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Research Article

On the Growth of Nonoscillatory Solutions for Difference Equations with Deviating Argument

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The half-linear difference equations with the deviating argument $\Delta(a_n|\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n) + b_n|x_{n+q}|^\alpha \operatorname{sgn} x_{n+q} = 0$, $q \in \mathbb{Z}$ are considered. We study the role of the deviating argument q, especially as regards the growth of the nonoscillatory solutions and the oscillation. Moreover, the problem of the existence of the intermediate solutions is completely resolved for the classical half-linear equation ($q = 1$). Some analogies or discrepancies on the growth of the nonoscillatory solutions for the delayed and advanced equations are presented; and the coexistence with different types of nonoscillatory solutions is studied.

1. Introduction

Consider the half-linear difference equations with the deviating argument

$$\Delta\left(a_n\left|\Delta x_n\right|^\alpha sgn\Delta x_n\right) + b_n\left|x_{n+q}\right|^\alpha sgn x_{n+q} = 0, \tag{1.1}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\alpha > 0$, $q \in \mathbb{Z}$ and $a = \{a_n\}$, $b = \{b_n\}$ are positive real sequences for $n \geq 0$ such that

$$Y_a = \sum_{n=0}^{\infty}\left(\frac{1}{a_n}\right)^{1/\alpha} = \infty, \qquad Y_b = \sum_{n=0}^{\infty} b_n < \infty. \tag{1.2}$$

A large number of papers deal with the oscillatory and asymptotic properties of several particular cases of (1.1), such as the classical half-linear equation

$$\Delta\left(a_n\left|\Delta x_n\right|^\alpha sgn\Delta x_n\right) + b_n\left|x_{n+1}\right|^\alpha sgn x_{n+1} = 0 \tag{H}$$

and the equations with the advanced or delayed argument $\tau \in \mathbb{Z}$, $\tau \geq 1$,

$$\Delta\left(a_n\left|\Delta x_n\right|^{\alpha} sgn\Delta x_n\right) + b_n\left|x_{n+1+\tau}\right|^{\alpha} sgn x_{n+1+\tau} = 0, \tag{H+}$$

$$\Delta\left(a_n\left|\Delta x_n\right|^{\alpha} sgn\Delta x_n\right) + b_n|x_{n+1-\tau}|^{\alpha} sgn x_{n+1-\tau} = 0, \tag{H−}$$

see, for example, [1–9], the monographs [10, 11], and references therein.

By solution of (1.1) we mean a nontrivial sequence satisfying (1.1) for large n. As usual, a solution $x = \{x_n\}$ of (1.1) is said to be *nonoscillatory* if there exists a large n_x such that $x_n x_{n+1} > 0$ for $n \geq n_x$, otherwise it is said to be *oscillatory*. Equation (1.1) is characterized by the *homogeneity property*, which means that if x is a solution of (1.1), then also λx is its solution for any constant λ.

It is well-known that the deviating argument τ plays an important role in the oscillation. For instance, for (H) the Sturm-type separation property holds and so all its solutions are either nonoscillatory or oscillatory, see, for example, [11, Section 8.2]. In general, this property is no true anymore for (H+) and (H−); and the coexistence of oscillatory and nonoscillatory solutions can occur even in the linear case, that is when $\alpha = 1$, as we illustrate below. Nevertheless, in [4] some comparison criteria, which link the nonoscillation of (H) with the existence of nonoscillatory solutions of the advanced equation (H+), or the delayed equation (H−), are given. In particular, if $a \equiv 1$, the nonoscillation of (H) is equivalent to the existence of nonoscillatory solutions for (H+) or (H−), see [4, Corollary 8].

Nonoscillatory solutions of (1.1) can be classified as *subdominant, intermediate, or dominant solutions*, according to their asymptotic behavior, see below for the definition. As it is claimed in [12, page 241], the existence of the intermediate solutions is a difficult problem also in the continuous case. Moreover, another well-known problem is their possible coexistence with different types of nonoscillatory solutions, see, for example, [13, page 213]. In [14], both problems have been completely resolved for the half-linear differential equation

$$\left(a(t)\left|x'\right|^{\alpha} sgn x'\right)' + b(t)|x|^{\alpha} sgn x = 0, \tag{1.3}$$

using the extension of the Wronskian. Such approach cannot be used for (H) because the monotonicity of the corresponding Casoratian-type function remains an open problem.

The aim of this paper is to study intermediate solutions for (1.1) and the role of the deviating argument q, especially as regards the growth of the nonoscillatory solutions and the oscillation. The problem of the existence of intermediate solutions is completely resolved when $q = 1$, that is for the half-linear equation (H). When $q \neq 1$ some analogies or discrepancies on the growth of the nonoscillatory solutions, due to the presence of the deviating argument, are presented and also the coexistence with different types of nonoscillatory solutions is studied. Roughly speaking, if $a_n \equiv 1$, the deviating argument has no effect, that is (1.1) has the same types of nonoscillatory solutions for any q. On the other hand, if a is rapidly increasing, or decreasing, for large n, the delay may change the type of nonoscillatory solutions as well as the oscillation, as examples below show.

2. Main results

For any solution x of (1.1) we denote by $x^{[1]} = \{x_n^{[1]}\}$ its quasidifference, where

$$x_n^{[1]} = a_n\left|\Delta x_n\right|^{\alpha} sgn\Delta x_n. \tag{2.1}$$

In view of (1.2), any nonoscillatory solution x of (1.1) is eventually monotone and verifies $x_n x_n^{[1]} > 0$ for large n; we denote this property by saying that x is of class $\mathbb{M}^+$. Let x be a solution of (1.1) in the class $\mathbb{M}^+$; then for large n either x is positive increasing and $x^{[1]}$ positive decreasing or x is negative decreasing and $x^{[1]}$ negative increasing. So, we can divide the class $\mathbb{M}^+$ into the three subclasses:

