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Research Article

Stability of Solutions for a Family of Nonlinear Difference Equations

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We consider the family of nonlinear difference equations: $x_{n+1} = (\sum_{i=1}^{3} f_i(x_n, \ldots, x_{n-k}) + f_4(x_n, \ldots, x_{n-k}) f_5(x_n, \ldots, x_{n-k}))/(f_1(x_n, \ldots, x_{n-k}) f_2(x_n, \ldots, x_{n-k}) + \sum_{i=3}^{5} f_i(x_n, \ldots, x_{n-k}))$, $n = 0, 1, \ldots$, where $f_i \in C((0,+\infty)^{k+1}, (0,+\infty))$, for $i \in \{1,2,4,5\}$, $f_3 \in C([0,+\infty)^{k+1}, (0,+\infty))$, $k \in \{1,2,\ldots\}$ and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0 \in (0,+\infty)$. We give sufficient conditions under which the unique equilibrium $\overline{x} = 1$ of these equations is globally asymptotically stable, which extends and includes corresponding results obtained in the cited references.

1. Introduction

In [1], Papaschinopoulos and Schinas investigated the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1} = \frac{\sum_{i\in \mathbb{Z}_k - \{j-1,j\}} x_{n-i} + x_{n-j}x_{n-j+1} + 1}{\sum_{i\in \mathbb{Z}_k} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $k \in \{1, 2, 3, \ldots\}$, $\{j, j-1\} \subset \mathbb{Z}_k \equiv \{0, 1, \ldots, k\}$, and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0 \in \mathbb{R}_+ \equiv (0, +\infty)$.

Moreover, Kruse and Nesemann [2] studied the global asymptotic stability of the unique equilibrium of a discrete dynamical system, and as a special result they proved that the unique equilibrium $\overline{x} = 1$ of the Putnam difference equation

$$x_{n+1}=\frac{x_n+x_{n-1}+x_{n-2}x_{n-3}}{x_nx_{n-1}+x_{n-2}+x_{n-3}},\quad n=0,1,\ldots,\tag{1.2}$$

is globally asymptotically stable, where the initial values $x_{-3},x_{-2},x_{-1},x_0\in\mathbb{R}_+$.

In [3], Çinar et al. investigated the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1}=\frac{x_n\sum_{i=1}^{k}x_{n-i}+1}{x_n+x_{n-1}+x_n\sum_{i=2}^{k}x_{n-i}},\quad n=0,1,\ldots,\tag{1.3}$$

where $k\in\{1,2,3,\ldots\}$ and the initial values $x_{-k},x_{-k+1},\ldots,x_0\in\mathbb{R}_+$. For closely related results, see [4–10].

In this paper, we consider the family of nonlinear difference equations:

$$x_{n+1}=\frac{\sum_{i=1}^{3}f_i(x_n,\ldots,x_{n-k})+f_4(x_n,\ldots,x_{n-k})f_5(x_n,\ldots,x_{n-k})}{f_1(x_n,\ldots,x_{n-k})f_2(x_n,\ldots,x_{n-k})+\sum_{i=3}^{5}f_i(x_n,\ldots,x_{n-k})},\quad n=0,1,\ldots,\tag{1.4}$$

where $f_i\in C((0,+\infty)^{k+1},(0,+\infty))$, for $i\in\{1,2,4,5\}$, $f_3\in C([0,+\infty)^{k+1},[0,+\infty))$, $k\in\{1,2,\ldots\}$, and the initial values $x_{-k},x_{-k+1},\ldots,x_0\in(0,+\infty)$. Our main result is the following theorem.

Theorem 1.1. *Let $u^*=\max\{u,1/u\}$, for any $u\in\mathbb{R}_+$. If $[f_i(u_0,u_1,\ldots,u_k)]^*\le\max\{u_0^*,u_1^*,\ldots,u_k^*\}$, for $i=1,2,4,5$, then $\overline{x}=1$ is the unique positive equilibrium of (1.4) which is globally asymptotically stable.*

2. The proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To do this, we need the following lemma.

Lemma 2.1. *Let $(a,b,c,d)\in\mathbb{R}_+^4-\{(1,1,1,1)\}$, $e\in[0,\infty)$, and $\alpha=\max\{a^*,b^*,c^*,d^*\}$. Then,*

$$\frac{1}{\alpha}<\frac{c+d+e+ab}{cd+e+a+b}<\alpha.\tag{2.1}$$

Proof. Since $(a,b,c,d)\in\mathbb{R}_+^4-\{(1,1,1,1)\}$, $e\in[0,\infty)$, and $\alpha=\max\{a^*,b^*,c^*,d^*\}$, we have $\alpha>1$ and either $\alpha\ge\beta>1/\alpha$ or $\alpha>\beta\ge1/\alpha$, for every $\beta\in\{a,b,c,d\}$. If $c<1$ or $d<1$, then

$$\alpha cd+\alpha a+\alpha b+\alpha e>ab+c+d+e.\tag{2.2}$$

It follows that

$$\frac{c+d+e+ab}{cd+e+a+b}<\alpha.\tag{2.3}$$

If $c\ge1$ and $d\ge1$, then $\alpha\ge c>1$ or $\alpha>c\ge1$ and $\alpha\ge d>1$ or $\alpha>d\ge1$. Thus, we have the following inequalities:

$$\begin{aligned}&\alpha(a+b)\ge2ab,\\&\alpha cd+\alpha a\ge\alpha c+1>2c,\\&\alpha cd+\alpha b\ge\alpha d+1>2d.\end{aligned}\tag{2.4}$$

