Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2008, Article ID 845121, 9 pages
doi:10.1155/2008/845121

Research Article

Existence of Positive Solutions for a Class of m-Point Boundary Value Problems

Xuemei Zhang[1,2] and Weigao Ge[1]

[1] *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China*
[2] *Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China*

Correspondence should be addressed to Xuemei Zhang, zxm74@sina.com

Received 22 August 2007; Revised 29 November 2007; Accepted 8 January 2008

Recommended by Svatoslav Stanek

This paper investigates the existence of positive solutions for a class of second-order singular m-point Sturm-Liouville-type boundary value problems by using fixed point theorem in cones. The results significantly extend and improve many known results even for nonsingular cases.

1. Introduction

The study of multipoint BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. Since then many authors studied more general nonlinear multipoint boundary value problems. We refer readers to [2–5] and the references therein.

Motivated by works mentioned above, in this paper, we study the existence of positive solutions for the following second-order singular m -point boundary value problem:

$$\begin{gathered}
\big(p(t)x'(t)\big)' + g(t)f\big(t,x(t)\big) = 0, \quad 0<t<1,\\
ax(0) - b\lim_{t\to 0^+} p(t)x'(t) = \sum_{i=1}^{m-2} a_i x(\xi_i),\\
cx(1) + d\lim_{t\to 1^-} p(t)x'(t) = \sum_{i=1}^{m-2} b_i x(\xi_i),
\end{gathered} \tag{1.1}$$

where $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $\rho := ad + ac\int_0^1 (1/p(s))ds + bc > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,+\infty)(i = 1,2,\ldots,m-2)$, $p \in C([0,1],[0,+\infty))$, $g \in C((0,1),[0,+\infty))$, and g may be singular at $t = 0$ and/or at $t = 1$.

Under the assumption that f is sublinear, that is, $f_0 = \lim_{x\to 0}(f(x)/x^{\alpha}) = \infty$ and $f_{\infty} = \lim_{x\to 0}(f(x)/x^{\beta}) = 0$, or that f is superlinear, that is, $f_0 = 0, f_{\infty} = \infty$, there are many results available in literature (see, e.g., [2–4]). However, we remark that $\alpha = \beta = 1$ in [2–4]. So it is interesting and important to discuss the existence of positive solutions for BVP (1.1) when $0 < \alpha < 1, 0 < \beta < 1$, or $\alpha > 1, \beta > 1$. Many difficulties occur when we deal with them, for example, the construction of cone and operator. So we need to introduce some new tools and methods to investigate the existence of positive solutions for BVP (1.1). Moreover, the methods used in this paper are different from those in [2–4] and the results obtained in this paper generalize some results in [2–4] to some degree.

To obtain positive solutions of (1.1), the following fixed point theorem in cones is fundamental.

Lemma 1.1 (see [6, 7]). *Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous, where P is a cone in E. Suppose that one of the two conditions is satisfied.*

(i) *There exists $x_0 \in P \setminus \{\theta\}$ such that $x - Ax \neq tx_0$, for all $x \in P \cap \partial\Omega_2$, $t \geq 0$, and $Ax \neq \mu x$, for all $x \in P \cap \partial\Omega_1$, $\mu \geq 1$.*

(ii) *There exists $x_0 \in P \setminus \{\theta\}$ such that $x - Ax \neq tx_0$, for all $x \in P \cap \partial\Omega_1$, $t \geq 0$, and $Ax \neq \mu x$, for all $x \in P \cap \partial\Omega_2$, $\mu \geq 1$.*

Then, A has at least one fixed point in $P \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

2. Preliminaries

The basic space used in this paper is $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{0\leq t\leq 1}|x(t)|$. Let K be a cone of E, and let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, $K_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$, where $0 < r < R$.

