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Research Article

Reducibility and Stability Results for Linear System of Difference Equations

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We first give a theorem on the reducibility of linear system of difference equations of the form $x(n+1) = A(n)x(n)$. Next, by the means of Floquet theory, we obtain some stability results. Moreover, some examples are given to illustrate the importance of the results.

1. Introduction

Consider the homogeneous linear system of difference equations

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{N} = \{0,1,2,\ldots\}, \tag{1.1}$$

where $A(n) = (a_{ij}(n))$ is a $k \times k$ nonsingular matrix with real entries and $x(n) = (x_1(n), x_2(n),\ldots,x_k(n))^T \in \mathbb{R}^k$.

If for some $n_0 \geq 0$,

$$x(n_0) = x_0 \tag{1.2}$$

is specified, then (1.1) is called an initial value problem (IVP). The solution of this IVP is given by

$$x(n,n_0,x_0) = \left(\prod_{i=n_0}^{n-1} A(i)\right)x_0 := \Phi(n)x_0, \tag{1.3}$$

where $\Phi(n)$ is the fundamental matrix defined by

$$\prod_{i=n_0}^{n-1} A(i) = \begin{cases} A(n-1)A(n-2)\cdots A(n_0), & \text{if } n > n_0, \\ I, & \text{if } n = n_0. \end{cases} \tag{1.4}$$

However, (1.1) is called reducible to equation

$$y(n+1) = B(n)y(n), \tag{1.5}$$

if there is a nonsingular matrix $H(n)$ with real entries such that

$$x(n) = H(n)y(n). \tag{1.6}$$

Let $S(n)$ be a $k \times k$ matrix function whose entries are real-valued functions defined for $n \geq n_0$. Consider the system

$$z(n+1) = S(n)z(n), \quad n \geq n_0. \tag{1.7}$$

Let $H(n)$ be a fundamental matrix of (1.7) satisfying $H(n_0) = I$. This $H(n)$ can be used to transform (1.1) into (1.5).

Stability properties of (1.1) can be deduced by considering the reduced form (1.5) under some additional conditions. In this study, we first give a theorem on the reducibility of (1.1) into the form of (1.5) and then obtain asymptotic stability of the zero solution of (1.1).

2. Reducible systems

In this section, we give a theorem on the structure of the matrix $S(n)$, and provide an example for illustration. The results in this section are discrete analogues of the ones given in [1].

Theorem 2.1. *The homogeneous linear difference system* (1.1) *is reducible to* (1.5) *under the transformation* (1.6) *if and only if there exists a $k \times k$ regular real matrix $S(n)$ such that*

$$\begin{gathered} A(n+1)S(n) = S(n+1)A(n) + S(n+1)S(n)H(n)(\Delta B(n))H^{-1}(n), \\ A(n_0) = S(n_0)B(n_0) \end{gathered} \tag{2.1}$$

hold.

Proof. Let $S(n)$ and $H(n)$ be defined as above. Under the transformation (1.6), (1.1) becomes

$$H(n+1)y(n+1) = A(n)H(n)y(n), \tag{2.2}$$

and after reorganizing, we get

$$y(n+1) = H^{-1}(n)S^{-1}(n)A(n)H(n)y(n). \tag{2.3}$$

Thus, (1.1) is reducible to (1.5) with

$$B(n) = H^{-1}(n)S^{-1}(n)A(n)H(n). \tag{2.4}$$

Clearly, $B(n)$ is the unique solution of the IVP:

$$\begin{gathered} \Delta B(n) = F(n), \\ B(n_0) = S^{-1}(n_0)A(n_0), \end{gathered} \tag{2.5}$$

where $F(n) := \Delta(H^{-1}(n)S^{-1}(n)A(n)H(n))$.

This problem is equivalent to solving (2.1). $\square$

Corollary 2.2. *The homogeneous linear system of difference equation* (1.1) *is reducible to*

$$y(n+1) = By(n), \tag{2.6}$$

with a constant matrix B under transformation (1.6) *if and only if there exists a $k \times k$ regular real matrix $S(n)$ defined for $n \geq n_0$, such that*

$$A(n+1)S(n) = S(n+1)A(n), \tag{2.7}$$

$$A(n_0) = S(n_0)B \tag{2.8}$$

hold.

Below, we give an example for Corollary 2.2 in the special case $k = 2$. To obtain the matrix $H(n)$, we choose a suitable form of the matrix $S(n)$.

Example 2.3. Consider the system

$$x(n+1) = \begin{bmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{bmatrix} x(n), \tag{2.9}$$

where

(i) $a_{ij}(n)$ are real-valued functions defined for $n \geq n_0$ such that $a_{ij}(n) \neq 0$ for all $i, j = 1, 2$,

(ii) $\det A \neq 0$ for all $n \geq n_0$,

(iii) $\Theta(n) := a_{12}(n+1)a_{22}(n) + a_{12}(n)a_{11}(n+1) \neq 0$.

