot, \cdot, \cdot)$ and $\widehat{W}_i$, respectively. This set of functions do not generate an algebra in $\mathcal{X}$, and hence, if used as an approximator for $\delta_i(\cdot, \cdot)$, $i = 1, \ldots, m$, will generate additional conservatism in the ultimate bound guarantees provided by the neural network controller. In particular, since each component of $\sigma_i(\cdot, \cdot)$ and $\widehat{\sigma}_i(\cdot, \cdot, \cdot)$ takes values between 0 and 1, it follows that

$$\|\sigma_i(x, z) - \widehat{\sigma}_i(x, z, \widehat{W}_i)\| \leq \sqrt{s_i}, \quad (x, z, \widehat{W}_i) \in \mathcal{D}_{cx} \times \mathcal{D}_{cz} \times \mathbb{R}^{s_i}, \ i = 1, \ldots, m. \quad (4.10)$$

This upper bound is used in the proof of Theorem 4.1 below.

For the remainder of the paper we assume that there exists a gain matrix $K \in \mathbb{R}^{m \times n_x}$ such that $A + BK$ is nonnegative and asymptotically stable, where A and B have the forms of (2.6) and (2.5), respectively. Now, partitioning the state in (4.1) as $x = [x_1^T, x_2^T]^T$, where $x_1 \in \mathbb{R}^m$ and $x_2 \in \mathbb{R}^{n_x - m}$, and using (4.3), it follows that (4.1) and (4.2) can be written as

$$x_1(k + 1) = A_{11}x_1(k) + A_{12}x_2(k) + \Delta f(x_1(k), x_2(k), z(k)) + B_u G_n(x_1(k), x_2(k), z(k))u(k),$$
$$x_1(0) = x_{10}, \quad k \in \overline{\mathbb{Z}}_+,$$
$$(4.11)$$

$$x_2(k + 1) = A_{21}x_1(k) + A_{22}x_2(k), \quad x_2(0) = x_{20}, \quad (4.12)$$

$$z(k + 1) = f_z(x_1(k), x_2(k), z(k)), \quad z(0) = z_0. \quad (4.13)$$

Thus, since $A + BK$ is nonnegative and asymptotically stable, it follows from Theorem 2.4 that the solution $x_2(k) \equiv x_{2e} \in \mathbb{R}_+^{n_x - m}$ of (4.12) with $x_1(k) \equiv x_{1e} \in \mathbb{R}_+^m$, where x_{1e} and x_{2e} satisfy $x_{2e} = A_{21}x_{1e} + A_{22}x_{2e}$, is globally exponentially stable, and hence, (4.12) is input-to-state stable

at $x_2(k) \equiv x_{2e}$ with $x_1(k) - x_{1e}$ viewed as the input. Thus, in this paper we assume that the dynamics (4.12) can be included in (4.2) so that $n_x = m$. In this case, the input matrix (4.3) is given by

$$G(x,z) = B_u G_n(x,z) \tag{4.14}$$

so that $B = B_u$. Now, for a given desired set point $(x_e, z_e) \in \mathbb{R}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ and for some $\epsilon_1, \epsilon_2 > 0$, our aim is to design a control input $u(k)$, $k \in \overline{\mathbb{Z}}_+$, such that $\|x(k) - x_e\| < \epsilon_1$ and $\|z(k) - z_e\| < \epsilon_2$ for all $k \geq K$, where $K \in \overline{\mathbb{Z}}_+$, and $x(k) \geqgeq 0$ and $z(k) \geqgeq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \mathbb{R}_+^{n_x} \times \mathbb{R}_+^{n_z}$. However, since in many applications of nonnegative systems and, in particular, compartmental systems, it is often necessary to regulate a subset of the nonnegative state variables which usually include a central compartment, here we only require that $\|x(k) - x_e\| < \epsilon_1$, $k \geq K$.

Theorem 4.1. *Consider the discrete-time nonlinear uncertain dynamical system $\mathcal{G}$ given by (4.1) and (4.2) where $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by (4.4) and (4.14), respectively, $f_x(\cdot, \cdot)$ is nonnegative with respect to x, $f_z(\cdot, \cdot)$ is nonnegative with respect to z, and $\Delta f(\cdot, \cdot)$ is nonnegative with respect to x and belongs to $\mathcal{F}$. For a given $x_e \in \mathbb{R}_+^{n_x}$ assume there exist nonnegative vectors $z_e \in \overline{\mathbb{R}}_+^{n_z}$ and $u_e \in \overline{\mathbb{R}}_+^{n_x}$ such that (4.6) and (4.7) hold. Furthermore, assume that (4.2) is input-to-state stable at $z(k) \equiv z_e$ with $x(k) - x_e$ viewed as the input. Finally, let $K \in \mathbb{R}^{n_x \times n_x}$ be such that $-K$ is nonnegative and $A_s \triangleq A + B_u K$ is nonnegative and asymptotically stable. Then the neuroadaptive feedback control law*

$$u(k) = G_n^{-1}(x(k), z(k))\left[K(x(k) - x_e) - \widehat{W}^T(k)\widehat{\sigma}(x(k), z(k), \widehat{W}(k))\right], \tag{4.15}$$

where

$$\widehat{W}(k) \triangleq \text{block-diag}\left[\widehat{W}_1(k), \dots, \widehat{W}_{n_x}(k)\right], \tag{4.16}$$

$\widehat{W}_i(k) \in \mathbb{R}^{s_i}$, $k \in \overline{\mathbb{Z}}_+$, $i = 1, \dots, n_x$, and $\widehat{\sigma}(x, z, \widehat{W}) \triangleq [\widehat{\sigma}_1^T(x, z, \widehat{W}_1), \dots, \widehat{\sigma}_{n_x}^T(x, z, \widehat{W}_{n_x})]^T$ *with* $\widehat{\sigma}_{i(j)}(x, z, \widehat{W}_i) = 0$ *whenever* $\widehat{W}_{i(j)} > 0$, $i = 1, \dots, n_x$, $j = 1, \dots, s_i$,—*with update law*

$$\widehat{W}_i(k+1) = \widehat{W}_i(k) + \frac{q_i \|P^{1/2}(x(k) - x_e)\|}{1 + \|P^{1/2}(x(k) - x_e)\|^2}\left[\widetilde{e}_i(k)\widehat{\sigma}_i(x(k), z(k), \widehat{W}_i(k)) - \gamma_i\widehat{W}_i(k)\right],$$

$$\widehat{W}_i(0) = \widehat{W}_{i0}, \quad i = 1, \dots, n_x, \tag{4.17}$$

where $P \triangleq \text{diag}[p_1, \dots, p_{n_x}] > 0$ satisfies

$$P = A_s^T P A_s + R \tag{4.18}$$

for positive definite $R \in \mathbb{R}^{n_x \times n_x}$, q_i and γ_i are positive constants satisfying $b_i q_i s_i < 2$ and $q_i \gamma_i \leq 1$, $i = 1, \dots, n_x$, and $\widetilde{e}(k) \triangleq x(k+1) - x_e - A_s(x(k) - x_e) = [\widetilde{e}_1(k), \widetilde{e}_2(k), \dots, \widetilde{e}_{n_x}(k)]^T$—guarantees that there exists a positively invariant set $\mathfrak{D}_\alpha \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \times \mathbb{R}^{s \times n_x}$ such that $(x_e, z_e, W) \in \mathfrak{D}_\alpha$, where $W \triangleq \text{block-diag}[W_1, \dots, W_{n_x}]$, and the solution $(x(k), z(k), \widehat{W}(k))$, $k \in \overline{\mathbb{Z}}_+$, of the closed-loop system
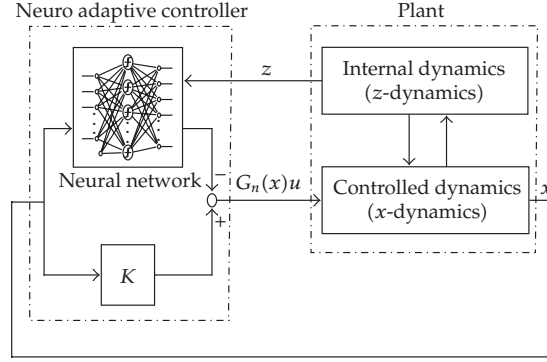
Figure 1: Block diagram of the closed-loop system.

given by (4.1), (4.2), (4.15), and (4.17) is ultimately bounded for all $(x(0), z(0), \widehat{W}(0)) \in \mathfrak{D}_\alpha$ with ultimate bound $\|P^{1/2}(x(k) - x_e)\| < \varepsilon$, $k \in \overline{\mathbb{Z}}_+$, where

$$\varepsilon \triangleq \sqrt{e^\eta - 1}, \tag{4.19}$$

$$\eta \triangleq \alpha_x^2 + \eta_w + \frac{1}{2}\left(\frac{\alpha}{a} + \beta\right) + 2\left(1 + \frac{1}{c}\right)(\alpha + \xi\eta_w), \qquad \alpha_x \triangleq \max\left\{\frac{a\beta + \alpha}{a(\mu_1 - c\mu_2)}, \frac{1+c}{c(2 - a - \xi)}\right\},$$

$$\alpha \triangleq \sum_{i=1}^{n_x} p_i b_i^2 (\varepsilon_i^* + \sqrt{s_i} w_i^*)^2, \qquad \beta \triangleq \sum_{i=1}^{n_x} p_i b_i \gamma_i w_i^{*2}, \quad \eta_w > (2 + \zeta)(\alpha + a\beta)/(2a\zeta), \tag{4.20}$$

$\mu_1 \triangleq \lambda_{\min}(R)/\lambda_{\max}(P)$, $\mu_2 \triangleq \lambda_{\max}(A_s^T P A_s)/\lambda_{\min}(P)$, $\xi \triangleq \max\{b_1 q_1 s_1, \ldots, b_{n_x} q_{n_x} s_{n_x}\}$, $\zeta \triangleq \min\{q_1\gamma_1, \ldots, q_{n_x}\gamma_{n_x}\}$, and a and c are positive constants satisfying $a < 2-\xi$ and $c\mu_2 < \mu_1$, respectively. Furthermore, $x(k) \geq\geq 0$ and $z(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

Proof. See Appendix A.   □

A block diagram showing the neuroadaptive control architecture given in Theorem 4.1 is shown in Figure 1. It is important to note that the adaptive control law (4.15) and (4.17) does not require the explicit knowledge of the optimal weighting matrix W and constants $\delta(x_e, z_e)$ and u_e. All that is required is the existence of the nonnegative vectors z_e and u_e such that the equilibrium conditions (4.6), and (4.7) hold. Furthermore, in the case where $B_u = \text{diag}[b_1, \ldots, b_{n_x}]$ is an *unknown* positive diagonal matrix but $b_i \leq \overline{b}$, $i = 1, \ldots, n_x$, where $\overline{b}$ is known, we can take the gain matrix K to be diagonal so that $K = \text{diag}[k_1, \ldots, k_{n_x}]$, where k_i is such that $-1/\overline{b} \leq k_i < 0$, $i = 1, \ldots, n_x$. In this case, taking A in (4.4) to be the identity matrix, A_s is given by $A_s = \text{diag}[1 + b_1 k_1, \ldots, 1 + b_{n_x} k_{n_x}]$ which is clearly nonnegative and asymptotically stable, and hence, any positive diagonal matrix P satisfies (4.18). Finally, it is important to note that the control input signal $u(k)$, $k \in \overline{\mathbb{Z}}_+$, in Theorem 4.1 can be negative depending on the values of $x(k)$, $k \in \overline{\mathbb{Z}}_+$. However, as is required for nonnegative and compartmental dynamical systems the closed-loop plant states remain nonnegative.