$$\begin{gathered}\mathbb{M}^+_{\infty,\ell} = \{x \in \mathbb{M}^+ : \lim_n |x_n| = \infty,\ \lim_n x_n^{[1]} = \ell_x,\ 0 < |\ell_x| < \infty\},\\ \mathbb{M}^+_{\infty,0} = \{x \in \mathbb{M}^+ : \lim_n |x_n| = \infty,\ \lim_n x_n^{[1]} = 0\},\\ \mathbb{M}^+_{\ell,0} = \{x \in \mathbb{M}^+ : \lim_n x_n = \ell_x,\ \lim_n x_n^{[1]} = 0,\ 0 < |\ell_x| < \infty\},\end{gathered} \tag{2.2}$$

see also [6, 9]. Following the terminology of [15] in the continuous case, solutions in $\mathbb{M}^+_{\infty,\ell}$, $\mathbb{M}^+_{\infty,0}$, $\mathbb{M}^+_{\ell,0}$ are called *dominant solutions, intermediate solutions,* and *subdominant solutions*, respectively. This terminology is justified by the fact that if $x \in \mathbb{M}^+_{\infty,\ell}$, $y \in \mathbb{M}^+_{\infty,0}$, $z \in \mathbb{M}^+_{\infty,\ell}$, then $|x_n| > |y_n| > |z_n|$ for large n.

The series

$$\begin{gathered}S_\alpha = \sum_{n=0}^{\infty}\left(\frac{1}{a_n}\right)^{1/\alpha}\left(\sum_{k=n}^{\infty} b_k\right)^{1/\alpha},\\ T_\alpha(q) = \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}\right)^{\alpha},\end{gathered} \tag{2.3}$$

where the convention $\sum_{n_1}^{n_2} u_i = 0$ if $n_1 > n_2$ is used, play an important role in the classification of nonoscillatory solutions of (1.1). The possible cases for the behavior of these series are the following

$$(C_1)\ S_\alpha < \infty, \qquad T_\alpha(q) < \infty; \tag{2.4}$$

$$(C_2)\ S_\alpha = \infty, \qquad T_\alpha(q) < \infty; \tag{2.5}$$

$$(C_3)\ S_\alpha < \infty, \qquad T_\alpha(q) = \infty; \tag{2.6}$$

$$(C_4)\ S_\alpha = \infty, \qquad T_\alpha(q) = \infty. \tag{2.7}$$

Observe that, when $q = 1$, the case (C_2) is possible only when $\alpha > 1$, while the case (C_3) is possible only when $\alpha < 1$ [2, Theorem 4]. When $q \neq 1$, in view of the fact $T_\alpha(1) \le T_\alpha(q)$ for $q > 1$, the case (C_2) is not possible when $\alpha < 1$, $q > 1$ and the case (C_3) is not possible when $\alpha > 1$, $q < 1$.

If $q < 1$ [$q > 1$], it is easy to give an example of (1.1) satisfying the case (C_2) [the case (C_3)] for any α, see Example 5.3 below.

The main results of this paper are the following.

Theorem 2.1. *For* (H) *we have:*

(i_1) *if* $S_\alpha < \infty$, $T_\alpha(1) < \infty$, *then* $\mathbb{M}^+_{\ell,0} \neq \varnothing$, $\mathbb{M}^+_{\infty,0} = \varnothing$, $\mathbb{M}^+_{\infty,\ell} \neq \varnothing$;

(i_2) *if* $S_\alpha = \infty$, $T_\alpha(1) < \infty$, *then* $\mathbb{M}^+_{\ell,0} = \varnothing$, $\mathbb{M}^+_{\infty,0} \neq \varnothing$, $\mathbb{M}^+_{\infty,\ell} \neq \varnothing$;

(i_3) *if* $S_\alpha < \infty$, $T_\alpha(1) = \infty$, *then* $\mathbb{M}^+_{\ell,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} = \emptyset$;

(i_4) *if* $S_\alpha = \infty$, $T_\alpha(1) = \infty$ *and* (H) *is nonoscillatory, then* $\mathbb{M}^+_{\ell,0} = \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} = \emptyset$.

Theorem 2.2. *For* (1.1) *with* $q \neq 1$ *we have:*

(i_1) *if* $S_\alpha < \infty$, $T_\alpha(q) < \infty$, *then* $\mathbb{M}^+_{\ell,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} = \emptyset$, $\mathbb{M}^+_{\infty,\ell} \neq \emptyset$;

(i_2) *if* $S_\alpha = \infty$, $T_\alpha(q) < \infty$, *then* $\mathbb{M}^+_{\ell,0} = \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} \neq \emptyset$;

(i_3) *if* $S_\alpha < \infty$, $T_\alpha(q) = \infty$ *and*

$$\liminf_n a_n > 0, \tag{2.8}$$

then $\mathbb{M}^+_{\ell,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} = \emptyset$;

(i_4) *if* $S_\alpha = \infty$, $T_\alpha(q) = \infty$, *the condition* (2.8) *is verified and* (H) *is nonoscillatory, then* $\mathbb{M}^+_{\ell,0} = \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} = \emptyset$.

Theorems 2.1, 2.2 will be proved in the following sections. Observe that Theorem 2.1 is a discrete counterpart of [14, Theorems 4, 6, 7] for (1.3), even if the approach here used is completely different.

3. Unbounded solutions when $T_\alpha(q) < \infty$

In this section we study the growth of unbounded solutions of (1.1) when $T_\alpha(q) < \infty$. The following holds.

Theorem 3.1. *Assume*

$$S_\alpha < \infty, \qquad T_\alpha(q) < \infty. \tag{3.1}$$

Then for (1.1) *we have* $\mathbb{M}^+_{\infty,0} = \emptyset$.