We also assume that for all $n \geq n_0$,

$$\Theta(n-1)\left[\frac{a_{21}(n+1)}{a_{12}(n-1)} - \frac{a_{22}(n+1)a_{11}(n+1)}{a_{12}(n+1)a_{12}(n-1)}\right] + \Theta(n)\left[\frac{a_{11}(n)a_{22}(n+1)}{a_{12}(n)a_{12}(n+1)} + \frac{a_{11}(n)a_{11}(n+2)}{a_{12}(n)a_{12}(n+2)}\right]$$
$$-\Theta(n+1)\left[\frac{a_{11}(n)a_{11}(n+1)}{a_{12}(n+1)a_{12}(n+2)} + \frac{a_{21}(n)}{a_{12}(n+2)}\right] = 0. \tag{2.10}$$

It is easy to verify that if we take

$$S(n) = \begin{bmatrix} s_{11}(n) & 0 \\ s_{21}(n) & s_{22}(n) \end{bmatrix}, \tag{2.11}$$

where

$$s_{11}(n) = \frac{\Theta(n-1)}{a_{12}(n-1)}, \tag{2.12}$$

$$s_{22}(n) = \frac{\Theta(n)}{a_{12}(n+1)}, \tag{2.13}$$

$$s_{21}(n) = \frac{a_{11}(n)\Theta(n)}{a_{12}(n)a_{12}(n+1)} - \frac{a_{11}(n+1)\Theta(n-1)}{a_{12}(n-1)a_{12}(n+1)}, \tag{2.14}$$

then (2.7) holds. Moreover, from (2.8) we have

$$B = S^{-1}(n_0)A(n_0). \tag{2.15}$$

In case $s_{21}(n) = 0$ for every $n \geq n_0$, that is,

$$\frac{a_{11}(n)\Theta(n)}{a_{12}(n)a_{12}(n+1)} - \frac{a_{11}(n+1)\Theta(n-1)}{a_{12}(n-1)a_{12}(n+1)} = 0, \quad n \geq n_0, \tag{2.16}$$

the relations (2.10), (2.12), and (2.13) take the form

$$\begin{gathered} \frac{a_{12}(n+1)a_{22}(n)}{a_{11}(n+1)a_{12}(n)} = \frac{a_{22}(n+1)a_{21}(n)}{a_{11}(n)a_{21}(n+1)} = \alpha, \\ s_{11}(n) = \alpha_1 a_{11}(n), \\ s_{22}(n) = \alpha_2 a_{22}(n), \end{gathered} \tag{2.17}$$

where $\alpha \neq 0$ is a real constant and α_1, α_2 are arbitrary real constants such that $\alpha_1/\alpha_2 = \alpha$.

Corollary 2.4. *If there exists a $k \times k$ regular constant matrix S such that*

$$A(n+1)S = SA(n), \tag{2.18}$$

then (1.1) *reduces to* (2.6) *with $B = S^{-1}A(n_0)$.*

It should be noted that in case the constant matrices S and B commute, that is, $SB = BS$, then $A(n)$ must be a constant matrix as well.

3. Stability of linear systems

It turns out that to obtain a stability result, one needs take $S(n)$, a periodic matrix [2]. Indeed, this allows using the Floquet theory for linear periodic system (1.7).

We need the following three well-known theorems [3–5].

Theorem 3.1. *Let $\Phi(n)$ be the fundamental matrix of* (1.1) *with $\Phi(n_0) = I$.*

The zero solution of (1.1) *is*

(i) *stable if and only if there exists a positive constant M such that*

$$\|\Phi(n)\| \leq M \quad \text{for } n \geq n_0 \geq 0; \tag{3.1}$$

(ii) *asymptotically stable if and only if*

$$\lim_{n\to\infty} \|\Phi(n)\| = 0, \tag{3.2}$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^{k\times k}$.

Theorem 3.2. *Consider system* (1.1) *with* $A(n) = A$, *a constant regular matrix. Then its zero solution is*

(i) *stable if and only if* $\rho(A) \leq 1$ *and the eigenvalues of unit modulus are semisimple;*

(ii) *asymptotically stable if and only if* $\rho(A) < 1$, *where* $\rho(A) = \max\{|\lambda| : \lambda$ *is an eigenvalue of* $A\}$ *is the spectral radius of* A.

Consider the linear periodic system

$$z(n+1) = S(n)z(n), \tag{3.3}$$

where $n \in \mathbb{Z}$, $S(n+N) = S(n)$, for some positive integer N.