Next, we generalize Theorem 4.1 to the case where the input matrix is not necessarily nonnegative. For this result $\text{row}_i(K)$ denotes the ith row of $K \in \mathbb{R}^{n_x \times n_x}$.

Theorem 4.2. *Consider the discrete-time nonlinear uncertain dynamical system $\mathcal{G}$ given by (4.1) and (4.2), where $f_x(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are given by (4.4) and (4.14), respectively, $f_x(\cdot,\cdot)$ is nonnegative with respect to x, $f_z(\cdot,\cdot)$ is nonnegative with respect to z, and $\Delta f(\cdot,\cdot)$ is nonnegative with respect to x and belongs to $\mathcal{F}$. For a given $x_e \in \mathbb{R}_+^{n_x}$, assume there exist a nonnegative vector $z_e \in \overline{\mathbb{R}}_+^{n_z}$ and a vector $u_e \in \mathbb{R}^{n_x}$ such that (4.6) and (4.7) hold with $f_x(x_e, z_e) \leq\leq x_e$. Furthermore, assume that (4.2) is input-to-state stable at $z(k) \equiv z_e$ with $x(k) - x_e$ viewed as the input. Finally, let $K \in \mathbb{R}^{n_x \times n_x}$ be such that $(\text{sgn } b_i)\text{row}_i(K) \leq\leq 0$, $i = 1, \dots, n_x$, and $A_s \triangleq A + B_u K$ is nonnegative and asymptotically stable. Then the neuroadaptive feedback control law (4.15), where $\widehat{W}(k)$ is given by (4.16) with $\widehat{W}_i(k) \in \mathbb{R}^{s_i}$, $k \in \overline{\mathbb{Z}}_+$, $i = 1, \dots, n_x$, and $\widehat{\sigma}(x, z, \widehat{W}) \triangleq [\widehat{\sigma}_1^{\mathrm{T}}(x, z, \widehat{W}_1), \dots, \widehat{\sigma}_{n_x}^{\mathrm{T}}(x, z, \widehat{W}_{n_x})]^{\mathrm{T}}$ with $\widehat{\sigma}_{i(j)}(x, z, \widehat{W}_i) = 0$ whenever $\widehat{W}_{i(j)} > 0$, $i = 1, \dots, n_x$, $j = 1, \dots, s_i$,—with update law*

$$\widehat{W}_i(k+1) = \widehat{W}_i(k) + \frac{q_i \|P^{1/2}(x(k) - x_e)\|}{1 + \|P^{1/2}(x(k) - x_e)\|^2}\left[(\text{sgn } b_i)\widetilde{e}_i(k)\widehat{\sigma}_i\big(x(k), z(k), \widehat{W}_i(k)\big) - \gamma_i \widehat{W}_i(k)\right],$$

$$\widehat{W}_i(0) = \widehat{W}_{i0}, \quad i = 1, \dots, n_x,$$

$$(4.21)$$

where $P \triangleq \text{diag}[p_1, \dots, p_{n_x}] > 0$ satisfies (4.18), q_i and γ_i are positive constants satisfying $|b_i|q_i s_i < 2$ and $q_i \gamma_i \leq 1$, $i = 1 \dots, n_x$, $\widetilde{e}(k) \triangleq x(k+1) - x_e - A_s(x(k) - x_e) = [\widetilde{e}_1(k), \widetilde{e}_2(k), \dots, \widetilde{e}_{n_x}(k)]^{\mathrm{T}}$—guarantees that there exists a positively invariant set $\mathfrak{D}_\alpha \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \times \mathbb{R}^{s \times n_x}$ such that $(x_e, z_e, W) \in \mathfrak{D}_\alpha$, where $W \triangleq \text{block-diag}[W_1, \dots, W_{n_x}]$, and the solution $(x(k), z(k), \widehat{W}(k))$, $k \in \overline{\mathbb{Z}}_+$, of the closed-loop system given by (4.1), (4.2), (4.15), and (4.21) is ultimately bounded for all $(x(0), z(0), \widehat{W}(0)) \in \mathfrak{D}_\alpha$ with ultimate bound $\|P^{1/2}(x(k) - x_e)\| < \varepsilon$, $k \geq K$, where ε is given by (4.19) with b_i replaced by $|b_i|$ in β and ξ, $i = 1, \dots, n_x$. Furthermore, $x(k) \geq\geq 0$ and $z(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

Proof. The proof is identical to the proof of Theorem 4.1 with Q replaced by $Q = \text{diag}[q_1/p_1|b_1|, \dots, q_{n_x}/p_{n_x}|b_{n_x}|]$.   □

Finally, in the case where B_u is an *unknown* diagonal matrix but the sign of each diagonal element is known and $0 < |b_i| \leq \overline{b}$, $i = 1, \dots, n_x$, where $\overline{b}$ is known, we can take the gain matrix K to be diagonal so that $K = \text{diag}[k_1, \dots, k_{n_x}]$, where k_i is such that $-1/\overline{b} \leq (\text{sgn } b_i)k_i < 0$, $i = 1, \dots, n_x$. In this case, taking A in (4.4) to be the identity matrix, A_s is given by $A_s = \text{diag}[1 + b_1 k_1, \dots, 1 + b_{n_x} k_{n_x}]$ which is nonnegative and asymptotically stable.

Example 4.3. Consider the nonlinear uncertain system (4.1) with

$$f_x(x, z) = \begin{bmatrix} x_1 + x_2 + ax_1 \sin \pi x_2 \\ 0.5x_1 + 0.25x_2 \end{bmatrix}, \qquad G(x, z) = \begin{bmatrix} \dfrac{b}{1 + x_1^2 + x_2^2} \\ 0 \end{bmatrix}, \qquad (4.22)$$

where $a, b \in \mathbb{R}$ are unknown. For simplicity of exposition, here we assume that there is no internal dynamics. Note that $f_x(x, z)$ and $G(x, z)$ in (4.22) can be written in the form of (4.4) and (4.3) with $A = \begin{bmatrix} 0.1 & 0.1 \\ 0.5 & 0.25 \end{bmatrix}$, $\Delta f(x) = [ax_1 \sin \pi x_2, 0]^{\mathrm{T}}$, $B_u = b$, and $G_n(x) = 1/(1 + x_1^2 + x_2^2)$. Furthermore, note that $\Delta f(x, z)$ is unknown and belongs to $\mathcal{F}$. Since for $x_e = [0.5, 1]^{\mathrm{T}}$ there exists $u_e \in \overline{\mathbb{R}}_+$ such that (4.6) is satisfied, it follows from Theorem 4.2 that the neuroadaptive feedback control law (4.15) with $K = [-0.1, 0]$ and update law (4.21)

(a)

(b)
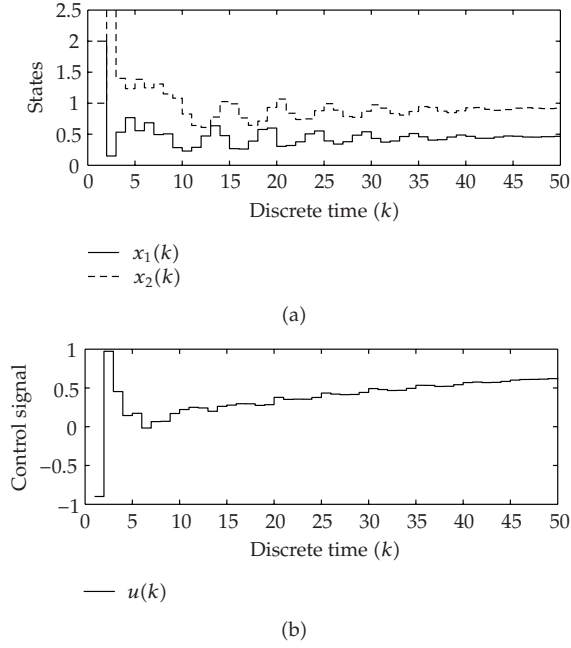
Figure 2: State trajectories and control signal versus time.

guarantees that the closed-loop systems trajectory is ultimately bounded and remains in the nonnegative orthant of the state space for nonnegative initial conditions. With $a = 0.9$, $b = 1$, $\sigma_1(x, z) = [1/(1 + e^{-cx_1}), \ldots, 1/(1 + e^{-6cx_1}), 1/(1 + e^{-cx_2}), \ldots, 1/(1 + e^{-6cx_2})]^{\mathrm{T}}$, $c = 0.5$, $q_1 = 0.1$, $\gamma_1 = 0.1$, and initial conditions $x(0) = [2, 1]^{\mathrm{T}}$ and $W(0) = [0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^{12}$, Figure 2 shows the state trajectories versus time and the control signal versus time.

5. Neuroadaptive control for discrete-time nonlinear nonnegative uncertain systems with nonnegative control

As discussed in the introduction, control (source) inputs of drug delivery systems for physiological and pharmacological processes are usually constrained to be nonnegative as are the system states. Hence, in this section we develop neuroadaptive control laws for discrete-time nonnegative systems with nonnegative control inputs. In general, unlike linear nonnegative systems with asymptotically stable plant dynamics, a given set point $x_e \in \mathbb{R}_+^n$ for a discrete-time nonlinear nonnegative dynamical system

$$x(k + 1) = f(x(k)) + u(k), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+, \tag{5.1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, may not be asymptotically stabilizable with a constant control $u(k) \equiv u_e \in \overline{\mathbb{R}}_+^n$. Hence, we assume that the set point $x_e \in \mathbb{R}_+^n$ satisfying $x_e = f(x_e) + u_e$ is a unique equilibrium point in the nonnegative orthant with $u(k) \equiv u_e$ and is

also asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$. This implies that the equilibrium solution $x(k) \equiv x_e$ to (5.1) with $u(k) \equiv u_e$ is asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$.

In this section, we assume that A in (4.4) is nonnegative and asymptotically stable, and hence, without loss of generality (see [19, Proposition 3.1]), we can assume that A is an asymptotically stable compartmental matrix [19]. Furthermore, we assume that the control inputs are injected directly into m separate compartments so that B_u and $G_n(x, z)$ in (4.14) are such that $B_u = \mathrm{diag}[b_1, \ldots, b_{n_x}]$ is a positive diagonal matrix and $G_n(x, z) = \mathrm{diag}[g_{n_1}(x, z), \ldots, g_{n_{n_x}}(x, z)]$, where $g_{n_i} : \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \to \mathbb{R}_+$, $i = 1, \ldots, m$, is a known positive diagonal matrix function. For compartmental systems, this assumption is not restrictive since control inputs correspond to control inflows to each individual compartment. For the statement of the next theorem, recall the definitions of W and $\widehat{W}(k)$, $k \in \overline{\mathbb{Z}}_+$, given in Theorem 4.1.