Proof. By contradiction, assume that $x \in \mathbb{M}^+_{\infty,0}$. Without loss of generality, let n_0 be large so that $n_0 + \min\{q, 0\} \geq 0$ and

$$x_n > 0, \quad x_{n+q} > 0, \quad \Delta x_n > 0, \quad \Delta x_{n+q} > 0 \quad \text{for } n \geq n_0. \tag{3.2}$$

Set $n_q = n_0 + q$; then $x_{n+q} > x_{n_q}$ for $n > n_0$. By summation of (1.1) we obtain, for $n > n_0$,

$$x_n^{[1]} = \sum_{k=n}^{\infty} b_k \left(x_{k+q}\right)^\alpha \tag{3.3}$$

and so

$$x_{n+q} - x_{n_q} = \sum_{k=n_q}^{n+q-1} \left(\frac{1}{a_k}\right)^{1/\alpha} \left(\sum_{i=k}^{n} b_i \left(x_{i+q}\right)^\alpha + \sum_{i=n+1}^{\infty} b_i \left(x_{i+q}\right)^\alpha\right)^{1/\alpha}. \tag{3.4}$$

Putting

$$\sigma_\alpha = \begin{cases} 1, & \text{if } \alpha \geq 1, \\ 2^{(1-\alpha)/\alpha}, & \text{if } \alpha < 1, \end{cases} \tag{3.5}$$

and using (3.3) and the inequality

$$(X+Y)^{1/\alpha} \le \sigma_\alpha\left(X^{1/\alpha}+Y^{1/\alpha}\right), \tag{3.6}$$

we obtain the following, for $n > n_0$:

$$\begin{aligned} x_{n+q}-x_{n_q} &\le \sigma_\alpha \sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}\left(\sum_{i=k}^{n} b_i\left(x_{i+q}\right)^{\alpha}\right)^{1/\alpha} + \sigma_\alpha \sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}\left(\sum_{i=n+1}^{\infty} b_i\left(x_{i+q}\right)^{\alpha}\right)^{1/\alpha} \\ &\le \sigma_\alpha x_{n+q} \sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}\left(\sum_{i=k}^{n} b_i\right)^{1/\alpha} + \sigma_\alpha\left(x_{n+1}^{[1]}\right)^{1/\alpha}\sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}, \end{aligned} \tag{3.7}$$

and so

$$x_{n+q}-x_{n_q} \le \sigma_\alpha x_{n+q} \sum_{k=n_q}^{\infty}\left(\frac{1}{a_k}\right)^{1/\alpha}\left(\sum_{i=k}^{\infty} b_i\right)^{1/\alpha} + \sigma_\alpha\left(x_{n+1}^{[1]}\right)^{1/\alpha}\sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}. \tag{3.8}$$

Therefore,

$$1 \le \frac{x_{n_q}}{x_{n+q}} + \sigma_\alpha \sum_{k=n_q}^{\infty}\left(\frac{1}{a_k}\right)^{1/\alpha}\left(\sum_{i=k}^{\infty} b_i\right)^{1/\alpha} + \frac{\sigma_\alpha\left(x_{n+1}^{[1]}\right)^{1/\alpha}}{x_{n+q}}\sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha}. \tag{3.9}$$

In view of $S_\alpha < \infty$, fixed ε, $0 < \varepsilon < 1$, we can choose n_q large so that

$$\sigma_\alpha \sum_{k=n_q}^{\infty}\left(\frac{1}{a_k}\right)^{1/\alpha}\left(\sum_{i=k}^{\infty} b_i\right)^{1/\alpha} < \varepsilon/2. \tag{3.10}$$

Since x is unbounded, there exists $N > n_0$ such that for $n \ge N$,

$$\frac{x_{n_q}}{x_{n+q}} < \varepsilon/2. \tag{3.11}$$

Hence, from (3.9) we obtain that there exists $\gamma > 0$ such that for $n \ge N$,

$$\left(x_{n+1}^{[1]}\right)^{1/\alpha}\sum_{k=n_q}^{n+q-1}\left(\frac{1}{a_k}\right)^{1/\alpha} \ge \gamma x_{n+q}, \tag{3.12}$$

where $\gamma = (1-\varepsilon)/\sigma_\alpha$. Summing (1.1) we obtain

$$x_N^{[1]} - x_{n+1}^{[1]} = \sum_{k=N}^{n} b_k\left(x_{k+q}\right)^{\alpha}, \tag{3.13}$$

that is, in view of (3.12),

$$\gamma^{\alpha}\left(x_N^{[1]} - x_{n+1}^{[1]}\right) \le \sum_{k=N}^{n} b_k x_{k+1}^{[1]} \left(\sum_{i=n_q}^{k+q-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\right)^{\alpha}. \tag{3.14}$$

Since $x^{[1]}$ is decreasing, we have

$$\gamma^{\alpha}\left(x_N^{[1]} - x_{n+1}^{[1]}\right) \le x_N^{[1]} \sum_{k=N}^{\infty} b_k \left(\sum_{i=n_q}^{k+q-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\right)^{\alpha}. \tag{3.15}$$

In view of $T_\alpha(q) < \infty$, let N be large so that

$$\sum_{k=N}^{\infty} b_k \left(\sum_{i=n_q}^{k+q-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\right)^{\alpha} \le \gamma^{\alpha}/2. \tag{3.16}$$

Thus, from (3.15) we obtain $(\gamma^{\alpha}/2)x_N^{[1]} \le \gamma^{\alpha} x_{n+1}^{[1]}$, which is a contradiction because $x^{[1]}$ tends to zero as $n \to \infty$. □

The following result extends [3, Theorem 2], where the existence of intermediate solutions has been proved for $q \ge 0$.

Theorem 3.2. *(i_1) Equation* (1.1) *has solutions in the class* $\mathbb{M}^{+}_{\infty,\ell}$ *if and only if* $T_\alpha(q) < \infty$.
(i_2) Equation (1.1) *has solutions in* $\mathbb{M}^{+}_{\infty,0}$ *if* $S_\alpha = \infty$ *and* $T_\alpha(q) < \infty$.

Proof (outline). If $q \ge 0$, claim (i_1) follows, for example, from [6, Theorem 3] with minor changes and claim (i_2) from [3, Theorem 2]. Now we sketch the existence part of claims (i_1), (i_2) when $q < 0$.