From the literature, we know that if $\Psi(n)$, with $\Psi(n_0) = I$ is a fundamental matrix of (3.3), then there exists a constant C matrix, whose eigenvalues are called the Floquet exponents, and periodic matrix $P(n)$ with period N such that $\Psi(n) = P(n)C^{n-n_0}$.

Theorem 3.3. *The zero solution of* (3.3) *is*

(i) *stable if and only if the Floquet exponents have modulus less than or equal to one; those with modulus of one are semisimple;*

(ii) *asymptotically stable if and only if all the Floquet exponents lie inside the unit disk.*

In view of Theorems 3.1, 3.2, and 3.3, we obtain from Corollary 2.2 the following new stability criteria for (1.1).

Theorem 3.4. *The zero solution of* (1.1) *is stable if and only if there exists a* $k \times k$ *regular periodic matrix* $S(n)$ *satisfying* (2.8) *such that*

(i) *the Floquet exponents of* $S(n)$ *have modulus less than or equal to one; those with modulus of one are semisimple;*

(ii) $\rho(S^{-1}(n_0)A(n_0)) \leq 1$; *those eigenvalues of* $S^{-1}(n_0)A(n_0)$ *of unit modulus are semisimple.*

Theorem 3.5. *The zero solution of* (1.1) *is asymptotically stable if and only if there exists a* $k \times k$ *regular periodic matrix* $S(n)$ *satisfying* (2.8) *such that either*

(i) *all the Floquet exponents of* $S(n)$ *lie inside the unit disk and* $\rho(S^{-1}(n_0)A(n_0)) \leq 1$; *those eigenvalues of* $S^{-1}(n_0)A(n_0)$ *of unit modulus are semisimple; or*

(ii) *the Floquet exponents of* $S(n)$ *have modulus less than or equal to one; those with modulus of one are semisimple; and* $\rho(S^{-1}(n_0)A(n_0)) < 1$.

Remark 3.6. Let $S(n)$ be periodic with period N. The Floquet exponents mentioned in Theorem 3.3 are the eigenvalues of C, where $C^N = S(N-1)S(N-2)\cdots S(0)$.

Example 3.7. Consider the system

$$x(n+1) = \begin{bmatrix} (-1)^n & \beta^{n+1} \\ \beta^{-n} & (-1)^n \end{bmatrix} x(n), \quad 0 < \beta < 1. \tag{3.4}$$

Note that the conditions of Example 2.3 are all satisfied. It follows that

$$S(n)=\begin{bmatrix}(-1)^n\beta & 0\\ 0 & (-1)^{n+1}\end{bmatrix},\quad N=2. \tag{3.5}$$

Now,

$$C^2=S(1)S(0)=\begin{bmatrix}-\beta^2 & 0\\ 0 & -1\end{bmatrix}, \tag{3.6}$$

for which the eigenvalues are $\lambda_1=-1$, $\lambda_2=-\beta^2$.

On the other hand, for

$$B=S^{-1}(0)A(0)=\begin{bmatrix}\dfrac{1}{\beta} & 1\\ -1 & -1\end{bmatrix}, \tag{3.7}$$

$\rho(B)<1$ if $2/3<\beta<1$, and $\rho(B)=1$ if $\beta=2/3$.

Applying Theorems 3.4 and 3.5, we see that the zero solution of (3.4) is asymptotically stable if $2/3<\beta<1$, and is stable if $\beta=2/3$.

In fact, the unique solution of (3.4) satisfying $x(0)=x_0$ is

$$x(n)=H(n)B^nx_0=\frac{1}{\mu_2-\mu_1}\begin{bmatrix}Q & (-1)^{\frac{n(n-1)}{2}}\beta^n(\mu_2^n-\mu_1^n)\\ (-1)^{n(n+1)/2}(\mu_1^n-\mu_2^n) & \mathcal{M}\end{bmatrix}x_0, \tag{3.8}$$

where $\mu_1=(-\gamma-\sqrt{\gamma^2-4\gamma})/2$, $\mu_2=(-\gamma+\sqrt{\gamma^2-4\gamma})/2$, $\gamma=1-1/\beta$, $Q=(-1)^{n(n-1)/2}\beta^n(\mu_1^n(\mu_2-1/\beta)-\mu_2^n(\mu_1-1/\beta))$, and $\mathcal{M}=(-1)^{n(n+1)/2}(\mu_2^n(\mu_2+1)-\mu_1^n(\mu_1+1))$.

It is easy to see that $\lim_{n\to\infty}\|x(n)\|=0$ if $2/3<\beta<1$, and $x(n)$ is bounded if $\beta=2/3$.

Remark 3.8. In the computation of $H(n)B^n$, $H(n)$ is calculated by using Example 2.3, and B^n is obtained by the method given in [6, 7].

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