Theorem 5.1. *Consider the discrete-time nonlinear uncertain dynamical system $\mathcal{G}$ given by (4.1) and (4.2), where $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by (4.4) and (4.14), respectively, A is nonnegative and asymptotically stable, $f_x(\cdot, \cdot)$ is nonnegative with respect to x, $f_z(\cdot, \cdot)$ is nonnegative with respect to z, and $\Delta f(\cdot, \cdot)$ is nonnegative with respect to x and belongs to $\mathcal{F}$. For a given $x_e \in \mathbb{R}_+^{n_x}$ assume there exist positive vectors $z_e \in \mathbb{R}_+^{n_z}$ and $u_e \in \mathbb{R}_+^{n_x}$ such that (4.6) and (4.7) hold and the set point $(x_e, z_e) \in \mathbb{R}_+^{n_x} \times \mathbb{R}_+^{n_z}$ is asymptotically stable with constant control $u(k) \equiv u_e \in \mathbb{R}_+^{n_x}$ for all $x_0 \in \mathbb{R}_+^n$. In addition, assume that (4.2) is input-to-state stable at $z(k) \equiv z_e$ with $x(k) - x_e$ viewed as the input. Then the neuroadaptive feedback control law*

$$u_i(k) = \max\{0, \widehat{u}_i(k)\}, \quad i = 1, \ldots, n_x, \tag{5.2}$$

where

$$\widehat{u}_i(k) = -g_{n_i}^{-1}(x(k), z(k))\widehat{W}_i^{\mathrm{T}}(k)\sigma_i(x(k), z(k)), \tag{5.3}$$

and $\widehat{W}_i(k) \in \mathbb{R}^{s_i}$, $k \in \overline{\mathbb{Z}}_+$, $i = 1, \ldots, n_x$,—with update law

$$\widehat{W}_i(k+1) = \widehat{W}_i(k) + \frac{q_i\|P^{1/2}(x(k) - x_e)\|}{1 + \|P^{1/2}(x(k) - x_e)\|^2}\left[\gamma\widetilde{e}_i(k)\sigma_i(x(k), z(k)) - \widehat{W}_i(k)\right],$$

$$\widehat{W}_i(0) = \widehat{W}_{i0}, \quad i = 1, \ldots, n_x, \tag{5.4}$$

where $P \triangleq \mathrm{diag}[p_1, \ldots, p_{n_x}] > 0$ satisfies

$$P = A^{\mathrm{T}}PA + R \tag{5.5}$$

for positive definite $R \in \mathbb{R}^{n_x \times n_x}$, γ and q_i are positive constants satisfying $b_i q_i \gamma < 1$ and $q_i \leq 1 - b_i s_i \gamma$, $i = 1, \ldots, n_x$, $\widetilde{e}(k) \triangleq x(k+1) - x_e - A(x(k) - x_e) = [\widetilde{e}_1(k), \ldots, \widetilde{e}_{n_x}(k)]^{\mathrm{T}}$—guarantees that there exists a positively invariant set $\mathfrak{D}_\alpha \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \times \mathbb{R}^{s \times n_x}$ such that $(x_e, z_e, W) \in \mathfrak{D}_\alpha$ and the solution $(x(k), z(k), \widehat{W}(k))$, $k \in \overline{\mathbb{Z}}_+$, of the closed-loop system given by (4.1), (4.2), (5.2), and (5.4) is ultimately bounded for all $(x(0), z(0), \widehat{W}(0)) \in \mathfrak{D}_\alpha$ with ultimate bound $\|P^{1/2}(x(k) - x_e)\| < \varepsilon$, $k \geq K$, where $\varepsilon \triangleq \sqrt{e^\eta - 1}$,

$$\eta \triangleq \alpha_x^2 + \eta_w + \frac{1}{2}\left(\frac{\alpha\gamma}{a} + \beta\right) + 2\left(1 + \frac{1}{c}\right)(\alpha + \xi\eta_w), \qquad \alpha_x \triangleq \max\left\{\frac{\alpha\gamma + a\beta}{a(\mu_1 - c\mu_2)}, \frac{1 + c}{c\gamma(1 - a - \gamma\xi)}\right\},$$

$$\alpha \triangleq \sum_{i=1}^{n_x} p_i b_i^2 \varepsilon_i^{*2}, \qquad \beta \triangleq \sum_{i=1}^{n_x} p_i b_i^2 w_i^{*2}, \quad \eta_w > (2 + \zeta)(\alpha\gamma + a\beta)/(2a\zeta),$$

$$\tag{5.6}$$

$\mu_1 \triangleq \lambda_{\min}(R)/\lambda_{\max}(P)$, $\mu_2 \triangleq \lambda_{\max}(A^{\mathrm{T}}PA)/\lambda_{\min}(P)$, $\xi \triangleq \max\{b_1 q_1 s_1, \ldots, b_{n_x} q_{n_x} s_{n_x}\}$, $\zeta \triangleq \min\{q_1, \ldots, q_{n_x}\}$, and a and c are positive constants satisfying $a < 1 - \gamma\xi$ and $c\mu_2 < \mu_1$. Furthermore, $u(k) \geq\geq 0$, $x(k) \geq\geq 0$, and $z(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

Proof. See Appendix B.   □

6. Conclusion

In this paper, we developed a neuroadaptive control framework for adaptive set-point regulation of discrete-time nonlinear uncertain nonnegative and compartmental systems. Using Lyapunov methods, the proposed framework was shown to guarantee ultimate boundedness of the error signals corresponding to the physical system states and the neural network weighting gains while additionally guaranteeing the nonnegativity of the closed-loop system states associated with the plant dynamics.

Appendices

A. Proof of Theorem 4.1

In this appendix, we prove Theorem 4.1. First, note that with $u(k)$, $k \in \overline{\mathbb{Z}}_+$, given by (4.15), it follows from (4.1), (4.4), and (4.14) that

$$x(k + 1) = Ax(k) + \Delta f(x(k), z(k)) + B_{\mathrm{u}} K(x(k) - x_{\mathrm{e}}) - B_{\mathrm{u}} \widehat{W}^{\mathrm{T}}(k)\widehat{\sigma}(x(k), z(k), \widehat{W}(k)),$$
$$x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+. \tag{A.1}$$

Now, defining $e_x(k) \triangleq x(k) - x_{\mathrm{e}}$ and $e_z(k) \triangleq z(k) - z_{\mathrm{e}}$, using (4.5)–(4.7), (4.9), and $A_{\mathrm{s}} = A + B_{\mathrm{u}}K$, it follows from (4.2) and (A.1) that

$$\begin{aligned}
e_x(k + 1) &= A_{\mathrm{s}} e_x(k) + (A - I)x_{\mathrm{e}} + \Delta f(x(k), z(k)) - B_{\mathrm{u}} \widehat{W}^{\mathrm{T}}(k)\widehat{\sigma}(x(k), z(k), \widehat{W}(k)) \\
&= A_{\mathrm{s}} e_x(k) + B_{\mathrm{u}}\left[\delta(x(k), z(k)) - \delta(x_{\mathrm{e}}, z_{\mathrm{e}}) - G_{\mathrm{n}}(x_{\mathrm{e}}, z_{\mathrm{e}})u_{\mathrm{e}} - \widehat{W}^{\mathrm{T}}(k)\sigma(x(k), z(k))\right] \\
&\quad + B_{\mathrm{u}} \widehat{W}^{\mathrm{T}}(k)\left[\sigma(x(k), z(k)) - \widehat{\sigma}(x(k), z(k), \widehat{W}(k))\right] \\
&= A_{\mathrm{s}} e_x(k) + B_{\mathrm{u}}\left[W^{\mathrm{T}}\sigma(x(k), z(k)) + \varepsilon(x(k), z(k)) - \widehat{W}^{\mathrm{T}}(k)\sigma(x(k), z(k))\right] \\
&\quad + B_{\mathrm{u}} \widehat{W}^{\mathrm{T}}(k)\left[\sigma(x(k), z(k)) - \widehat{\sigma}(x(k), z(k), \widehat{W}(k))\right] \\
&= A_{\mathrm{s}} e_x(k) - B_{\mathrm{u}} \widetilde{W}^{\mathrm{T}}(k)\widehat{\sigma}(x(k), z(k), \widehat{W}(k)) \\
&\quad + B_{\mathrm{u}}\left[\varepsilon(x(k), z(k)) - W^{\mathrm{T}}\widetilde{\sigma}(x(k), z(k), \widehat{W}(k))\right] \\
&= A_{\mathrm{s}} e_x(k) - B_{\mathrm{u}} \widetilde{W}^{\mathrm{T}}(k)\widehat{\sigma}(x(k), z(k), \widehat{W}(k)) + B_{\mathrm{u}} r(k), \quad e_x(0) = x_0 - x_{\mathrm{e}}, \ k \in \overline{\mathbb{Z}}_+,
\end{aligned} \tag{A.2}$$

$$e_z(k + 1) = \widetilde{f}_z(e_x(k), e_z(k)), \qquad e_z(0) = z_0 - z_{\mathrm{e}}, \tag{A.3}$$

where $\widetilde{f}_z(e_x, e_z) \triangleq f_z(e_x + x_{\mathrm{e}}, e_z + z_{\mathrm{e}})$, $\varepsilon(x, z) \triangleq [\varepsilon_1(x, z), \ldots, \varepsilon_{n_x}(x, z)]^{\mathrm{T}}$, $\sigma(x, z) \triangleq [\sigma_1^{\mathrm{T}}(x, z), \ldots, \sigma_{n_x}^{\mathrm{T}}(x, z)]^{\mathrm{T}}$, $\widetilde{W}(k) \triangleq \widehat{W}(k) - W$, $\widetilde{\sigma}(x, z, \widehat{W}) \triangleq \widehat{\sigma}(x, z, \widehat{W}) - \sigma(x, z)$, and $r \triangleq \varepsilon(x, z) - W^{\mathrm{T}}\widetilde{\sigma}(x, z, \widehat{W}_i) = [r_1, \ldots, r_{n_x}]^{\mathrm{T}}$. Furthermore, since A_{s} is nonnegative and asymptotically stable,

it follows from Theorem 2.3 that there exist a positive *diagonal* matrix $P = \text{diag}[p_1, \ldots, p_{n_x}]$ and a positive-definite matrix $R \in \mathbb{R}^{n_x \times n_x}$ such that (4.18) holds.

Next, to show that the closed-loop system given by (4.17), (A.2), and (A.3) is ultimately bounded with respect to $\widetilde{W}$, consider the Lyapunov-like function

$$V_{\mathrm{w}}\big(e_x, e_z, \widetilde{W}\big) = \text{tr}\,\widetilde{W}(k) Q^{-1} \widetilde{W}(k)^{\mathrm{T}}, \tag{A.4}$$

where $Q \triangleq \text{diag}[\hat{q}_1, \ldots, \hat{q}_{n_x}] = \text{diag}[q_1/p_1 b_1, \ldots, q_{n_x}/p_{n_x} b_{n_x}]$. Note that (A.4) satisfies (3.3) with $x_1 = [\hat{q}_1^{-1/2}\widetilde{W}_1^{\mathrm{T}}, \ldots, \hat{q}_{n_x}^{-1/2}\widetilde{W}_{n_x}^{\mathrm{T}}]^{\mathrm{T}}$, $x_2 = [e_x^{\mathrm{T}}, e_z^{\mathrm{T}}]^{\mathrm{T}}$, $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^2$, where $\|x_1\|^2 = \text{tr}\,\widetilde{W}Q^{-1}\widetilde{W}^{\mathrm{T}}$. Furthermore, $\alpha(\|x_1\|)$ is a class-$\mathcal{K}_\infty$ function. Now, using (4.17) and (A.2), it follows that the difference of $V_{\mathrm{w}}(e_x, e_z, \widetilde{W})$ along the closed-loop system trajectories is given by

$$\Delta V_{\mathrm{w}}\big(e_x(k), e_z(k), \widetilde{W}(k)\big) \triangleq \text{tr}\,\widetilde{W}(k+1)Q^{-1}\widetilde{W}^{\mathrm{T}}(k+1) - \text{tr}\,\widetilde{W}(k)Q^{-1}\widetilde{W}^{\mathrm{T}}(k)$$