Assume $T_\alpha(q) < \infty$ and let $\delta \in \{0, 1\}$. Using the comparison criterion, the series

$$\sum_{k=1}^{\infty} b_k \left(\delta + \sum_{i=1}^{k+q-1} \frac{1}{(a_i)^{1/\alpha}}\right)^{\alpha} \tag{3.17}$$

converges, too. So, choose n_0 large so that $n_0 + q \ge 2$ and

$$\sum_{k=n_0}^{\infty} b_k \left(\delta + \sum_{i=1}^{k+q-1} \frac{1}{(a_i)^{1/\alpha}}\right)^{\alpha} < 1, \qquad \sum_{j=1}^{n_0+q-1} \frac{1}{(a_j)^{1/\alpha}} \ge 1. \tag{3.18}$$

Set $n_q = n_0 + q$ and denote by $\mathbb{X}$ the Fréchet space of the real sequences defined for $n \ge n_q$, endowed with the topology of convergence on finite subsets of $\mathbb{N}_{n_q} = \{n \in \mathbb{N},\ n \ge n_q\}$. Consider the subset

$$\Omega = \left\{ u = \{u_n\} \in \mathbb{X} : 1 \le u_n \le \sum_{j=1}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \right\} \tag{3.19}$$

and define the operator $\mathcal{T} : \Omega \to \mathbb{X}$ given by $\mathcal{T}(u) = y$, where

$$y_n = 1 \quad \text{if } n_q \le n \le n_0, \qquad y_n = 1 + \sum_{j=n_0}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \left(\delta + \sum_{i=j}^{\infty} b_i (u_{i+q})^{\alpha} \right)^{1/\alpha}, \quad \text{if } n > n_0. \tag{3.20}$$

For $i \ge n_0$ it results $i + q \ge n_q$ and so we have for $u \in \Omega$

$$u_{i+q} \le \sum_{k=1}^{i+q-1} \frac{1}{(a_k)^{1/\alpha}}. \tag{3.21}$$

Then, in view of (3.18) we obtain the following for $n > n_0$:

$$\begin{aligned} y_n &\le 1 + \sum_{j=n_0}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \left(\sum_{i=j}^{\infty} b_i \left(\delta + \sum_{k=1}^{i+q-1} \frac{1}{(a_k)^{1/\alpha}} \right)^{\alpha} \right)^{1/\alpha} \\ &\le 1 + \left(\sum_{j=n_0}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \right) \left(\sum_{i=n_0}^{\infty} b_i \left(\delta + \sum_{k=1}^{i+q-1} \frac{1}{(a_k)^{1/\alpha}} \right)^{\alpha} \right)^{1/\alpha} \\ &\le 1 + \sum_{j=n_0}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \le \sum_{j=1}^{n-1} \frac{1}{(a_j)^{1/\alpha}}. \end{aligned} \tag{3.22}$$

If $n_q \le n \le n_0$, from (3.18) we have

$$y_n = 1 \le \sum_{j=1}^{n_q-1} \frac{1}{(a_j)^{1/\alpha}} \le \sum_{j=1}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \tag{3.23}$$

and so $\mathcal{T}$ maps Ω into itself. In virtue of the Ascoli theorem, any bounded set in $\mathbb{X}$ is relatively compact and so, because $\mathcal{T}(\Omega)$ is bounded according to the topology of $\mathbb{X}$, the compactness follows. Let $\{v^{(k)}\}$ be a sequence in Ω, converging on finite subsets of $\mathbb{N}_{n_q}$ to $v^{(\infty)} \in \Omega$. Using the discrete analogue of the Lebesgue dominated convergence theorem, the sequence $\{\mathcal{T}(v^{(k)})\}$ converges on finite subsets of $\mathbb{N}_{n_q}$ to $\mathcal{T}(v^{(\infty)})$; and so the continuity of $\mathcal{T}$ is proved. So, by applying the Tychonov fixed point theorem, there exists a solution x of (1.1) which satisfies for large n

$$\begin{gathered} x_n \ge 1 + \sum_{j=n_0}^{n-1} \frac{1}{(a_j)^{1/\alpha}} \left(\delta + \sum_{i=j}^{\infty} b_i \right)^{1/\alpha}, \\ \delta + \sum_{i=n}^{\infty} b_i \le x_n^{[1]} \le \delta + \sum_{i=n}^{\infty} b_i \left(\sum_{k=1}^{i+q-1} \frac{1}{(a_k)^{1/\alpha}} \right)^{\alpha}. \end{gathered} \tag{3.24}$$

If $\delta = 1$, then $\lim_n x_n^{[1]} = 1$ and so $x \in \mathbb{M}^{+}_{\infty,\ell}$. If $\delta = 0$ and $S_\alpha = \infty$, then $x \in \mathbb{M}^{+}_{\infty,0}$ and the proof is complete. □

4. Bounded solutions and proof of Theorem 2.1

It is well-known that the existence of bounded solutions of (1.1) depends on the convergence of the series S_α, and so the deviating argument does not play any role. This property follows, for example, from [9, Theorem 4.2] or [6, Theorem 2], in which the case $q \geq 0$ is considered, but the used argument can be easily modified for any $q \in \mathbb{Z}$. More precisely the following holds.

Proposition 4.1. *Equation* (1.1) *has solutions in the class* $\mathbb{M}^+_{\ell,0}$ *if and only if* $S_\alpha < \infty$.

When $q = 1$, bounded solutions of (H) are uniquely determined up to a constant factor. To prove this, we use a uniqueness result, stated in [1, Theorem 1], and the so-called *reciprocity principle* (see, e.g., [11, Section 8.2.2]), which links solutions of (H) with ones of the *reciprocal equation* ($n \geq 1$):

$$\Delta\left(B_n\left|\Delta y_n\right|^{1/\alpha} sgn \Delta y_n\right) + A_n\left|y_{n+1}\right|^{1/\alpha} sgn y_{n+1} = 0, \tag{4.1}$$

where

$$B_n = \frac{1}{\left(b_{n-1}\right)^{1/\alpha}}, \qquad A_n = \frac{1}{\left(a_n\right)^{1/\alpha}}. \tag{4.2}$$

Indeed, let y be a solution of (4.1) and denote by $y^{[1]}$ its quasidifference, that is $y_n^{[1]} = B_n|\Delta y_n|^{1/\alpha}$ sgn Δy_n. Then the sequence x, where $x_n = -y_n^{[1]}$, is a solution of (H) and satisfies

$$x_n^{[1]} = y_n, \qquad sgn y_n y_n^{[1]} = -sgn x_n x_n^{[1]}. \tag{4.3}$$

Proposition 4.2. *If* $Y_a = \infty$ *and* $S_\alpha < \infty$, *then for any fixed* $c \neq 0$, (H) *has a unique solution* $x \in \mathbb{M}^+_{\ell,0}$ *such that* $\lim_n x_n = c$.