To establish the existence of multiple positive solutions in E of problem (1.1), let us list the following assumptions:

(H_1) $p \in C([0,1],[0,+\infty))$ and $0 < \int_0^1 (dt/p(t)) < +\infty$;

(H_2) $g \in C((0,1),[0,+\infty))$, $g(t)\not\equiv 0$, on any subinterval of (0,1) and $\int_0^1 g(t)dt < +\infty$;

(H_3) $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ and $f(t,0) = 0$ uniformly with respect to t on $[0,1]$;

(H_4) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$, where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix}, \tag{2.1}$$

$$\psi(t) = b + a\int_0^t \frac{1}{p(r)}dr, \quad \phi(t) = d + c\int_t^1 \frac{1}{p(r)}dr, \quad t \in [0,1],$$

are linearly independent solutions of the equation $(p(t)x'(t))' = 0$.

We remark that (H_2) implies that g may be singular at $t = 0$ and/or at $t = 1$ and (2.1) shows that ψ is nondecreasing on $[0,1]$ and ϕ is nonincreasing on $[0,1]$.

Lemma 2.1. *Assume that $\Delta \neq 0$. Then for any $y \in E$, the boundary value problem*

$$\begin{aligned} &\big(p(t)x'(t)\big)' + y(t) = 0, \quad 0 < t < 1, \\ &ax(0) - b\lim_{t\to 0^+} p(t)x'(t) = \sum_{i=1}^{m-2} a_i x(\xi_i), \\ &cx(1) + d\lim_{t\to 1^-} p(t)x'(t) = \sum_{i=1}^{m-2} b_i x(\xi_i) \end{aligned} \tag{2.2}$$

has a unique solution $x(t)$, and $x(t)$ can be expressed in the form

$$x(t) = \int_0^1 G(t,s)y(s)ds + A\big(y(\cdot)\big)\psi(t) + B\big(y(\cdot)\big)\phi(t), \tag{2.3}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} \left(b + a\int_0^s \frac{1}{p(r)}dr\right)\left(d + c\int_t^1 \frac{1}{p(r)}dr\right), & \text{if } 0 \le s \le t \le 1, \\ \left(b + a\int_0^t \frac{1}{p(r)}dr\right)\left(d + c\int_s^1 \frac{1}{p(r)}dr\right), & \text{if } 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$A\big(y(\cdot)\big) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,t)y(t)dt & \rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,t)y(t)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix}, \tag{2.5}$$

$$B\big(y(\cdot)\big) := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,t)y(t)dt \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,t)y(t)dt. \end{vmatrix}. \tag{2.6}$$

Proof. The proof follows by routine calculations.

It is not difficult to show that $A(y(\cdot))$ and $B(y(\cdot))$ have the following properties. □

Proposition 2.2. *From* (2.5)*, one has*

$$\big|A\big(y(t)\big)\big| \le \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,t)dt & \rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,t)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \|y\| := \widetilde{A}\|y\|. \tag{2.7}$$

Proposition 2.3. *From* (2.6)*, one has*

$$\big|B\big(y(t)\big)\big| \le \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,t)dt \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,t)dt \end{vmatrix} \|y\| := \widetilde{B}\|y\|. \tag{2.8}$$

By (2.4), one remarks that there exists $\tau > 0$ such that for $t, s \in J_\theta$,

$$G(t,s) \geq \tau, \tag{2.9}$$

where $\theta \in (0, 1/2)$, $J_\theta = [\theta, 1-\theta]$.

3. Main results

In this section, we apply Lemma 1.1 to establish the existence of positive solutions for BVP (1.1). We consider the following two cases for $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha > 1$, $\beta > 1$. The case $0 < \alpha < 1$, $0 < \beta < 1$ is treated in the following theorem.

Theorem 3.1. *Suppose* (H_1)–(H_4) *and* f *satisfies the following conditions:*

(H_5) $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $\rho = ad + ac\int_0^1 (1/p(s))ds + bc > 0$;

(H_6) *there exists* $0 < \alpha < 1$ *such that* $0 < \liminf_{x\to 0^+} \min_{t\in[0,1]} (f(t,x)/x^\alpha) \leq +\infty$;

(H_7) *there exists* $0 < \beta < 1$ *such that* $0 \leq \limsup_{x\to+\infty} \max_{t\in[0,1]} (f(t,x)/x^\beta) < +\infty$.