$$= \sum_{i=1}^{n_x} \frac{2p_i b_i \|P^{1/2}e_x(k)\|}{1 + \|P^{1/2}e_x(k)\|^2} \big[\widetilde{e}_i(k)\hat{\sigma}_i(x(k), z(k), \widehat{W}_i(k)) - \gamma_i \widehat{W}_i(k)\big]^{\mathrm{T}} \widetilde{W}_i(k)$$

$$+ \sum_{i=1}^{n_x} \frac{p_i b_i q_i \|P^{1/2}e_x(k)\|^2}{\big(1 + \|P^{1/2}e_x(k)\|^2\big)^2} \big\|\widetilde{e}_i(k)\hat{\sigma}_i(x(k), z(k), \widehat{W}_i(k)) - \gamma_i \widehat{W}_i(k)\big\|^2$$

$$= \sum_{i=1}^{n_x} \frac{2p_i b_i \|P^{1/2}e_x(k)\| r_i(k)\widetilde{e}_i(k)}{1 + \|P^{1/2}e_x(k)\|^2} - \sum_{i=1}^{n_x} \frac{2p_i \|P^{1/2}e_x(k)\| \widetilde{e}_i^2(k)}{1 + \|P^{1/2}e_x(k)\|^2}$$

$$- \sum_{i=1}^{n_x} \frac{2p_i b_i \gamma_i \|P^{1/2}e_x(k)\| \widehat{W}_i^{\mathrm{T}}(k)\widetilde{W}_i(k)}{1 + \|P^{1/2}e_x(k)\|^2}$$

$$+ \sum_{i=1}^{n_x} \frac{p_i b_i q_i \|P^{1/2}e_x(k)\|^2}{\big(1 + \|P^{1/2}e_x(k)\|^2\big)^2} \big\|\widetilde{e}_i(k)\hat{\sigma}_i(x(k), z(k), \widehat{W}_i(k)) - \gamma_i \widehat{W}_i(k)\big\|^2. \tag{A.5}$$

Next, using

$$\sum_{i=1}^{n_x} 2p_i b_i r_i \widetilde{e}_i \leq a^{-1} \sum_{i=1}^{n_x} p_i b_i^2 r_i^2 + a\|P^{1/2}\widetilde{e}\|^2, \tag{A.6}$$

$$\sum_{i=1}^{n_x} p_i b_i q_i \|\widetilde{e}_i \hat{\sigma}_i(x, z, \widehat{W}_i) - \gamma_i \widehat{W}_i\|^2 \leq \sum_{i=1}^{n_x} 2p_i b_i q_i s_i \widetilde{e}_i^2 + \sum_{i=1}^{n_x} 2p_i b_i q_i \gamma_i^2 \|\widehat{W}_i\|^2, \tag{A.7}$$

$$2\widetilde{W}_i^{\mathrm{T}}\widehat{W}_i = \|\widetilde{W}_i\|^2 + \|\widehat{W}_i\|^2 - \|W_i\|^2, \tag{A.8}$$

$$\sum_{i=1}^{n_x} \frac{2p_i b_i q_i s_i \|P^{1/2}e_x\|^2 \widetilde{e}_i^2}{\big(1 + \|P^{1/2}e_x\|^2\big)^2} \leq \frac{\xi \|P^{1/2}\widetilde{e}\|^2 \|P^{1/2}e_x\|}{1 + \|P^{1/2}e_x\|^2}, \tag{A.9}$$

it follows that

$$\Delta V_{\mathrm{w}}\big(e_x(k), e_z(k), \widetilde{W}(k)\big)$$

$$\leq \frac{a^{-1}\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}\sum_{i=1}^{n_x} p_i b_i^2 r_i^2(k) + \frac{a\big\|P^{1/2}\widetilde{e}(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}$$

$$-\frac{2\big\|P^{1/2}\widetilde{e}(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} - \sum_{i=1}^{n_x}\frac{p_i b_i \gamma_i\big\|\widetilde{W}_i(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}$$

$$-\sum_{i=1}^{n_x}\frac{p_i b_i \gamma_i\big\|\widehat{W}_i(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} + \sum_{i=1}^{n_x}\frac{p_i b_i \gamma_i\big\|W_i\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}$$

$$+\sum_{i=1}^{n_x}\frac{2p_i b_i q_i s_i\big\|P^{1/2}e_x(k)\big\|^2\widetilde{e}_i^2(k)}{\big(1+\big\|P^{1/2}e_x(k)\big\|^2\big)^2} + \sum_{i=1}^{n_x}\frac{2p_i b_i q_i \gamma_i^2\big\|P^{1/2}e_x(k)\big\|^2\big\|\widehat{W}_i(k)\big\|^2}{\big(1+\big\|P^{1/2}e_x(k)\big\|^2\big)^2}$$

$$\leq \frac{(\alpha/a)\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} + \frac{a\big\|P^{1/2}\widetilde{e}(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}$$

$$-\frac{2\big\|P^{1/2}\widetilde{e}(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} - \sum_{i=1}^{n_x}\frac{p_i b_i \gamma_i\big\|\widetilde{W}_i(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} \qquad (A.10)$$

$$-\sum_{i=1}^{n_x}\frac{p_i b_i \gamma_i\big\|\widehat{W}_i(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} + \frac{\beta\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}$$

$$+\frac{\xi\big\|P^{1/2}\widetilde{e}(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2} + \sum_{i=1}^{n_x}\frac{2p_i b_i q_i \gamma_i^2\big\|P^{1/2}e_x(k)\big\|^2\big\|\widehat{W}_i(k)\big\|^2}{\big(1+\big\|P^{1/2}e_x(k)\big\|^2\big)^2}$$

$$= -\frac{\big\|P^{1/2}\widetilde{e}(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}(2-a-\xi)$$

$$-\frac{\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}\left(\sum_{i=1}^{n_x} p_i b_i \gamma_i\big\|\widetilde{W}_i(k)\big\|^2 - \frac{\alpha}{a} - \beta\right)$$

$$-\sum_{i=1}^{n_x}\frac{p_i b_i \gamma_i\big\|\widehat{W}_i(k)\big\|^2\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}\left(1 - \frac{2q_i \gamma_i\big\|P^{1/2}e_x(k)\big\|}{1+\big\|P^{1/2}e_x(k)\big\|^2}\right).$$

Furthermore, note that since, by assumption, $2 - a - \xi > 0$ and $q_i \gamma_i \leq 1$, $i = 1, \ldots, n_x$, it follows that

$$1 - \frac{2q_i \gamma_i\big\|P^{1/2}e_x\big\|}{1+\big\|P^{1/2}e_x\big\|^2} \geq 0, \quad i = 1, \ldots, n_x. \qquad (A.11)$$

Hence,

$$\Delta V_{\mathrm{w}}\big(e_x(k), e_z(k), \widetilde{W}(k)\big) \leq -\frac{\big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\left(\sum_{i=1}^{n_x} p_i b_i \gamma_i \big\|\widetilde{W}_i(k)\big\|^2 - \frac{\alpha}{a} - \beta\right)$$

$$\leq -\frac{\big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\left(\zeta\sum_{i=1}^{n_x} \big\|\widehat{q}_i^{-1/2}\widetilde{W}_i(k)\big\|^2 - \frac{\alpha}{a} - \beta\right). \tag{A.12}$$

Now, for

$$\zeta\sum_{i=1}^{n_x} \big\|\widehat{q}_i^{-1/2}\widetilde{W}_i\big\|^2 > \frac{\alpha}{a} + \beta, \tag{A.13}$$

it follows that $\Delta V_{\mathrm{w}}(e_x(k), e_z(k), \widetilde{W}(k)) \leq 0$ for all $k \in \overline{\mathbb{Z}}_+$, that is, $\Delta V_{\mathrm{w}}(e_x(k), e_z(k), \widetilde{W}(k)) \leq 0$ for all $(e_x(k), e_z(k), \widetilde{W}(k)) \in (\mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s}) \setminus \widetilde{\mathfrak{D}}_{\mathrm{w}}$ and $k \in \overline{\mathbb{Z}}_+$, where

$$\widetilde{\mathfrak{D}}_{\mathrm{w}} \triangleq \left\{(e_x, e_z, \widetilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s} : \sum_{i=1}^{n_x} \big\|\widehat{q}_i^{-1/2}\widetilde{W}_i\big\|^2 \leq \frac{\alpha + a\beta}{a\zeta}\right\}. \tag{A.14}$$

Furthermore, it follows from (A.12) that

$$\Delta V_{\mathrm{w}}(e_x, e_z, \widetilde{W}) \leq \frac{\big\|P^{1/2}e_x\big\|}{1 + \big\|P^{1/2}e_x\big\|^2}\left(\frac{\alpha}{a} + \beta\right) \leq \frac{1}{2}\left(\frac{\alpha}{a} + \beta\right), \quad (e_x, e_z, \widetilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s}. \tag{A.15}$$

Hence, it follows from (A.4) and (A.15) that

$$\sup_{(\widetilde{W}, e_x, e_z) \in \overline{\mathcal{B}}_\mu(0) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}} \big(V_{\mathrm{w}}(e_x, e_z, \widetilde{W}) + \Delta V_{\mathrm{w}}(e_x, e_z, \widetilde{W})\big) \leq \left(\frac{1}{2} + \frac{1}{\zeta}\right)\left(\frac{\alpha}{a} + \beta\right), \tag{A.16}$$

where $\mu^2 = (\alpha + a\beta)/(a\zeta)$. Thus, it follows from Theorem 3.2 that the closed-loop system given by (4.17), (A.2), and (A.3) is globally bounded with respect to $\widetilde{W}$ uniformly in $(e_x(0), e_z(0))$, and for every $\widetilde{W}_i(0) \in \mathbb{R}^{s_i}$, $\sum_{i=1}^{n_x} \big\|\widehat{q}_i^{-1/2}\widetilde{W}_i(k)\big\|^2 \leq \varepsilon_{\mathrm{w}}$, $k \in \overline{\mathbb{Z}}_+$, where

$$\varepsilon_{\mathrm{w}} \triangleq \max\{\overline{\eta}_{\mathrm{w}}, \delta_{\mathrm{w}}\}, \tag{A.17}$$

$\overline{\eta}_{\mathrm{w}} \geq (2 + \zeta)(\alpha + a\beta)/(2a\zeta)$, and $\delta_{\mathrm{w}} \triangleq \sum_{i=1}^{n_x} \big\|\widehat{q}_i^{-1/2}\widetilde{W}_i(0)\big\|^2$. Furthermore, to show that $\sum_{i=1}^{n_x} \big\|\widehat{q}_i^{-1/2}\widetilde{W}_i(k)\big\|^2 < \varepsilon_1$, $k \geq K$, suppose there exists $k^* \in \overline{\mathbb{Z}}_+$ such that $e_x(k) = 0$ for all $k \geq k^*$. In this case, $\widetilde{W}(k+1) = \widetilde{W}(k)$, $k \geq k^*$, which implies $\widetilde{W}(k) = \widetilde{W}(k^*)$, $k \geq k^*$. Alternatively, suppose there does not exist $k^* \in \overline{\mathbb{Z}}_+$ such that $e_x(k) = 0$ for all $k \geq k^*$. In this case, there exists an infinite set $\overline{\mathbb{Z}}_+^* \triangleq \{k \in \overline{\mathbb{Z}}_+ : e_x(k) \neq 0\} \subset \overline{\mathbb{Z}}_+$. Now, with (A.13) satisfied, it follows that $\Delta V_{\mathrm{w}}(e_x(k), e_z(k), \widetilde{W}(k)) < 0$ for all $k \in \overline{\mathbb{Z}}_+^*$, that is, $\Delta V_{\mathrm{w}}(e_x(k), e_z(k), \widetilde{W}(k)) < 0$ for all $(e_x(k), e_z(k), \widetilde{W}(k)) \in (\mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s}) \setminus \widetilde{\mathfrak{D}}_{\mathrm{w}}$ and $k \in \overline{\mathbb{Z}}_+^*$, where $\widetilde{\mathfrak{D}}_{\mathrm{w}}$ is given by (A.14). Furthermore, note that $\Delta V_{\mathrm{w}}(e_x(k), e_z(k), \widetilde{W}(k)) = 0$, $k \in \overline{\mathbb{Z}}_+ \setminus \overline{\mathbb{Z}}_+^*$, and (A.16) holds. Hence, it follows from Theorem 3.3 that the closed-loop system given by (4.17), (A.2), and (A.3) is

globally ultimately bounded with respect to $\widetilde{W}$ uniformly in $(e_x(0), e_z(0))$ with ultimate bound given by $\sqrt{\eta_w}$, where $\eta_w > (2 + \zeta)(\alpha + a\beta)/(2a\zeta)$.