Proof. First, observe that $Y_b < \infty$, because $S_\alpha < \infty$. Consider (4.1), then

$$S_\alpha = \sum_{n=0}^{\infty}\left(\frac{1}{a_n}\right)^{1/\alpha}\left(\sum_{k=n+1}^{\infty} b_{k-1}\right)^{1/\alpha} = \sum_{n=0}^{\infty} A_n\left(\sum_{k=n+1}^{\infty}\left(\frac{1}{B_k}\right)^{\alpha}\right)^{1/\alpha} < \infty. \tag{4.4}$$

By applying [1, Theorem 1] to (4.1), there exists a unique solution y of (4.1) such that $\lim_n y_n = 0$, $\lim_n y_n^{[1]} = -c$. Applying the reciprocity principle, (H) has a unique solution x such that $\lim_n x_n = c$ and $\lim_n x_n^{[1]} = 0$. Since (1.2) holds, $x \in \mathbb{M}^+_{\ell,0}$. □

Using the above results, we can prove our first main result.

Proof of Theorem 2.1. The claims (i_k), $k = 1, 2, 4$, follow from Theorems 3.1, 3.2 and Proposition 4.1.

Claim (i_3). Let x be the unique solution of (H), defined for $n \geq N$, satisfying $x \in \mathbb{M}^+_{\ell,0}$ and $\lim_n x_n = 1$. Let y be a solution of (H) such that $y_N = x_N$ and $y_{N+1} \not\equiv x_{N+1}$. If $y \in \mathbb{M}^+_{\ell,0}$, we have

$$\lim_n y_n = y_\infty \neq 1, \quad 0 < \left|y_\infty\right| < \infty. \tag{4.5}$$

In view of the homogeneity property, also $z = (y_\infty)^{-1}y$ is a solution of (H), and $\lim_n z_n = 1$, $\lim_n z_n^{[1]} = 0$. Then, from Proposition 4.2, we have $z \equiv x$, that is $y \equiv y_\infty x$. Hence $y_N = y_\infty x_N$ and so $y_\infty = 1$, which contradicts (4.5). Thus $y \notin \mathbb{M}^+_{\ell,0}$. Since $T_\alpha(1) = \infty$ and all solutions of (H) are nonoscillatory, from Theorem 3.2 the assertion follows. □

In the proof of Theorem 2.1 we used the Sturm separation property saying that oscillatory and nonoscillatory solutions cannot coexist for (H). Nevertheless, when $q \neq 1$, such a property can fail, as the following example shows.

Example 4.3. Consider the difference equations

$$\Delta^2 x_n + b_n x_{n+1} = 0, \tag{4.6}$$

$$\Delta^2 y_n + b_n y_{n+3} = 0, \tag{4.7}$$

where

$$b_n = 2^{-6n-12}\left(2^{4n+6} + 2^{2n+3} + 1\right). \tag{4.8}$$

Since $S_1 < \infty$, by Proposition 4.1 both equations have bounded nonoscillatory solutions. Thus, because the Sturm separation property holds for (4.6), this equation has all solutions nonoscillatory. However (4.7) has the oscillatory solution $y = \{(-1)^n 2^{n(n+1)}\}$. Similarly,

$$\Delta^2 z_n + B_n z_{n-5} = 0, \quad B_n = 3^{-10n+25}\left(3^{-4n-4} + 2\cdot 3^{-2n-1} + 1\right) \tag{4.9}$$

has the oscillatory solution $z = \{(-1)^n 3^{-n^2}\}$ and, again in view of Proposition 4.1, a bounded nonoscillatory solution, while the corresponding linear equation $\Delta^2 x_n + B_n x_{n+1} = 0$ is nonoscillatory, as it follows from Proposition 4.1 and the Sturm separation property.

As far as we know in the literature criteria assuring that all solutions of (1.1) with $q \neq 1$ are nonoscillatory are not available. This fact yields a strong difficulty to prove the existence of intermediate solutions of (1.1) when $T_\alpha(q) = \infty$, $q \neq 1$. These difficulties can be overcome by making a comparison result for intermediate solutions of (1.1) with $q \neq 1$ and (H), as it is described in the following section.

5. Comparison result and proof of Theorem 2.2

Clearly, the convergence of the series $T_\alpha(q)$ depends on the deviating argument q. The following example illustrates this fact and shows how the presence of the deviating argument can modify the growth of nonoscillatory solutions.

Example 5.1. Let $0 < \alpha < 1$ and define the sequences a, b so that

$$\sum_{i=0}^{k}\left(\frac{1}{a_i}\right)^{1/\alpha} = 2^{k^2}, \qquad b_k = 2^{-\alpha k^2}\frac{1}{k+1}. \tag{5.1}$$

Hence

$$T_\alpha(0) = \sum_{k=1}^{\infty} b_k\left(\sum_{i=0}^{k-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\right)^{\alpha} = \sum_{k=1}^{\infty}\frac{1}{k+1}2^{-\alpha(2k-1)} < \infty, \tag{5.2}$$

while

$$T_\alpha(1) = \sum_{k=0}^{\infty} \frac{1}{k+1} 2^{-\alpha k^2} 2^{\alpha k^2} = \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty. \tag{5.3}$$