Then BVP (1.1) *has at least one positive solution.*

Proof. By Lemma 2.1, $x \in C^2[0,1]$ is a solution of problem (1.1) if and only if $x \in C[0,1]$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)g(s)f(s,x(s))ds + A\big(g(\cdot)f(\cdot,x(\cdot))\big)\varphi(t) + B\big(g(\cdot)f(\cdot,x(\cdot))\big)\phi(t), \tag{3.1}$$

where $G(t,s)$ is defined by (2.4), and the definitions of $A(g(\cdot)f(\cdot,x(\cdot)))$ and $B(g(\cdot)f(\cdot,x(\cdot)))$ are similar tothose of $A(y(\cdot))$ and $B(y(\cdot))$, respectively. For the sake of applying Lemma 1.1, we construct a cone K in E by $K = \{x \in E : x(t) \geq 0,\ t \in [0,1]\}$.

Define $T : K \to K$ by

$$(Tx)(t) = \int_0^1 G(t,s)g(s)f(s,x(s))ds + A\big(g(\cdot)f(\cdot,x(\cdot))\big)\varphi(t) + B\big(g(\cdot)f(\cdot,x(\cdot))\big)\phi(t), \tag{3.2}$$

then $T : K \to K$ is completely continuous.

Define $w : [0,1] \to R$ by

$$cw(t) = \begin{cases} 1, & t \in [\theta, 1-\theta], \\ 0, & t \notin \left[\dfrac{\theta}{8}, 1 - \dfrac{7\theta}{8}\right], \\ \dfrac{8}{7\theta}\left(t - \dfrac{\theta}{8}\right), & t \in \left[\dfrac{\theta}{8}, \theta\right], \\ -\dfrac{8}{\theta}\left(t - 1 + \dfrac{7\theta}{8}\right), & t \in \left[1-\theta, 1 - \dfrac{7\theta}{8}\right]. \end{cases} \tag{3.3}$$

Obviously, w is a nonnegative continuous function, that is, $w \in K$, and $\|w\| = 1$.

Suppose that there is a $\varepsilon_1 > 0$ such that

$$x - Tx \neq 0, \quad \forall x \in K,\ 0 < \|x\| \leq \varepsilon_1 \tag{3.4}$$

(if not, then the conclusion holds). The conditions (H_6) and $f(t,0) = 0$ imply that there exist $\sigma > 0$, $\varepsilon_2 > 0$ such that

$$f(t,x) \geq \sigma x^{\alpha}, \quad 0 \leq x \leq \varepsilon_2. \tag{3.5}$$

Let

$$\varepsilon_3 = \min\left\{\varepsilon_1, \varepsilon_2, \left(\tau\sigma\int_{\theta}^{1-\theta} g(s)ds\right)^{1/(1-\alpha)}\right\}. \tag{3.6}$$

Choose $0 < r < \varepsilon_3$. We now show that

$$x - Tx \neq \zeta w, \quad \forall x \in \partial K_r, \ \zeta \geq 0. \tag{3.7}$$

In fact, if there exist $x_1 \in \partial K_r$, $\zeta_1 \geq 0$ such that $x_1 - Tx_1 = \zeta_1 w$, then (3.7) implies that $\zeta_1 > 0$. On the other hand, $x_1 = \zeta_1 w + Tx_1 \geq \zeta_1 w$. So we can choose $\zeta^* = \sup\{\zeta \mid x_1 \geq \zeta w\}$, then $\zeta_1 \leq \zeta^* < +\infty$, $x_1 \geq \zeta^* w$. Therefore,

$$\zeta^* = \zeta^*\|w\| \leq \|x_1\| = r < \varepsilon_3 \leq \left(\tau\sigma\int_{\theta}^{1-\theta} g(s)ds\right)^{1/(1-\alpha)}. \tag{3.8}$$