Next, to show ultimate boundedness of the error dynamics, consider the Lyapunov-like function

$$V_e(e_x, e_z, \widetilde{W}) = \ln\left(1 + e_x^T P e_x\right) + \operatorname{tr} \widetilde{W} Q^{-1} \widetilde{W}^T. \tag{A.18}$$

Note that (A.18) satisfies

$$\alpha\left(\left\| [x_1^T, x_2^T]^T \right\|\right) \leq V_e(x_1, x_2, x_3) \leq \beta\left(\left\| [x_1^T, x_2^T]^T \right\|\right), \tag{A.19}$$

with $x_1 = P^{1/2} e_x$, $x_2 = [\hat{q}_1^{-1/2} \widetilde{W}_1^T, \ldots, \hat{q}_{n_x}^{-1/2} \widetilde{W}_{n_x}^T]^T$, $x_3 = e_z$, $\alpha(\|[x_1^T, x_2^T]^T\|) = \ln(1 + \|[x_1^T, x_2^T]^T\|^2)$, and $\beta(\|[x_1^T, x_2^T]^T\|) = \|[x_1^T, x_2^T]^T\|^2$, where $\|[x_1^T, x_2^T]^T\|^2 = e_x^T P e_x + \operatorname{tr} \widetilde{W} Q^{-1} \widetilde{W}^T$. Furthermore, $\alpha(\cdot)$ is a class-$\mathcal{K}_\infty$ function. Now, using (4.18), (A.10), and the definition of $\tilde{e}$, it follows that the difference of $V_e(e_x, e_z, \widetilde{W})$ along the closed-loop system trajectories is given by

$$\Delta V_e(e_x(k), e_z(k), \widetilde{W}(k)) \triangleq V_e(e_x(k+1), e_z(k+1), \widetilde{W}(k+1)) - V_e(e_x(k), e_z(k), \widetilde{W}(k))$$

$$= \ln\left[\frac{1 + e_x^T(k+1) P e_x(k+1)}{1 + e_x^T(k) P e_x(k)}\right]$$

$$+ \operatorname{tr} \widetilde{W}(k+1) Q^{-1} \widetilde{W}^T(k+1) - \operatorname{tr} \widetilde{W}(k) Q^{-1} \widetilde{W}^T(k)$$

$$\leq \frac{e_x^T(k+1) P e_x(k+1) - e_x^T(k) P e_x(k)}{1 + e_x^T(k) P e_x(k)} + \Delta V_w(e_x(k), e_z(k), \widetilde{W}(k))$$

$$= -\frac{e_x^T(k) R e_x(k)}{1 + e_x^T(k) P e_x(k)} + \frac{2 e_x^T(k) A_s^T P \tilde{e}(k)}{1 + e_x^T(k) P e_x(k)} + \frac{\tilde{e}^T(k) P \tilde{e}(k)}{1 + e_x^T(k) P e_x(k)}$$

$$+ \Delta V_w(e_x(k), e_z(k), \widetilde{W}(k))$$

$$\leq -\frac{\left\| R^{1/2} e_x(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{2 e_x^T(k) A_s^T P \tilde{e}(k)}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{\left\| P^{1/2} \tilde{e}(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2}$$

$$- \frac{\left\| P^{1/2} \tilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} (2 - a - \xi)$$

$$- \frac{\left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(\sum_{i=1}^{n_x} p_i b_i \gamma_i \|\widetilde{W}_i(k)\|^2 - \alpha a^{-1} - \beta\right)$$

$$- \sum_{i=1}^{n_x} \frac{p_i b_i \gamma_i \|\widehat{W}_i(k)\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(1 - \frac{2 q_i \gamma_i \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2}\right), \tag{A.20}$$

where in (A.20) we used $\ln a - \ln b = \ln(a/b)$ and $\ln(1 + c) \leq c$ for $a, b > 0$ and $c > -1$. Now, noting $a < 2 - \xi$ and $c\mu_2 < \mu_1$, using the inequalities

$$\mu_1 \left\| P^{1/2} e_x \right\|^2 \leq \left\| R^{1/2} e_x \right\|^2,$$

$$2 e_x^{\mathrm{T}} A_s^{\mathrm{T}} P \widetilde{e} \leq c\mu_2 \left\| P^{1/2} e_x \right\|^2 + c^{-1} \left\| P^{1/2} \widetilde{e} \right\|^2,$$

(A.21)

and rearranging terms in (A.20) yields

$$\Delta V_{\mathrm{e}}\big(e_x(k), e_z(k), \widetilde{W}(k)\big)$$

$$\leq -\frac{\mu_1 \left\| P^{1/2} e_x(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{c\mu_2 \left\| P^{1/2} e_x(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{c^{-1} \left\| P^{1/2} \widetilde{e}(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2}$$

$$+ \frac{(\alpha/a) \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{a \left\| P^{1/2} \widetilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2}$$

$$+ \frac{\beta \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} - \frac{\big(2 \left\| P^{1/2} e_x(k) \right\| - 1\big) \left\| P^{1/2} \widetilde{e}(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2}$$

$$+ \frac{\xi \left\| P^{1/2} \widetilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} - \sum_{i=1}^{n_x} \frac{p_i b_i \gamma_i \left\| \widetilde{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2}$$

$$- \sum_{i=1}^{n_x} \frac{p_i b_i \gamma_i \left\| \widehat{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(1 - \frac{2 q_i \gamma_i \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \right)$$

$$\leq -\frac{\left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left[(\mu_1 - c\mu_2) \left\| P^{1/2} e_x(k) \right\| - \beta - \alpha a^{-1} \right]$$

(A.22)

$$- \frac{\left\| P^{1/2} \widetilde{e}(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left[(2 - a - \xi) \left\| P^{1/2} e_x(k) \right\| - 1 - c^{-1} \right]$$

$$- \sum_{i=1}^{n_x} \frac{p_i b_i \gamma_i \left\| \widetilde{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2}$$

$$- \sum_{i=1}^{n_x} \frac{p_i b_i \gamma_i \left\| \widehat{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(1 - \frac{2 q_i \gamma_i \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \right)$$

$$\leq -\frac{\left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left[(\mu_1 - c\mu_2) \left\| P^{1/2} e_x(k) \right\| - \beta - \alpha a^{-1} \right]$$

$$- \frac{\left\| P^{1/2} \widetilde{e}(k) \right\|^2}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left[(2 - a - \xi) \left\| P^{1/2} e_x(k) \right\| - 1 - c^{-1} \right].$$

Now, for

$$\left\| P^{1/2} e_x(k) \right\| > \max \left\{ \frac{a\beta + \alpha}{a(\mu_1 - c\mu_2)}, \frac{1 + c}{c(2 - a - \xi)} \right\} \triangleq \alpha_x,$$
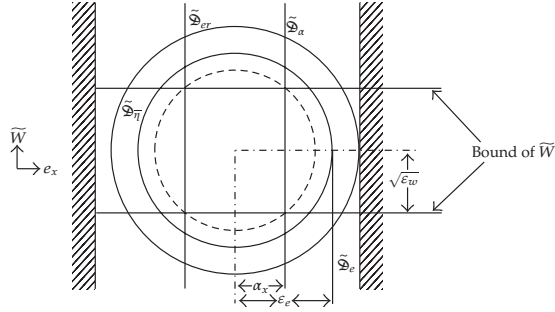
(A.23)

Figure 3: Visualization of sets used in the proof of Theorem 4.1.

it follows that $\Delta V_{\mathrm{e}}(e_x(k), e_z(k), \widetilde{W}(k)) \leq -W(e_x(k))$ for all $k \in \overline{\mathbb{Z}}_+$, where

$$W(e_x) \triangleq \frac{\|P^{1/2} e_x\|}{1 + \|P^{1/2} e_x\|^2} \left[(\mu_1 - c\mu_2) \|P^{1/2} e_x\| - \beta - \alpha a^{-1} \right], \tag{A.24}$$

or, equivalently, $\Delta V_{\mathrm{e}}(e_x(k), e_z(k), \widetilde{W}(k)) \leq -W(e_x(k))$ for all $(e_x(k), e_z(k), \widetilde{W}(k)) \in \widetilde{\mathfrak{D}}_{\mathrm{e}} \setminus \widetilde{\mathfrak{D}}_{\mathrm{er}}$, $k \in \overline{\mathbb{Z}}_+$, where (see Figure 3)

$$\widetilde{\mathfrak{D}}_{\mathrm{e}} \triangleq \left\{ (e_x, e_z, \widetilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s} : x \in \mathfrak{D}_x \right\}, \tag{A.25}$$

$$\widetilde{\mathfrak{D}}_{\mathrm{er}} \triangleq \left\{ (e_x, e_z, \widetilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s} : \|P^{1/2} e_x\| \leq \alpha_x \right\}. \tag{A.26}$$

Next, we show that $\|x_1(k)\| < \varepsilon_{\mathrm{e}}$, $k \in \overline{\mathbb{Z}}_+$. Since $\|x_2(k)\|^2 \leq \varepsilon_{\mathrm{w}}$ for all $k \in \overline{\mathbb{Z}}_+$, it follows that, for $x_1(k) \in \overline{\mathcal{B}}_{\alpha_x}(0)$, $k \in \overline{\mathbb{Z}}_+$,

$$\begin{aligned}
V_{\mathrm{e}}\big(x_1(k), x_2(k), x_3(k)\big) &+ \Delta V_{\mathrm{e}}\big(x_1(k), x_2(k), x_3(k)\big) \\
&\leq \alpha_x^2 + \varepsilon_{\mathrm{w}} + \Delta V_{\mathrm{e}}\big(x_1(k), x_2(k), x_3(k)\big) \\
&\leq \alpha_x^2 + \varepsilon_{\mathrm{w}} + \frac{1}{2}\left(\frac{\alpha}{a} + \beta\right) + \left(1 + \frac{1}{c}\right)\|P^{1/2} \widetilde{e}(k)\|^2 \\
&\leq \alpha_x^2 + \varepsilon_{\mathrm{w}} + \frac{1}{2}\left(\frac{\alpha}{a} + \beta\right) + \left(1 + \frac{1}{c}\right)(2\alpha + 2\xi\varepsilon_{\mathrm{w}}) \\
&\triangleq \overline{\eta}.
\end{aligned} \tag{A.27}$$