Moreover, if $0 < X < 1$, we have $(i \geq 0)$

$$-\Delta\left(X^{i^2}\right) = X^{i^2} - X^{(i+1)^2} = X^{i^2}\left(1 - X^{2i-1}\right) > (1-X)X^{i^2}, \tag{5.4}$$

and so, taking $X = 2^{-\alpha}$, we obtain

$$\sum_{i=k}^{\infty} 2^{-\alpha i^2} < -\beta \sum_{i=k}^{\infty} \Delta\left(2^{-\alpha i^2}\right) = \beta 2^{-\alpha k^2}, \tag{5.5}$$

where $\beta = (1 - 2^{-\alpha})^{-1}$. Hence

$$\begin{aligned} S_\alpha &= \sum_{k=0}^{\infty} \left(\frac{1}{a_k}\right)^{1/\alpha} \left(\sum_{i=k}^{\infty} b_i\right)^{1/\alpha} \\ &\leq \sum_{k=0}^{\infty} 2^{k^2} \left(\frac{1}{k+1}\right)^{1/\alpha} \left(\sum_{i=k}^{\infty} 2^{-\alpha i^2}\right)^{1/\alpha} \\ &\leq \beta^{1/\alpha} \sum_{k=0}^{\infty} 2^{k^2} \left(\frac{1}{k+1}\right)^{1/\alpha} 2^{-k^2} < \infty. \end{aligned} \tag{5.6}$$

Taking into account that $T_\alpha(p) \leq T_\alpha(q)$ for $p \leq q$, we have

$$S_\alpha < \infty, \qquad T_\alpha(q) < \infty \quad \text{for } q \leq 0, \; T_\alpha(1) = \infty \tag{5.7}$$

and so the case (C_3) and (C_1) occurs for (H) and (H−) for $\tau \geq 1$, respectively. Therefore, the delayed argument changes the growth of unbounded solutions: all unbounded solutions of (H) are intermediate in virtue of Theorem 2.1(i_3), while all unbounded solutions of (H−) are dominant for any delay τ in virtue of Theorems 3.1 and 3.2.

Now we state a comparison result for intermediate solutions of (1.1).

Theorem 5.2. *Assume* (2.8). *If* (1.1) *has solutions in the class* $\mathbb{M}^+_{\infty,0}$ *for a fixed* q, *then the same occurs for any* q.

Proof. Jointly with (1.1), consider the nonlinear difference equations

$$\Delta\left(a_n\left|\Delta y_n\right|^\alpha sgn \Delta y_n\right) + b_n\left|x_{n+q-1}\right|^\alpha sgn y_{n+q-1} = 0, \tag{5.8}$$

$$\Delta\left(a_n\left|\Delta x_n\right|^\alpha sgn \Delta x_n\right) + b_n\left|x_{n+q+1}\right|^\alpha sgn x_{n+q+1} = 0. \tag{5.9}$$

It is sufficient to prove that if $\mathbb{M}^+_{\infty,0} \neq \varnothing$ for (1.1), then the same holds for (5.9) and (5.8). Put $\overline{q} = \min\{0, q\}$. Let z be a solution of (1.1) in the class $\mathbb{M}^+_{\infty,0}$ and let n_0 be a positive integer so that $n_0 + \overline{q} \geq 1$. Set

$$\overline{n} = n_0 + \overline{q} - 1, \qquad \tilde{n} = \overline{n} + 1 = n_0 + \overline{q}. \tag{5.10}$$

Then

$$0 \le \overline{n} < \tilde{n} \le n_0. \tag{5.11}$$

Without loss of generality, assume $z_i > 0$, $z_i^{[1]} > 0$ for $i \ge \overline{n}$ and

$$a_i > h, \tag{5.12}$$

where h is a positive constant. Since $z^{[1]}$ is positive decreasing for $n \ge \overline{n}$, we have $z_n^{[1]} \le z_{\overline{n}}^{[1]}$ for $n \ge \overline{n}$, that is,

$$z_{n+1} - z_n \le \left(\frac{z_{\overline{n}}^{[1]}}{a_n}\right)^{1/\alpha} \le \left(\frac{z_{\overline{n}}^{[1]}}{h}\right)^{1/\alpha} = h_1. \tag{5.13}$$

Summing twice (1.1), we obtain, for $n > n_0$,

$$z_n = z_{n_0} + \sum_{i=n_0}^{n-1} \left(\frac{1}{a_i}\right)^{1/\alpha} \left(\sum_{k=i}^{\infty} b_k \left(z_{k+q}\right)^{\alpha}\right)^{1/\alpha}. \tag{5.14}$$

Step 1 ($\mathbb{M}^+_{\infty,0} \neq \varnothing$ for (1.1) $\Rightarrow$ $\mathbb{M}^+_{\infty,0} \neq \varnothing$ for (5.8)). Denote by $\mathbb{X}$ the Fréchet space of the real sequences defined for $n \ge \overline{n}$, endowed with the topology of convergence on finite subsets of $\mathbb{N}_{\overline{n}} = \{n \in \mathbb{N},\ n \ge \overline{n}\}$ and consider the set $\Omega \subset \mathbb{X}$ defined by

$$\Omega = \left\{v = \left\{v_n\right\} \in \mathbb{X} : z_{n+1} \le v_n \le M z_{n+1}\right\}, \tag{5.15}$$

where

$$M = 1 + \frac{h_1}{z_{n_0}}. \tag{5.16}$$

Let $\mathcal{T} : \Omega \to \mathbb{X}$ be the map given by

$$\begin{aligned} \mathcal{T}(v)_n &= z_{n+1} \quad \text{if } \overline{n} \le n \le n_0, \\ \mathcal{T}(v)_n &= d + \sum_{i=n_0}^{n-1} \left(\frac{1}{a_i}\right)^{1/\alpha} \left(\sum_{k=i}^{\infty} b_k \left(v_{k+q-1}\right)^{\alpha}\right)^{1/\alpha} \quad \text{if } n \ge n_0 + 1, \end{aligned} \tag{5.17}$$

where

$$d = z_{n_0} + h_1. \tag{5.18}$$

Let $\overline{n} \le n \le n_0$; clearly

$$z_{n+1} = \mathcal{T}(v)_n < M z_{n+1}. \tag{5.19}$$

Now let $n > n_0$. Since $n + q - 1 \ge n_0 + \overline{q} - 1 = \overline{n}$, we have for $v \in \Omega$ and $k \ge n_0$

$$z_{k+q} \le v_{k+q-1} \le M z_{k+q}. \tag{5.20}$$

Then, from (5.14) and (5.20), we get

$$\mathcal{T}(v)_n \le d + \sum_{i=n_0}^{n-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\left(\sum_{k=i}^{\infty} b_k\left(Mz_{k+q}\right)^{\alpha}\right)^{1/\alpha} = Mz_n - Mz_{n_0} + d. \tag{5.21}$$