Consequently, for any $t \in [0,1]$, (2.9) and (3.5) imply

$$\begin{aligned} x_1(t) &= \int_0^1 G(t,s)g(s)f(s,x_1(s))ds + A(g(\cdot)f(\cdot,x_1(\cdot)))\psi(t) \\ &\quad + B(g(\cdot)f(\cdot,x_1(\cdot)))\phi(t) + \zeta_1 w(t) \\ &\geq \int_0^1 G(t,s)g(s)\sigma[x_1(s)]^{\alpha}ds + \zeta_1 w(t) \\ &\geq \int_{\theta}^{1-\theta} G(t,s)g(s)\sigma(\zeta^*)^{\alpha}[w(s)]^{\alpha}ds + \zeta_1 w(t) \\ &\geq \tau\sigma(\zeta^*)^{\alpha}\int_{\theta}^{1-\theta} g(s)ds + \zeta_1 w(t) \\ &\geq (\zeta^* + \zeta_1)w(t), \end{aligned} \tag{3.9}$$

that is, $x_1(t) \geq (\zeta^* + \zeta_1)w(t)$, $t \in [0,1]$, which is a contradiction to the definition of ζ^*. Hence, (3.7) holds.

Now turning to (H_7), there exist $m > 0$, $\varepsilon_4 > 0$, for $t \in [0,1]$, $x \geq \varepsilon_4$, such that $f(t,x) \leq mx^{\beta}$. Letting $\mu = \max_{0\leq t\leq 1, 0\leq x\leq \varepsilon_4} f(t,x)$, then

$$0 \leq f(t,x) \leq mx^{\beta} + \mu. \tag{3.10}$$

Choosing $R > \varepsilon_4$ such that

$$\frac{\mu M}{R} + \frac{mM}{R^{1-\beta}} < 1, \tag{3.11}$$

where $M = \max_{t\in[0,1]}\int_0^1 G(t,s)g(s)ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|$, now we prove that

$$Tx \neq \lambda x, \quad \forall x \in \partial K_R,\ \lambda \geq 1. \tag{3.12}$$

If not, then there exist $x_0 \in \partial K_R$, $\lambda_0 \geq 1$ such that $Tx_0 = \lambda_0 x_0$. By (2.7), (2.8), and (3.10), then for any $t \in [0,1]$, we have

$$\begin{aligned}\lambda_0 x_0(t) &= \int_0^1 G(t,s)g(s)f(s,x_0(s))ds + A\big(g(\cdot)f(\cdot,x_0(\cdot))\big)\psi(t) + B\big(g(\cdot)f(\cdot,x_0(\cdot))\big)\phi(t)\\ &\leq \big(\mu + m\|x_0\|^{\beta}\big)\left[\int_0^1 G(t,s)g(s)ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\right];\end{aligned} \tag{3.13}$$

so $R \leq \lambda_0 R = \lambda_0\|x_0\| \leq (\mu + m\|x_0\|^{\beta})[\int_0^1 G(t,s)g(s)ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|]$, that is,

$$\frac{\mu M}{R} + \frac{mM}{R^{1-\beta}} \geq 1, \tag{3.14}$$

which is a contradiction to (3.11). So (3.12) holds.

By (ii) of Lemma 1.1, (3.7) and (3.12) yield that T has a fixed point $x \in K_{r,R}$, $r \leq \|x\| \leq R$. Thus it follows that BVP (1.1) has at least one positive solution x with $r \leq \|x\| \leq R$. The proof is complete. □

The following theorem deals with the case $\alpha > 1$, $\beta > 1$.

Theorem 3.2. *Suppose* (H_1)–(H_4) *and* f *satisfy the following conditions:*

(H_8) $a \geq 0$, $b > 0$, $c \geq 0$, $d > 0$, $\rho = ad + ac\int_0^1(1/p(s))ds + bc > 0$;

(H_9) *there exists* $\alpha > 1$ *such that* $0 < \liminf_{x\to+\infty}\min_{t\in[0,1]}(f(t,x)/x^{\alpha}) \leq +\infty$;

(H_{10}) *there exists* $\beta > 1$ *such that* $0 \leq \limsup_{x\to 0^+}\max_{t\in[0,1]}(f(t,x)/x^{\beta}) < +\infty$.