Now, let $\delta \in (0, \alpha_x]$ and assume $\|x_{10}\| \leq \delta$. If $\|x_1(k)\| \leq \alpha_x$, $k \in \overline{\mathbb{Z}}_+$, then it follows that $\|x_1(k)\| \leq \alpha_x \leq \alpha^{-1}\big(\beta\big(\sqrt{\alpha_x^2 + \varepsilon_{\mathrm{w}}}\big)\big) \leq \alpha^{-1}(\overline{\eta})$, $k \in \overline{\mathbb{Z}}_+$. Alternatively, if there exists $K^* > 0$ such that $\|x_1(K^*)\| > \alpha_x$, then, since $\|x_{10}\| \leq \alpha_x$, it follows that there exists $\kappa \leq K^*$, such that $\|x_1(\kappa-1)\| \leq \alpha_x$ and $\|x_1(k)\| > \alpha_x$, where $k \in \{\kappa, \ldots, K^*\}$. Hence, it follows that

$$\begin{aligned}
\alpha\big(\|x_1(K^*)\|\big) &\leq \alpha\big(\|[x_1^{\mathrm{T}}(K^*), x_2^{\mathrm{T}}(K^*)]^{\mathrm{T}}\|\big) \\
&\leq V_{\mathrm{e}}\big(x_1(K^*), x_2(K^*), x_3(K^*)\big) \\
&\leq V_{\mathrm{e}}\big(x_1(\kappa), x_2(\kappa), x_3(\kappa)\big) \\
&= \Delta V_{\mathrm{e}}\big(x_1(\kappa-1), x_2(\kappa-1), x_3(\kappa-1)\big) + V_{\mathrm{e}}\big(x_1(\kappa-1), x_2(\kappa-1), x_3(\kappa-1)\big) \\
&\leq \overline{\eta},
\end{aligned} \tag{A.28}$$

which implies that $\|x_1(K^*)\| \leq \alpha^{-1}(\overline{\eta})$. Next, let $\delta \in (\alpha_x, \gamma)$, where $\gamma \triangleq \sup\{r > 0 : \mathcal{B}_{\alpha^{-1}(\beta(r))}(0) \subset \widetilde{\mathfrak{D}}_e\}$ and assume $x_{10} \in \overline{\mathcal{B}}_\delta(0)$ and $\|x_1(K)\| > \alpha_x$. Now, for every $\widehat{k} > 0$ such that $\|x_1(k)\| \geq \alpha_x$, $k \in \{0, \ldots, \widehat{k}\}$, it follows that

$$
\begin{aligned}
\alpha\big(\|x_1(k)\|\big) &\leq \alpha\big(\big\|\,[x_1^{\mathrm{T}}(k), x_2^{\mathrm{T}}(k)]^{\mathrm{T}}\,\big\|\big) \\
&\leq V_e\big(x_1(k), x_2(k), x_3(k)\big) \\
&\leq V_e\big(x_{10}, x_{20}, x_{30}\big) \\
&\leq \beta\big(\big\|\,[x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}\,\big\|\big) \\
&\leq \beta\big(\sqrt{\delta^2 + \varepsilon_{\mathrm{w}}}\big),
\end{aligned} \tag{A.29}
$$

which implies that $\|x_1(k)\| \leq \alpha^{-1}(\beta(\sqrt{\delta^2 + \varepsilon_{\mathrm{w}}}))$, $k \in \{0, \ldots, \widehat{k}\}$. Now, if there exists $K^* > 0$ such that $\|x_1(K^*)\| \leq \alpha_x$, then it follows as in the earlier case shown above that $\|x_1(k)\| \leq \alpha^{-1}(\overline{\eta})$, $k \geq K^*$. Hence, if $x_{10} \in \mathcal{B}_\delta(0)$, then

$$
\|x_1(k)\| \leq \alpha^{-1}\Big(\max\Big\{\overline{\eta}, \beta\big(\sqrt{\delta^2 + \varepsilon_{\mathrm{w}}}\big)\Big\}\Big) \triangleq \varepsilon_e, \quad k \in \overline{\mathbb{Z}}_+. \tag{A.30}
$$

Finally, repeating the above arguments with $\|x_2(k)\|^2 \leq \varepsilon_{\mathrm{w}}$, $k \in \overline{\mathbb{Z}}_+$, replaced by $\|x_2(k)\|^2 \leq \eta_{\mathrm{w}}$, $k \geq K > 0$, it can be shown that $\|x_1(k)\| < \varepsilon$, $k \geq K$, where $\varepsilon = \sqrt{e^\eta - 1}$.

Next, define

$$
\widetilde{\mathfrak{D}}_\alpha \triangleq \big\{(e_x, e_z, \widetilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s} : V_e(e_x, e_z, \widetilde{W}) \leq \alpha\big\}, \tag{A.31}
$$

where α is the maximum value such that $\widetilde{\mathfrak{D}}_\alpha \subseteq \widetilde{\mathfrak{D}}_e$, and define

$$
\widetilde{\mathfrak{D}}_{\overline{\eta}} \triangleq \big\{(e_x, e_z, \widetilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s} : V_e(e_x, e_z, \widetilde{W}) \leq \varepsilon_e^2\big\}, \tag{A.32}
$$

where ε_e is given by (A.30). Assume that $\widetilde{\mathfrak{D}}_{\overline{\eta}} \subset \widetilde{\mathfrak{D}}_\alpha$ (see Figure 3) (this assumption is standard in the neural network literature and ensures that in the error space $\widetilde{\mathfrak{D}}_e$ there exists at least one Lyapunov level set $\widetilde{\mathfrak{D}}_{\overline{\eta}} \subset \widetilde{\mathfrak{D}}_\alpha$. In the case where the neural network approximation holds in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, this assumption is automatically satisfied. See Remark A.1 for further details). Now, for all $(e_x, e_z, \widetilde{W}) \in \widetilde{\mathfrak{D}}_{\overline{\eta}} \cap (\widetilde{\mathfrak{D}}_e \setminus \widetilde{\mathfrak{D}}_{er})$, $\Delta V_e(e_x, e_z, \widetilde{W}) \leq 0$. Alternatively, for all $(e_x, e_z, \widetilde{W}) \in \widetilde{\mathfrak{D}}_{\overline{\eta}} \cap \widetilde{\mathfrak{D}}_{er}$, $V_e(e_x, e_z, \widetilde{W}) + \Delta V_e(e_x, e_z, \widetilde{W}) \leq \overline{\eta} \leq \varepsilon_e^2$. Hence, it follows that $\widetilde{\mathfrak{D}}_{\overline{\eta}}$ is positively invariant. In addition, since (A.3) is input-to-state stable with e_x viewed as the input, it follows from Proposition 3.4 that the solution $e_z(k)$, $k \in \overline{\mathbb{Z}}_+$, to (A.3) is ultimately bounded. Furthermore, it follows from [21, Theorem 1] that there exist a continuous, radially unbounded, positive-definite function $V_z : \mathbb{R}^{n_z} \to \mathbb{R}$, a class-$\mathcal{K}_\infty$ function $\gamma_1(\cdot)$, and a class-$\mathcal{K}$ function $\gamma_2(\cdot)$ such that

$$
\Delta V_z(e_z) \leq -\gamma_1\big(\|e_z\|\big) + \gamma_2\big(\|e_x\|\big). \tag{A.33}
$$

Since the upper bound for $\|e_x\|^2$ is given by $(e^\eta - 1)/\lambda_{\min}(P)$, it follows that the set is given by

$$
\mathfrak{D}_z \triangleq \Big\{ z \in \mathfrak{D}_{cz} : V_z(z - z_e) \leq \max_{\|z - z_e\| = \gamma_1^{-1}(\gamma_2(\sqrt{e^\eta - 1}/\lambda_{\min}(P^{1/2})))} V_z(z - z_e) \Big\}, \tag{A.34}
$$

is also positively invariant as long as $\mathfrak{D}_z \subset \mathfrak{D}_{cz}$ (see Remark A.1). Now, since $\widetilde{\mathfrak{D}}_{\overline{\eta}}$ and $\mathfrak{D}_z$ are positively invariant, it follows that

$$\mathfrak{D}_\alpha \triangleq \{(x, z, \widehat{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x \times s} : (x - x_e, z - z_e, \widehat{W} - W) \in \widetilde{\mathfrak{D}}_{\overline{\eta}}, z \in \mathfrak{D}_z\} \tag{A.35}$$

is also positively invariant. In addition, since (4.1), (4.2), (4.15), and (4.17) are ultimately bounded with respect to $(x, \widehat{W})$; and since (4.2) is input-to-state stable at $z(k) \equiv z_e$ with $x(k) - x_e$ viewed as the input then it follows from Proposition 3.4 that the solution $(x(k), z(k), \widehat{W}(k))$, $k \in \overline{\mathbb{Z}}_+$, of the closed-loop system (4.1), (4.2), (4.15), and (4.17) is ultimately bounded for all $(x(0), z(0), \widehat{W}(0)) \in \mathfrak{D}_\alpha$.

Finally, to show that $x(k) \geq\geq 0$ and $z(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ note that the closed-loop system (4.1), (4.15), and (4.17), is given by

$$\begin{aligned}
x(k+1) &= f_x(x(k), z(k)) + B_u K(x(k) - x_e) - B_u \widehat{W}^T(k) \widehat{\sigma}(x(k), z(k), \widehat{W}(k)) \\
&= (A + B_u K) x(k) + \Delta f(x(k), z(k)) - B_u \widehat{W}^T(k) \widehat{\sigma}(x(k), z(k), \widehat{W}(k)) - B_u K x_e \quad \text{(A.36)} \\
&= \tilde{f}(k, x(k), z(k)) + v, \quad x(0) = x_0, k \in \overline{\mathbb{Z}}_+,
\end{aligned}$$

where

$$\tilde{f}(k, x, z) \triangleq (A + B_u K) x + \Delta f(x, z) - B_u \widehat{W}^T(k) \widehat{\sigma}(x, z, \widehat{W}), \quad v \triangleq -B_u K x_e. \tag{A.37}$$

Note that $A + B_u K$ and $\Delta f(\cdot, \cdot)$ are nonnegative and, since $\widehat{\sigma}_{i(j)}(x, z, \widehat{W}_i) = 0$ whenever $\widehat{W}_{i(j)} > 0$, $i = 1, \ldots, n_x$, $j = 1, \ldots, s_i$, $-\widehat{W}^T \widehat{\sigma}(x, z, \widehat{W}) \geq\geq 0$. Hence, since $\tilde{f}(k, x, z)$ is nonnegative with respect to x pointwise-in-time, $f_z(x, z)$ is nonnegative with respect to z, and $v \geq\geq 0$, it follows from Proposition 2.9 that $x(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, and $z(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

Remark A.1. In the case where the neural network approximation holds in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, the assumptions $\widetilde{\mathfrak{D}}_{\overline{\eta}} \subset \widetilde{\mathfrak{D}}_\alpha$ and $\mathfrak{D}_z \subset \mathfrak{D}_{cz}$ invoked in the proof of Theorem 4.1 are automatically satisfied. Furthermore, in this case the control law (4.15) ensures global ultimate boundedness of the error signals. However, the existence of a global neural network approximator for an uncertain nonlinear map cannot in general be established. Hence, as is common in the neural network literature, for a given arbitrarily large compact set $\mathfrak{D}_{cx} \times \mathfrak{D}_{cz} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, we assume that there exists an approximator for the unknown nonlinear map up to a desired accuracy. Furthermore, we assume that in the error space $\widetilde{\mathfrak{D}}_e$ there exists at least one Lyapunov level set such that $\widetilde{\mathfrak{D}}_{\overline{\eta}} \subset \widetilde{\mathfrak{D}}_\alpha$. In the case where $\delta(\cdot, \cdot)$ is continuous on $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, it follows from the Stone-Weierstrass theorem that $\delta(\cdot, \cdot)$ can be approximated over an arbitrarily large compact set $\mathfrak{D}_{cx} \times \mathfrak{D}_{cz}$. In this case, our neuroadaptive controller guarantees semiglobal ultimate boundedness. An identical assumption is made in the proof of Theorem 5.1.