Since from (5.16) it results $Mz_{n_0} = z_{n_0} + h_1$, from (5.21) we obtain

$$\mathcal{T}(v)_n \le Mz_n - z_{n_0} - h_1 + d = Mz_n < Mz_{n+1}. \tag{5.22}$$

Moreover, using again (5.14) and (5.20), we obtain, for $n > n_0$,

$$\mathcal{T}(v)_n \ge d + \sum_{i=n_0}^{n-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\left(\sum_{k=i}^{\infty} b_k\left(z_{k+q}\right)^{\alpha}\right)^{1/\alpha} = z_n - z_{n_0} + d = z_n + h_1, \tag{5.23}$$

that is, in view of (5.13),

$$\mathcal{T}(v)_n \ge z_{n+1}. \tag{5.24}$$

Hence, from (5.19), (5.22), and (5.24) we have $\mathcal{T}(\Omega) \subset \Omega$. Reasoning as in the proof of Theorem 3.2, the continuity and compactness of $\mathcal{T}$ in Ω follows and so, by applying the Tychonov fixed point theorem, there exists y such that $y = \mathcal{T}(y)$. It is easy to verify that y is a solution of (5.8) for large n and, since $y \in \Omega$, we have $y \in \mathbb{M}^{+}_{\infty,0}$, that is, the assertion.

Step 2 ($\mathbb{M}^{+}_{\infty,0} \neq \emptyset$ for (1.1) $\Rightarrow$ $\mathbb{M}^{+}_{\infty,0} \neq \emptyset$ for (5.9)). Denote by $\mathbb{X}$ the Fréchet space of the real sequences defined for $n \ge \tilde{n}$, endowed with the topology of convergence on finite subsets of $\mathbb{N}_{\tilde{n}} = \{n \in \mathbb{N}, n \ge \tilde{n}\}$, and consider the set $\Omega \subset \mathbb{X}$ defined by

$$\Omega = \left\{w = \left\{w_n\right\} \in \mathbb{X} : z_{n-1} \le w_n \le Hz_{n-1}\right\}, \tag{5.25}$$

where $H > 1$ is a constant. Let $\mathcal{T} : \Omega \to \mathbb{X}$ be the map given by

$$\begin{aligned} \mathcal{T}(w)_n &= z_{n-1} \quad \text{if } \tilde{n} \le n \le n_0, \\ \mathcal{T}(w)_n &= z_{n_0} + \sum_{i=n_0}^{n-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\left(\sum_{k=i}^{\infty} b_k\left(w_{k+q+1}\right)^{\alpha}\right)^{1/\alpha} \quad \text{for } n \ge n_0 + 1. \end{aligned} \tag{5.26}$$

Let $\tilde{n} \le n \le n_0$; clearly

$$z_{n-1} = \mathcal{T}(w)_n \le Hz_{n-1}. \tag{5.27}$$

Now let $n > n_0$. For $k \ge n_0 + 1$ it results that $k + q + 1 \ge n_0 + \overline{q} = \overline{n} + 1 = \tilde{n}$ and so for $w \in \Omega$ and $k \ge n_0 + 1$:

$$z_{k+q} \le w_{k+q+1} \le Hz_{k+q}. \tag{5.28}$$

Taking into account (5.14) and (5.28), we have

$$\mathcal{T}(w)_n \ge z_{n_0} + \sum_{i=n_0}^{n-1}\left(\frac{1}{a_i}\right)^{1/\alpha}\left(\sum_{k=i}^{\infty} b_k\left(z_{k+q}\right)^{\alpha}\right)^{1/\alpha} = z_n > z_{n-1}. \tag{5.29}$$

Similarly, results for $n > n_0$:

$$\mathcal{T}(w)_n \le z_{n_0} + \sum_{i=n_0}^{n-1} \left(\frac{1}{a_i}\right)^{1/\alpha} \left(\sum_{k=i}^{\infty} b_k \left(Hz_{k+q}\right)^{\alpha}\right)^{1/\alpha} = z_{n_0} + Hz_n - Hz_{n_0}. \tag{5.30}$$

From (5.13) we obtain $z_{i-1} \ge z_i + h_1$ for $i \ge \overline{n} + 1 = \tilde{n}$ and so, since $n > \tilde{n}$, we have $z_{n-1} \ge z_n + h_1$. Consequently,

$$\mathcal{T}(w)_n \le Hz_{n-1} + z_{n_0} - H\left(h_1 + z_{n_0}\right). \tag{5.31}$$

Since $H > 1$, we have $H(h_1 + z_{n_0}) \ge z_{n_0}$ and so $\mathcal{T}(w)_n \le Hz_{n-1}$. Thus $\mathcal{T}(\Omega) \subset \Omega$. Since $\mathcal{T}$ is continuous and compact in Ω, by applying the Tychonov fixed point theorem, there exists x such that $x = \mathcal{T}(x)$. It is easy to verify that x is a solution of (5.9) for n large and, since $x \in \Omega$, we have $x \in \mathbb{M}^{+}_{\infty,0}$, that is, the assertion. □

Now we are able to prove Theorem 2.2.

Proof of Theorem 2.2. Claims (i_1), (i_2) follow from Theorems 3.1, 3.2, and Proposition 4.1.