Then BVP (1.1) *has at least one positive solution.*

Proof. By Lemma 2.1, $x \in C^2[0,1]$ is a solution of problem (1.1) if and only if $x \in C[0,1]$ is a solution of the integral equation (3.1). By (H_8), we know that $G(t,s)$ is positive and continuous function in $[0,1]\times[0,1]$. For the sake of applying Lemma 1.1, we construct a cone K^* in E by $K^* = \{x \in E : x \geq 0, \min_{t\in[0,1]} x(t) \geq \Gamma\|x\|\}$, where $\Gamma = \min\{l/L, \nabla/\Lambda\}$, $\nabla := \min\{\min_{t\in[0,1]}\phi(t), \min_{t\in[0,1]}\psi(t)\}$, $\Lambda := \max\{1,\|\phi\|,\|\psi\|\}$, $L = \max_{(t,s)\in[0,1]\times[0,1]}G(t,s)$, $l = \min_{(t,s)\in[0,1]\times[0,1]}G(t,s) > 0$. Obviously, $0 < \Gamma < 1$. Define $T^* : K^* \to K^*$ by

$$(T^*x)(t) = \int_0^1 G(t,s)g(s)f(s,x(s))ds + A\big(g(\cdot)f(\cdot,x(\cdot))\big)\psi(t) + B\big(g(\cdot)f(\cdot,x(\cdot))\big)\phi(t). \tag{3.15}$$

Then, for any $x \in K^*$, by (3.15), we obtain $T^*x \geq 0$ and

$$\begin{aligned}(T^*x)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds + A\big(g(\cdot)f(\cdot,x(\cdot))\big)\psi(t) + B\big(g(\cdot)f(\cdot,x(\cdot))\big)\phi(t)\\ &\leq \int_0^1 Lg(s)f(s,x(s))ds + \Lambda\big[A\big(g(\cdot)f(\cdot,x(\cdot))\big) + B\big(g(\cdot)f(\cdot,x(\cdot))\big)\big], \quad \text{for } t \in [0,1].\end{aligned} \tag{3.16}$$

On the other hand, we have for $t \in [0,1]$,

$$\begin{aligned}(T^*x)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds + A(g(\cdot)f(\cdot,x(\cdot)))\varphi(t) + B(g(\cdot)f(\cdot,x(\cdot)))\phi(t)\\ &\geq \int_0^1 lg(s)f(s,x(s))ds + \frac{\nabla}{\Lambda}\Lambda\big[A(g(\cdot)f(\cdot,x(\cdot))) + B(g(\cdot)f(\cdot,x(\cdot)))\big]\\ &= \frac{l}{L}\int_0^1 Lg(s)f(s,x(s))ds + \frac{\nabla}{\Lambda}\Lambda\big[A(g(\cdot)f(\cdot,x(\cdot))) + B(g(\cdot)f(\cdot,x(\cdot)))\big]\\ &\geq \Gamma\left[\int_0^1 Lg(s)f(s,x(s))ds + \Lambda\big[A(g(\cdot)f(\cdot,x(\cdot))) + B(g(\cdot)f(\cdot,x(\cdot)))\big]\right]\\ &\geq \Gamma\|T^*x\|.\end{aligned} \tag{3.17}$$

Therefore, $T^*K^* \subset K^*$ is completely continuous.

Define $w^*(t) = \int_0^1 G(t,s)g(s)ds$, then $w^* \in K^* \setminus \{\theta\}$ and

$$l\int_0^1 g(s)ds \leq w^*(t) \leq L\int_0^1 g(s)ds, \quad \forall t \in [0,1]. \tag{3.18}$$