B. Proof of Theorem 5.1

In this appendix, we prove Theorem 5.1. First, define $\widehat{W}_u(k) \triangleq \text{block-diag}[\widehat{W}_{u1}(k), \ldots, \widehat{W}_{u2}(k)]$, where

$$\widehat{W}_{ui}(k) = \begin{cases} 0, & \text{if } \widehat{u}_i(k) < 0, \\ \widehat{W}_i(k), & \text{otherwise,} \end{cases} \quad i = 1, \ldots, n_x. \tag{B.1}$$

Next, note that with $u(k)$, $k \in \overline{\mathbb{Z}}_+$, given by (5.2), it follows from (4.1), (4.4), and (4.14) that

$$x(k+1) = Ax(k) + \Delta f(x(k), z(k)) - B_u \widehat{W}_u^{\mathrm{T}}(k) \sigma(x(k), z(k)), \quad x(0) = x_0, \ k \in \overline{\mathbb{Z}}_+. \qquad \text{(B.2)}$$

Now, defining $e_x(k) \triangleq x(k) - x_e$ and $e_z(k) \triangleq z(k) - z_e$ and using (4.6), (4.7), and (4.9), it follows from (4.2) and (B.2) that

$$
\begin{aligned}
e_x(k+1) &= Ae_x(k) + (A - I)x_e + \Delta f(x(k), z(k)) - B_u \widehat{W}_u^{\mathrm{T}}(k) \sigma(x(k), z(k)) \\
&= Ae_x(k) + B_u \big[\delta(x(k), z(k)) - \delta(x_e, z_e) - G_n(x_e, z_e) u_e - \widehat{W}^{\mathrm{T}}(k) \sigma(x(k), z(k)) \big] \\
&\quad + B_u \big(\widehat{W}(k) - \widehat{W}_u(k) \big)^{\mathrm{T}} \sigma(x(k), z(k)) \\
&= Ae_x(k) - B_u \widetilde{W}^{\mathrm{T}}(k) \sigma(x(k), z(k)) + B_u \varepsilon(x(k), z(k)) \\
&\quad + B_u \big(\widehat{W}(k) - \widehat{W}_u(k) \big)^{\mathrm{T}} \sigma(x(k), z(k)), \quad e_x(0) = x_0 - x_e, \ k \in \overline{\mathbb{Z}}_+,
\end{aligned}
$$

$$\text{(B.3)}$$

$$e_z(k+1) = \widetilde{f}_z(e_x(k), e_z(k)), \quad e_z(0) = z_0 - z_e, \qquad \text{(B.4)}$$

where $\widetilde{f}_z(e_x, e_z) \triangleq f_z(e_x + x_e, e_z + z_e)$, and $\varepsilon(x, z) \triangleq [\varepsilon_1(x, z), \ldots, \varepsilon_{n_x}(x, z)]^{\mathrm{T}}$. Furthermore, since A is nonnegative and asymptotically stable, it follows from Theorem 2.3 that there exist a positive *diagonal* matrix $P = \mathrm{diag}[p_1, \ldots, p_{n_x}]$ and a positive-definite matrix $R \in \mathbb{R}^{n_x \times n_x}$ such that (5.5) holds.

Next, to show ultimate boundedness of the closed-loop system (5.4), (B.3), and (B.4), consider the Lyapunov-like function

$$V_{\mathrm{w}}(e_x, e_z, \widetilde{W}) = \mathrm{tr}\, \widetilde{W} Q^{-1} \widetilde{W}^{\mathrm{T}}, \qquad \text{(B.5)}$$

where $Q \triangleq \mathrm{diag}[\widehat{q}_1, \ldots, \widehat{q}_{n_x}] = \mathrm{diag}[q_1/p_1 b_1, \ldots, q_{n_x}/p_{n_x} b_{n_x}]$ and $\widetilde{W}(k) \triangleq \widehat{W}(k) - W$ with $W \triangleq$ block-diag$[W_1, \ldots, W_{n_x}]$. Note that (B.5) satisfies (3.3) with $x_1 = [\widehat{q}_1^{-1/2} \widetilde{W}_1^{\mathrm{T}}, \ldots, \widehat{q}_{n_x}^{-1/2} \widetilde{W}_{n_x}^{\mathrm{T}}]^{\mathrm{T}}$, $x_2 = [e_x^{\mathrm{T}}, e_z^{\mathrm{T}}]^{\mathrm{T}}$, $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^2$, where $\|x_1\|^2 = \mathrm{tr}\, \widetilde{W} Q^{-1} \widetilde{W}^{\mathrm{T}}$. Furthermore, $\alpha(\|x_1\|)$ is a class-$\mathcal{K}_\infty$ function. Now, using (5.4) and (B.3), it follows that the difference of $V_{\mathrm{w}}(e_x, e_z, \widetilde{W})$ along the closed-loop system trajectories is given by

$$
\begin{aligned}
\Delta V_{\mathrm{w}}&\big(e_x(k), e_z(k), \widetilde{W}(k)\big) \\
&\triangleq \mathrm{tr}\, \widetilde{W}(k+1) Q^{-1} \widetilde{W}^{\mathrm{T}}(k+1) - \mathrm{tr}\, \widetilde{W}(k) Q^{-1} \widetilde{W}^{\mathrm{T}}(k) \\
&= \sum_{i=1}^{n_x} \frac{2 p_i b_i \big\| P^{1/2} e_x(k) \big\|}{1 + \big\| P^{1/2} e_x(k) \big\|^2} \big(\gamma \widetilde{e}_i(k) \sigma_i(x(k), z(k)) - \widehat{W}_i(k) \big)^{\mathrm{T}} \widetilde{W}_i(k) \\
&\quad + \sum_{i=1}^{n_x} \frac{p_i b_i q_i \big\| P^{1/2} e_x(k) \big\|^2}{\big(1 + \big\| P^{1/2} e_x(k) \big\|^2 \big)^2} \big\| \gamma \widetilde{e}_i(k) \sigma_i(x(k), z(k)) - \widehat{W}_i(k) \big\|^2
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i=1}^{n_x} \frac{2p_i b_i \gamma \left\| P^{1/2} e_x(k) \right\| \varepsilon_i(x(k), z(k)) \tilde{e}_i(k)}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
&\quad - \sum_{i=1}^{n_x} \frac{2p_i \gamma \left\| P^{1/2} e_x(k) \right\| \tilde{e}_i^2(k)}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
&\quad + \sum_{i=1}^{n_x} \frac{2p_i b_i \gamma \left\| P^{1/2} e_x(k) \right\| \tilde{e}_i(k)}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \sigma_i^{\mathrm{T}}(x(k), z(k)) \left[\widehat{W}_i(k) - \widehat{W}_{ui}(k) \right] \\
&\quad - \sum_{i=1}^{n_x} \frac{2p_i b_i \left\| P^{1/2} e_x(k) \right\| \widehat{W}_i^{\mathrm{T}}(k) \widetilde{W}_i(k)}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
&\quad + \sum_{i=1}^{n_x} \frac{p_i b_i q_i \left\| P^{1/2} e_x(k) \right\|^2}{\left(1 + \left\| P^{1/2} e_x(k) \right\|^2 \right)^2} \left\| \gamma \tilde{e}_i(k) \sigma_i(x(k), z(k)) - \widehat{W}_i(k) \right\|^2.
\end{aligned}$$

$$(B.6)$$

Now, for each $i \in \{1, \ldots, n_x\}$ and for the two cases given in (B.1), the right-hand side of (B.6) gives the following:

(1) if $\hat{u}_i(k) < 0$, then $\widehat{W}_{ui}(k) = 0$. Now, using (A.8), (A.9), and the inequalities

$$\sum_{i=1}^{n_x} 2p_i b_i \gamma \varepsilon_i(x, z) \tilde{e}_i \le a^{-1} \alpha \gamma + a\gamma \left\| P^{1/2} \tilde{e} \right\|^2, \qquad (B.7)$$

$$\sum_{i=1}^{n_x} 2p_i b_i \gamma \tilde{e}_i \sigma_i^{\mathrm{T}}(x, z) \widehat{W}_i \le \sum_{i=1}^{n_x} p_i b_i^2 s_i \gamma \left\| \widehat{W}_i \right\|^2 + \gamma \left\| P^{1/2} \tilde{e} \right\|^2, \qquad (B.8)$$

$$\sum_{i=1}^{n_x} p_i b_i q_i \left\| \gamma \tilde{e}_i \sigma_i(x, z) - \widehat{W}_i \right\|^2 \le \sum_{i=1}^{n_x} 2p_i b_i q_i s_i \gamma^2 \tilde{e}_i^2 + \sum_{i=1}^{n_x} 2p_i b_i q_i \left\| \widehat{W}_i \right\|^2, \qquad (B.9)$$

$$\sum_{i=1}^{n_x} p_i b_i \left\| W_i \right\|^2 \le \beta, \qquad (B.10)$$

it follows that

$$\Delta V_{\mathrm{w}}\big(e_x(k), e_z(k), \widetilde{W}(k) \big)$$

$$\begin{aligned}
&\le \frac{(\alpha/a)\gamma \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{a\gamma \left\| P^{1/2} \tilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
&\quad - \frac{\gamma \left\| P^{1/2} \tilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{\beta \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \sum_{i=1}^{n_x} \frac{p_i b_i^2 \gamma s_i \left\| \widehat{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
&\quad + \frac{\gamma^2 \xi \left\| P^{1/2} \tilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} - \sum_{i=1}^{n_x} \frac{p_i b_i \left\| \widetilde{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
&\quad - \sum_{i=1}^{n_x} \frac{p_i b_i \left\| \widehat{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \sum_{i=1}^{n_x} \frac{2p_i b_i q_i \left\| P^{1/2} e_x(k) \right\|^2 \left\| \widehat{W}_i(k) \right\|^2}{\left(1 + \left\| P^{1/2} e_x(k) \right\|^2 \right)^2};
\end{aligned}$$

$$(B.11)$$

(2) otherwise, $\widehat{W}_{ui}(k) = \widehat{W}_i(k)$, and hence, using (A.8), (A.9), (B.7), (B.9), and (B.10), it follows that

$$
\begin{aligned}
\Delta V_w & \left(e_x(k), e_z(k), \widetilde{W}(k) \right) \\
\leq & \frac{(\alpha/a)\gamma \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{a\gamma \left\| P^{1/2} \widetilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
& - \frac{2\gamma \left\| P^{1/2} \widetilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \frac{\beta \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
& + \frac{\gamma^2 \xi \left\| P^{1/2} \widetilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} - \sum_{i=1}^{n_x} \frac{p_i b_i \left\| \widetilde{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \\
& - \sum_{i=1}^{n_x} \frac{p_i b_i \left\| \widehat{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} + \sum_{i=1}^{n_x} \frac{2 p_i b_i q_i \left\| P^{1/2} e_x(k) \right\|^2 \left\| \widehat{W}_i(k) \right\|^2}{\left(1 + \left\| P^{1/2} e_x(k) \right\|^2 \right)^2}.
\end{aligned}
\tag{B.12}
$$