Claim (i_3). From Theorem 3.2(i_1) and Proposition 4.1 we have $\mathbb{M}^{+}_{\ell,0} \neq \emptyset$, $\mathbb{M}^{+}_{\infty,\ell} = \emptyset$. To prove $\mathbb{M}^{+}_{\infty,0} \neq \emptyset$, let us show that $T_\alpha(1) = \infty$. Clearly, this is true if $q < 1$, because $T_\alpha(q) = \infty$. If $q > 1$, in view of (2.8) we have for large n

$$\begin{aligned} \sum_{k=0}^{n+q-1} \left(\frac{1}{a_k}\right)^{1/\alpha} &= \sum_{k=0}^{n} \left(\frac{1}{a_k}\right)^{1/\alpha} + \sum_{k=n+1}^{n+q-1} \left(\frac{1}{a_k}\right)^{1/\alpha} \\ &\le \sum_{k=0}^{n} \left(\frac{1}{a_k}\right)^{1/\alpha} + h^{-1/\alpha}(q-1), \end{aligned} \tag{5.32}$$

where h is given by (5.12), and so $T_\alpha(1) = \infty$. In view of Theorem 2.1(i_3), (H) has intermediate solutions. Thus, Theorem 5.2 yields $\mathbb{M}^{+}_{\infty,0} \neq \emptyset$ also for (1.1).

Claim (i_4). Again from Theorem 3.2(i_1) and Proposition 4.1, we have $\mathbb{M}^{+}_{\ell,0} = \mathbb{M}^{+}_{\infty,\ell} = \emptyset$. Reasoning as in the proof of Claim (i_3), we get $\mathbb{M}^{+}_{\infty,0} \neq \emptyset$. □

We close this section by an example illustrating the role of the delayed argument, when a is rapidly varying at infinity.

Example 5.3. Consider (1.1) with $b_n = 2^{-n^2+2n}$ and

$$a_{n+1} = 2^{-n^2}\left(2^{(2n+1)/\alpha} - 1\right)^{-\alpha} \quad \text{for } n \ge 0, \ a_0 = 1. \tag{5.33}$$

For any $\alpha > 0$ we have

$$\sum_{k=0}^{n} \left(\frac{1}{a_k}\right)^{1/\alpha} = 1 + \sum_{k=1}^{n} \Delta 2^{(n-1)^2/\alpha} = 2^{n^2/\alpha}, \tag{5.34}$$

and so $T_\alpha(1) = \infty$ and $T_\alpha(-1) < \infty$. Using the inequality $\sum_{i=n}^{\infty} 2^{-i^2+2i} \geq 2^{-n^2+2n}$, we obtain

$$S_\alpha \geq \sum_{n=0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\alpha} 2^{(-n^2+2n)/\alpha} = 1 + \sum_{n=1}^{\infty} \left(2^{2n/\alpha} - 2^{1/\alpha}\right) = \infty. \tag{5.35}$$

Thus, for $q \leq -1$ and any α, (1.1) satisfies the case (C_2) and so, by Theorem 2.2, it has intermediate and dominant solutions. Observe that if $q = 1$, the case (C_4) occurs, and, by Theorem 2.1, the half-linear equation (H) is either oscillatory or all its solutions are intermediate. So, this example shows that if q is negative, the case (C_2) can occur for any α.

Moreover, when $\alpha = 1$ the linear equation

$$\Delta\left(a_n \Delta x_n\right) + b_n x_{n+1} = 0 \tag{5.36}$$

is oscillatory (see, e.g., [10, Remark 1.11.9]), and the corresponding delayed linear equation

$$\Delta\left(a_n \Delta x_n\right) + b_n x_{n-\tau} = 0, \quad \tau \in \mathbb{N}, \tag{5.37}$$

has, by Theorem 2.2, intermediate and dominant solutions. In comparison with Example 4.3, where the linear equation is nonoscillatory and the deviating argument may produce oscillatory solutions, in this case we have an opposite effect to the oscillation: the linear equation is oscillatory and the delay produces nonoscillatory solutions.

6. Conclusion and open problems

(1) *The role of the deviating argument*. By Theorems 2.1, 2.2, the existence of subdominant solutions does not depend on the deviating argument q, the one of dominant solutions can depend on q. Especially because $T_\alpha(1-\tau) \leq T_\alpha(1) \leq T_\alpha(1+\tau)$, we obtain a possible discrepancy concerning unbounded solutions of (H) and an equation with deviating argument. For instance, if the case (C_1) holds, (H) has dominant solutions and no intermediate solutions, while it may occur that (H+) has intermediate solutions and no dominant solutions. Similarly, the delayed argument may cause the opposite phenomena: if the case (C_3) holds, (H) has intermediate solutions and no dominant solutions, while it may occur that (H−) has dominant solutions and no intermediate solutions. Clearly, the deviating argument can produce also the oscillation, or nonoscillation, as Examples 4.3 and 5.3 show.

(2) *Open problems*. As we noticed in Section 4, the nonoscillation of all solutions of (1.1) with $q \neq 1$ is an open problem.

For this reason, when $T_\alpha(q) = \infty$ $(q \neq 1)$, the existence of intermediate solutions has been obtained by means of a comparison result between (H) and (1.1), say Theorem 5.2. Such a result shows that the deviating argument does not have any influence on the existence of intermediate solutions, provided (2.8) holds. Especially, if the classical half-linear equation (H) has intermediate solutions and (2.8) holds, then also (H+) and (H−) have intermediate solutions for any $\tau \in \mathbb{Z}$, $\tau \geq 1$.

If (2.8) is not satisfied, Theorem 5.2 can fail, as Example 5.1 illustrates. Is condition (2.8) necessary for the validity of this theorem and, consequently, of Theorem 2.2? By means of an argument similar to the one given in the proof of Theorem 2.2(i_3) we have that (2.8) yields that the series $T_\alpha(p)$, $T_\alpha(q)$ have the same character, that is, are both convergent or divergent for any $p, q \in \mathbb{Z}$. Now, the following question arises: is the statement of these theorems valid by assuming, instead of (2.8), that $T_\alpha(p)$, $T_\alpha(q)$ are both convergent or divergent?

(3) *Future investigations*. The future studies will be addressed to extend the results of this paper, especially those concerning the intermediate solutions, to Emden-Fowler-type equations of the form

$$\Delta\left(a_n|\Delta x_n|^\alpha sgn\Delta x_n\right) + b_n|x_{n+q}|^\beta sgn x_{n+q} = 0, \quad \alpha \neq \beta. \tag{6.1}$$

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