It follows from (2.4) that

$$\alpha cd + \alpha a + \alpha b + \alpha e > ab + c + d + e, \tag{2.5}$$

which implies

$$\frac{c + d + e + ab}{cd + e + a + b} < \alpha. \tag{2.6}$$

By the symmetry, we have also that

$$\frac{1}{\alpha} < \frac{c + d + e + ab}{cd + e + a + b}. \tag{2.7}$$

This completes the proof. □

Proof of Theorem 1.1. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of (1.4) with the initial values x_{-k}, $x_{-k+1}, \ldots, x_0 \in \mathbb{R}_+$. For any $n > 0$, write

$$p_n = \max\{x_n^*, x_{n-1}^*, \ldots, x_{n-k}^*\}. \tag{2.8}$$

From Lemma 2.1, it follows that for any $n \geq 0$,

$$\begin{aligned} x_{n+1} &= \frac{\sum_{i=1}^{3} f_i(x_n, \ldots, x_{n-k}) + f_4(x_n, \ldots, x_{n-k}) f_5(x_n, \ldots, x_{n-k})}{f_1(x_n, \ldots, x_{n-k}) f_2(x_n, \ldots, x_{n-k}) + \sum_{i=3}^{5} f_i(x_n, \ldots, x_{n-k})} \\ &\leq \max\left\{\left[f_i(x_n, \ldots, x_{n-k})\right]^* : i = 1, 2, 4, 5\right\} \\ &\leq \max\left\{x_{n-i}^* : 0 \leq i \leq k\right\} = p_n, \\ x_n + 1 &= \frac{\sum_{i=1}^{3} f_i(x_n, \ldots, x_{n-k}) + f_4(x_n, \ldots, x_{n-k}) f_5(x_n, \ldots, x_{n-k})}{f_1(x_n, \ldots, x_{n-k}) f_2(x_n, \ldots, x_{n-k}) + \sum_{i=3}^{5} f_i(x_n, \ldots, x_{n-k})} \\ &\geq \frac{1}{\max\left\{\left[f_i(x_n, \ldots, x_{n-k})\right]^* : i = 1, 2, 4, 5\right\}} \\ &\geq \frac{1}{\max\left\{x_{n-i}^* : 0 \leq i \leq k\right\}} = \frac{1}{p_n}. \end{aligned} \tag{2.9}$$

By (2.9), we have that for any $n \geq 0$,

$$1 \leq x_{n+1}^* \leq p_n, \quad p_{n+1} \leq p_n. \tag{2.10}$$

From (2.10), we may assume that

$$\lim_{n\to\infty} p_n = M \geq 1. \tag{2.11}$$

Then,

$$\frac{1}{M} \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq M. \tag{2.12}$$

Since $p_n = \max\{x_n^*, x_{n-1}^*, \ldots, x_{n-k}^*\}$, there exists a sequence $l_s \to \infty$ such that

$$\lim_{s\to\infty} x_{l_s} = M \tag{2.13}$$

or

$$\lim_{s\to\infty} x_{l_s} = \frac{1}{M}. \tag{2.14}$$

We may suppose (by taking a subsequence) that for $1 \le i \le k+1$,

$$\lim_{s\to\infty} x_{l_s - i} = M_i. \tag{2.15}$$

From (2.12), it follows that $1/M \le M_i \le M$.

We claim that $M = 1$. Indeed, if $M > 1$, then $f_i(M_1, \ldots, M_{k+1}) \neq 1$, for some $i \in \{1,2,4,5\}$. If $\lim_{s\to\infty} x_{l_s} = M$, then it follows from Lemma 2.1 and (1.4) that

$$\begin{aligned} M &= \frac{\sum_{i=1}^{3} f_i(M_1, \ldots, M_{k+1}) + f_4(M_1, \ldots, M_{k+1}) f_5(M_1, \ldots, M_{k+1})}{f_1(M_1, \ldots, M_{k+1}) f_2(M_1, \ldots, M_{k+1}) + \sum_{i=3}^{5} f_i(M_1, \ldots, M_{k+1})} \\ &< \max\left\{\left[f_i(M_1, \ldots, M_{k+1})\right]^* : i = 1,2,4,5\right\} \\ &\le \max\left\{M_i : 1 \le i \le k+1\right\} \le M, \end{aligned} \tag{2.16}$$

which is a contradiction.