Hence, it follows from (B.6) that in either case

$$
\begin{aligned}
\Delta V_w \left(e_x(k), e_z(k), \widetilde{W}(k) \right) \leq & -\frac{\gamma \left\| P^{1/2} \widetilde{e}(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} (1 - a - \gamma\xi) \\
& - \frac{\left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(\sum_{i=1}^{n_x} p_i b_i \left\| \widetilde{W}_i(k) \right\|^2 - \frac{\alpha\gamma}{a} - \beta \right) \\
& - \sum_{i=1}^{n_x} \frac{p_i b_i \left\| \widehat{W}_i(k) \right\|^2 \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(1 - b_i s_i \gamma - \frac{2 q_i \left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \right).
\end{aligned}
\tag{B.13}
$$

Furthermore, note that since, by assumption, $1 - a - \gamma\xi > 0$ and $b_i q_i \gamma < 1$, $q_i \leq 1 - b_i s_i \gamma$, $i = 1, \ldots, n_x$, it follows that

$$
1 - b_i s_i \gamma - \frac{2 q_i \left\| P^{1/2} e_x \right\|}{1 + \left\| P^{1/2} e_x \right\|^2} \geq 0, \quad i = 1, \ldots, n_x.
\tag{B.14}
$$

Hence,

$$
\begin{aligned}
\Delta V_w \left(e_x(k), e_z(k), \widetilde{W}(k) \right) \leq & -\frac{\left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(\sum_{i=1}^{n_x} p_i b_i \left\| \widetilde{W}_i(k) \right\|^2 - \frac{\alpha\gamma}{a} - \beta \right) \\
\leq & -\frac{\left\| P^{1/2} e_x(k) \right\|}{1 + \left\| P^{1/2} e_x(k) \right\|^2} \left(\xi \sum_{i=1}^{n_x} \left\| \widehat{q}_i^{-1/2} \widetilde{W}_i(k) \right\|^2 - \frac{\alpha\gamma}{a} - \beta \right).
\end{aligned}
\tag{B.15}
$$

Now, it follows using similar arguments as in the proof of Theorem 4.1 that the closed-loop system (5.4), (B.3), and (B.4) is globally bounded with respect to $\widetilde{W}$ uniformly in $(e_x(0), e_z(0))$. If there does not exist $k^* \in \overline{\mathbb{Z}}_+$ such that $e_x(k) = 0$ for all $k \geq k^*$, it follows

using similar arguments as in the proof of Theorem 4.1 that the closed-loop system (5.4), (B.3), and (B.4) is globally ultimately bounded with respect to $\widetilde{W}$ uniformly in $(e_x(0), e_z(0))$ with ultimate bound given by $\sqrt{\eta_\mathrm{w}}$, where $\eta_\mathrm{w} > (2 + \zeta)(\alpha\gamma + a\beta)/(2a\zeta)$. Alternatively, if there exists $k^* \in \overline{\mathbb{Z}}_+$ such that $e_x(k) = 0$ for all $k \geq k^*$, then $\widetilde{W}(k) = \widetilde{W}(k^*)$ for all $k \geq k^*$.

Next, to show ultimate boundedness of the error dynamics, consider the Lyapunov-like function

$$V_\mathrm{e}(e_x, e_z, \widetilde{W}) = \ln\left(1 + e_x^\mathrm{T} P e_x\right) + \mathrm{tr}\,\widetilde{W} Q^{-1} \widetilde{W}^\mathrm{T}. \tag{B.16}$$

Note that (B.16) satisfies (A.19) with $x_1 = P^{1/2} e_x$, $x_2 = [\hat{q}_1^{-1/2}\widetilde{W}_1^\mathrm{T}, \ldots, \hat{q}_{n_x}^{-1/2}\widetilde{W}_{n_x}^\mathrm{T}]^\mathrm{T}$, $x_3 = e_z$, $\alpha(\|[x_1^\mathrm{T}, x_2^\mathrm{T}]^\mathrm{T}\|) = \ln(1 + \|[x_1^\mathrm{T}, x_2^\mathrm{T}]^\mathrm{T}\|^2)$, and $\beta(\|[x_1^\mathrm{T}, x_2^\mathrm{T}]^\mathrm{T}\|) = \|[x_1^\mathrm{T}, x_2^\mathrm{T}]^\mathrm{T}\|^2$, where $\|[x_1^\mathrm{T}, x_2^\mathrm{T}]^\mathrm{T}\|^2 = e_x^\mathrm{T} P e_x + \mathrm{tr}\,\widetilde{W} Q^{-1} \widetilde{W}^\mathrm{T}$. Furthermore, $\alpha(\cdot)$ is a class-$\mathcal{K}_\infty$ function. Now, using (5.5), (B.13), and the definition of $\tilde{e}$, it follows that the forward difference of $V_\mathrm{e}(e_x, e_z, \widetilde{W})$ along the closed-loop system trajectories is given by

$$\Delta V_\mathrm{e}(e_x(k), e_z(k), \widetilde{W}(k)) \triangleq V_\mathrm{e}(e_x(k+1), e_z(k+1), \widetilde{W}(k+1)) - V_\mathrm{e}(e_x(k), e_z(k), \widetilde{W}(k))$$

$$= \ln\left[\frac{1 + e_x^\mathrm{T}(k+1) P e_x(k+1)}{1 + e_x^\mathrm{T}(k) P e_x(k)}\right]$$

$$+ \mathrm{tr}\,\widetilde{W}(k+1) Q^{-1} \widetilde{W}^\mathrm{T}(k+1) - \mathrm{tr}\,\widetilde{W}(k) Q^{-1} \widetilde{W}^\mathrm{T}(k)$$

$$\leq \frac{e_x^\mathrm{T}(k+1) P e_x(k+1) - e_x^\mathrm{T}(k) P e_x(k)}{1 + e_x^\mathrm{T}(k) P e_x(k)} + \Delta V_\mathrm{w}(e_x(k), e_z(k), \widetilde{W}(k))$$

$$= -\frac{e_x^\mathrm{T}(k) R e_x(k)}{1 + e_x^\mathrm{T}(k) P e_x(k)} + \frac{2 e_x^\mathrm{T}(k) A^\mathrm{T} P \tilde{e}(k)}{1 + e_x^\mathrm{T}(k) P e_x(k)} + \frac{\tilde{e}_x^\mathrm{T}(k) P \tilde{e}_x(k)}{1 + e_x^\mathrm{T}(k) P e_x(k)}$$

$$+ \Delta V_\mathrm{w}(e_x(k), e_z(k), \widetilde{W}(k))$$

$$\leq -\frac{\|R^{1/2} e_x(k)\|^2}{1 + \|P^{1/2} e_x(k)\|^2} + \frac{2 e_x^\mathrm{T}(k) A^\mathrm{T} P \tilde{e}(k)}{1 + \|P^{1/2} e_x(k)\|^2} + \frac{\|P^{1/2} \tilde{e}_x(k)\|^2}{1 + \|P^{1/2} e_x(k)\|^2}$$

$$- \frac{\gamma \|P^{1/2} \tilde{e}(k)\|^2 \|P^{1/2} e_x(k)\|}{1 + \|P^{1/2} e_x(k)\|^2}(1 - a - \gamma\xi)$$

$$- \frac{\|P^{1/2} e_x(k)\|}{1 + \|P^{1/2} e_x(k)\|^2}\left(\sum_{i=1}^{n_x} p_i b_i \|\widetilde{W}_i(k)\|^2 - \frac{\alpha\gamma}{a} - \beta\right)$$

$$- \sum_{i=1}^{n_x} \frac{p_i b_i \|\widehat{W}_i(k)\|^2 \|P^{1/2} e_x(k)\|}{1 + \|P^{1/2} e_x(k)\|^2}\left(1 - b_i s_i \gamma - \frac{2 q_i \|P^{1/2} e_x(k)\|}{1 + \|P^{1/2} e_x(k)\|^2}\right), \tag{B.17}$$

where once again in (B.17) we used $\ln a - \ln b = \ln(a/b)$ and $\ln(1+c) \leq c$ for $a, b > 0$ and $c > -1$.

Next, using (A.21) and (B.17) yields

$$
\begin{aligned}
\Delta V_{\mathrm{e}}\big(e_x(k), e_z(k), \widetilde{W}(k)\big) &\leq -\frac{\mu_1 \big\|P^{1/2}e_x(k)\big\|^2}{1 + \big\|P^{1/2}e_x(k)\big\|^2} + \frac{c\mu_2 \big\|P^{1/2}e_x(k)\big\|^2}{1 + \big\|P^{1/2}e_x(k)\big\|^2} + \frac{c^{-1}\big\|P^{1/2}\widetilde{e}(k)\big\|^2}{1 + \big\|P^{1/2}e_x(k)\big\|^2} \\
&\quad + \frac{\gamma(\alpha/a)\big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2} + \frac{a\gamma\big\|P^{1/2}\widetilde{e}(k)\big\|^2 \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2} \\
&\quad + \frac{\beta\big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2} - \frac{\big(\gamma\big\|P^{1/2}e_x(k)\big\| - 1\big)\big\|P^{1/2}\widetilde{e}(k)\big\|^2}{1 + \big\|P^{1/2}e_x(k)\big\|^2} \\
&\quad + \frac{\gamma^2 \xi \big\|P^{1/2}\widetilde{e}(k)\big\|^2 \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2} - \sum_{i=1}^{n_x} \frac{p_i b_i \big\|\widetilde{W}_i(k)\big\|^2 \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2} \\
&\quad - \sum_{i=1}^{n_x} \frac{p_i b_i \big\|\widehat{W}_i(k)\big\|^2 \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\left(1 - b_i s_i \gamma - \frac{2 q_i \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\right) \\
&\leq -\frac{\big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\Big[(\mu_1 - c\mu_2)\big\|P^{1/2}e_x(k)\big\| - \beta - \alpha\gamma a^{-1}\Big] \\
&\quad - \frac{\big\|P^{1/2}\widetilde{e}(k)\big\|^2}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\Big[\gamma(1 - a - \gamma\xi)\big\|P^{1/2}e_x(k)\big\| - 1 - c^{-1}\Big] \\
&\quad - \sum_{i=1}^{n_x} \frac{p_i b_i \big\|\widetilde{W}_i(k)\big\|^2 \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2} \\
&\quad - \sum_{i=1}^{n_x} \frac{p_i b_i \big\|\widehat{W}_i(k)\big\|^2 \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\left(1 - b_i s_i \gamma - \frac{2 q_i \big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\right) \\
&\leq -\frac{\big\|P^{1/2}e_x(k)\big\|}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\Big[(\mu_1 - c\mu_2)\big\|P^{1/2}e_x(k)\big\| - \beta - \alpha\gamma a^{-1}\Big] \\
&\quad - \frac{\big\|P^{1/2}\widetilde{e}(k)\big\|^2}{1 + \big\|P^{1/2}e_x(k)\big\|^2}\Big[\gamma(1 - a - \gamma\xi)\big\|P^{1/2}e_x(k)\big\| - 1 - c^{-1}\Big].
\end{aligned}
$$

(B.18)

Now, using similar arguments as in the proof of Theorem 4.1 it follows that the solution $(x(k), z(k), \widehat{W}(k))$, $k \in \overline{\mathbb{Z}}_+$, of the closed-loop system (5.4), (B.3), and (B.4) is ultimately bounded for all $(x(0), z(0), \widehat{W}(0)) \in \mathfrak{D}_\alpha$ given by (A.35) and $\|P^{1/2}e_x(k)\| < \varepsilon$ for $k \geq K$.

Finally, $u(k) \geq\geq 0$, $k \geq 0$, is a restatement of (5.2). Now, since $G(x(k)) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, and $u(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, it follows from Proposition 2.8 that $x(k) \geq\geq 0$ and $z(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

Acknowledgments

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