Considering (H_9) and $f(t,0) = 0$, there exist $e > 0$, $\varepsilon_5 > 0$ such that

$$f(t,x) \geq ex^{\alpha}, \quad x \geq \varepsilon_5. \tag{3.19}$$

Choose $R^* > \max\{\varepsilon_5\Gamma^{-1}, \Gamma^{(-2\alpha-1)/(\alpha-1)}e^{-1/(\alpha-1)}Ll^{-\alpha/(\alpha-1)}(\int_0^1 g(s)ds)^{-1/(\alpha-1)}\}$. Suppose that $x \neq T^*x$ for $x \in \partial K^*_{R^*}$ (if not, then the conclusion holds). We now show that

$$x - T^*x \neq \delta w^*, \quad \forall x \in \partial K^*_{R^*},\ \delta \geq 0. \tag{3.20}$$

In fact, if there exist $x_0 \in \partial K^*_{R^*}$, $\delta_0 \geq 0$ such that $x_0 - T^*x_0 = \delta_0 w^*$, then $\delta_0 > 0$ (since $x \neq T^*x$ for any $x_0 \in \partial K^*_{R^*}$). Noticing $x_0 = \delta_0 w^* + T^*x_0 \geq \delta_0 w^*$, we can choose $\delta^{**} = \sup\{\delta^* \mid x_0 \geq \delta^* w^*\}$, then $\delta_0 \leq \delta^{**} < +\infty$, $x_0 \geq \delta^{**}w^*$.

From $x_0 \in K^*$, $\|x_0\| = R^*$, we have

$$\begin{aligned}x_0(t) &\geq \Gamma R^* > \Gamma^{-\alpha/(\alpha-1)}e^{-1/(\alpha-1)}Ll^{-\alpha/(\alpha-1)}\left(\int_0^1 g(s)ds\right)^{-1/(\alpha-1)}\\ &\geq \Gamma^{-\alpha/(\alpha-1)}e^{-1/(\alpha-1)}l^{-\alpha/(\alpha-1)}\left(\int_0^1 g(s)ds\right)^{-\alpha/(\alpha-1)} L\int_0^1 g(s)ds\\ &\geq \Gamma^{-\alpha/(\alpha-1)}e^{-1/(\alpha-1)}l^{-\alpha/(\alpha-1)}\left(\int_0^1 g(s)ds\right)^{-\alpha/(\alpha-1)} w^*(t).\end{aligned} \tag{3.21}$$

So, by the definition of δ^{**}, we have

$$\delta^{**} \geq \Gamma^{-\alpha/(\alpha-1)}e^{-1/(\alpha-1)}l^{-\alpha/(\alpha-1)}\left(\int_0^1 g(s)ds\right)^{-\alpha/(\alpha-1)}. \tag{3.22}$$

Noticing $x_0(t) \geq \Gamma R^* > \varepsilon_5$, $0 < \Gamma < 1$, then (3.19) and (3.22) imply

$$\begin{aligned}
x_0(t) &= \int_0^1 G(t,s)g(s)f(s,x_0(s))ds + A(g(\cdot)f(\cdot,x_0(\cdot)))\varphi(t) \\
&\quad + B(g(\cdot)f(\cdot,x_0(\cdot)))\phi(t) + \delta_0 w^*(t) \\
&\geq \int_0^1 G(t,s)g(s)e[x_0(s)]^\alpha ds + \delta_0 w^*(t) \\
&\geq \int_0^1 G(t,s)g(s)e(\delta^{**})^\alpha [w^*(s)]^\alpha ds + \delta_0 w^*(t) \\
&\geq \int_0^1 G(t,s)g(s)e\left(\delta^{**} l\int_0^1 g(s)ds\right)^\alpha ds + \delta_0 w^*(t) \\
&\geq \Gamma^\alpha \int_0^1 G(t,s)g(s)e\left(\delta^{**} l\int_0^1 g(s)ds\right)^\alpha ds + \delta_0 w^*(t) \\
&= \left[e\left(\delta^{**} l\Gamma\int_0^1 g(s)ds\right)^\alpha + \delta_0\right] w^*(t) \\
&\geq (\delta^{**} + \delta_0) w^*(t),
\end{aligned} \tag{3.23}$$

which is a contradiction to the definition of δ^{**}. Hence, (3.20) holds.

Next, turning to (H_{10}), there exist $\upsilon > 0$, $\varepsilon_6 > 0$, for $t \in J$, $0 \leq x \leq \varepsilon_6$, such that $0 \leq f(t,x) \leq \upsilon x^\beta$. Choosing $r^* > 0$ satisfies

$$r^* < \left\{\varepsilon_6, R^*, \left[L\int_0^1 g(s)ds + \widetilde{A}\|\varphi\| + \widetilde{B}\|\phi\|\right]^{-1/(\beta-1)} \upsilon^{-1/(\beta-1)}\right\}. \tag{3.24}$$

Now we prove that

$$T^* x \neq \mu^* x, \quad \forall x \in \partial K^*_{r^*},\ \mu^* \geq 1. \tag{3.25}$$

If not, then there exist $x_1 \in \partial K^*_{r^*}$, $\mu_1^* \geq 1$ such that $T^* x_1 = \mu_1^* x_1$ and

$$\begin{aligned}
x_1(t) &\leq \mu_1^* x_1(t) \\
&= \int_0^1 G(t,s)g(s)f(s,x_1(s))ds + A(g(\cdot)f(\cdot,x_1(\cdot)))\varphi(t) + B(g(\cdot)f(\cdot,x_1(\cdot)))\phi(t) \\
&\leq \upsilon\|x_1\|^\beta \left[L\int_0^1 g(s)ds + \widetilde{A}\|\varphi\| + \widetilde{B}\|\phi\|\right].
\end{aligned} \tag{3.26}$$

Therefore, $r^* = \|x_1\| \leq \upsilon\|x_1\|^\beta [L\int_0^1 g(s)ds + \widetilde{A}\|\varphi\| + \widetilde{B}\|\phi\|]$, that is, $r^* \geq [L\int_0^1 g(s)ds + \widetilde{A}\|\varphi\| + \widetilde{B}\|\phi\|]^{-1/(\beta-1)} \upsilon^{-1/(\beta-1)}$, which is a contradiction to (3.24). So (3.25) holds.

By (i) of Lemma 1.1, (3.20) and (3.25) yield that T^* has a fixed point $x \in K^*_{r^*,R^*}$, $r^* \leq \|x\| \leq R^*$. Thus it follows that BVP (1.1) has at least one positive solution x with $r^* \leq \|x\| \leq R^*$. The proof is complete. □

Remark 3.3. The condition about f in Theorem 3.1 is sublinear, while the condition about f in Theorem 3.2 is superlinear. Generally, the problems of superlinearity are more difficult to study than those of sublinearity. So we need stronger conditions in dealing with superlinear problems. For example, the condition (H_8) in Theorem 3.2 is stricter than (H_5) in Theorem 3.1.

Acknowledgments

The authors thank the referees for their careful reading of the manuscript and useful suggestions. This work is sponsored by the National Natural Science Foundation of China (10671012), the Doctoral Program Foundation of Education Ministry of China (20050007011), and the Science and Technology Innovation Foundation for postgraduate students of Beijing Information Technology Institute (GA200809).

References

[1] V. A. Il'in and E. I. Moiseev, "Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator," *Differential Equations*, vol. 23, no. 8, pp. 979–987, 1987.
[2] M. Feng and W. Ge, "Positive solutions for a class of m-point singular boundary value problems," *Mathematical and Computer Modelling*, vol. 46, no. 3-4, pp. 375–383, 2007.
[3] R. Ma and N. Castaneda, "Existence of solutions of nonlinear m-point poundary-value problems," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 556–567, 2001.
[4] R. Ma and H. Wang, "Positive solutions of nonlinear three-point boundary-value problems," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 1, pp. 216–227, 2003.
[5] C. P. Gupta, "A generalized multi-point boundary value problem for second order ordinary differential equations," *Applied Mathematics and Computation*, vol. 89, no. 1–3, pp. 133–146, 1998.
[6] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, NY, USA, 1988.
[7] D. J. Guo, J. X. Sun, and Z. L. Liu, *Functional Method of Ordinary Differential Equations*, Shandong Science and Technology Press, Jinan, China, 1994.