If $\lim_{s\to\infty} x_{l_s} = 1/M$, then it follows from Lemma 2.1 and (1.4) that

$$\begin{aligned} \frac{1}{M} &= \frac{\sum_{i=1}^{3} f_i(M_1, \ldots, M_{k+1}) + f_4(M_1, \ldots, M_{k+1}) f_5(M_1, \ldots, M_{k+1})}{f_1(M_1, \ldots, M_{k+1}) f_2(M_1, \ldots, M_{k+1}) + \sum_{i=3}^{5} f_i(M_1, \ldots, M_{k+1})} \\ &> \frac{1}{\max\left\{\left[f_i(M_1, \ldots, M_{k+1})\right]^* : i = 1,\ 2,\ 4,\ 5\right\}} \\ &\ge \frac{1}{\max\left\{M_i : 1 \le i \le k+1\right\}} \ge \frac{1}{M}, \end{aligned} \tag{2.17}$$

which is a contradiction. This completes the proof of the claim.

By (1.4) and (2.12), it follows that $\lim_{n\to\infty} x_n = 1$ and

$$1 = \frac{\sum_{i=1}^{3} f_i(1, \ldots, 1) + f_4(1, \ldots, 1) f_5(1, \ldots, 1)}{f_1(1, \ldots, 1) f_2(1, \ldots, 1) + \sum_{i=3}^{5} f_i(1, \ldots, 1)}. \tag{2.18}$$

Thus, $\overline{x} = 1$ is the unique positive equilibrium of (1.4).

For any $0 < \varepsilon < 1$, choose $\delta = \varepsilon/(\varepsilon+1)$ and let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.4) with the initial values $x_{-k}, x_{-k+1}, \ldots, x_0 \in (1-\delta, 1+\delta)$. Then, for any $-k \le i \le 0$, we have that $x_i < 1+\varepsilon$ and $1/x_i < 1/(1-\delta) = 1+\varepsilon$. By (2.9) it follows that for any $n \ge 0$,

$$1 - \varepsilon < \frac{1}{p_0} \le \frac{1}{p_n} \le x_{n+1} \le p_n \le p_0 < 1 + \varepsilon, \tag{2.19}$$

which implies that $\overline{x} = 1$ is globally asymptotically stable. This completes the proof. □

3. Example

In this section, we will give an application of Theorem 1.1.

Example 3.1. Consider the following equation:

$$x_{n+1} = \frac{x_{n-i} + x_{n-j} + g(x_n, \ldots, x_{n-k}) + x_{n-s}x_{n-t}}{x_{n-i}x_{n-j} + g(x_n, \ldots, x_{n-k}) + x_{n-s} + x_{n-t}}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $k \in \{1, 2, \ldots\}$, $i, j, s, t \in \{0, 1 \ldots, k\}$, the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in \mathbb{R}_+$, and $g \in C([0, +\infty)^{k+1}, [0, +\infty))$. Then, $\overline{x} = 1$ is the unique positive equilibrium of (3.1) which is globally asymptotically stable.

Proof. Let $f_1(u_0, u_1, \ldots, u_k) = u_i$, $f_2(u_0, u_1, \ldots, u_k) = u_j$, $f_3(u_0, u_1, \ldots, u_k) = g(u_0, u_1, \ldots, u_k)$, $f_4(u_0, u_1, \ldots, u_k) = u_s$, and $f_5(u_0, u_1, \ldots, u_k) = u_t$. It is easy to verify that $[f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0^*, u_1^*, \ldots, u_k^*\}$, for $i = 1, 2, 4, 5$. By Theorem 1.1, we know that $\overline{x} = 1$ is the unique positive equilibrium of (3.1) which is globally asymptotically stable. □

Remark 3.2. Let $k \geq 3$, $f_1(u_0, u_1, \ldots, u_k) = 1$, $f_2(u_0, u_1, \ldots, u_k) = u_t$, for some $t \in \mathbb{Z}_k - \{j-1, j\}$, $f_3(u_0, u_1, \ldots, u_k) = \sum_{i \in \mathbb{Z}_k - \{j-1, j, t\}} u_i$, $f_4(u_0, u_1, \ldots, u_k) = u_{j-1}$, and $f_5(u_0, u_1, \ldots, u_k) = u_j$. Then, (1.4) is (1.1), since $[f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0^*, u_1^*, \ldots, u_k^*\}$, for $i = 1, 2, 4, 5$. By Theorem 1.1, we know that the unique positive equilibrium $\overline{x} = 1$ of (1.1) is globally asymptotically stable.

Remark 3.3. Let $k = 3$, $f_1(u_0, u_1, u_2, u_3) = u_0$, $f_2(u_0, u_1, u_2, u_3) = u_1$, $f_3(u_0, u_1, u_2, u_3) = 0$, $f_4(u_0, u_1, u_2, u_3) = u_2$, and $f_5(u_0, u_1, u_2, u_3) = u_3$. Then, (1.4) is (1.2), since $[f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0^*, u_1^*, \ldots, u_k^*\}$, for $i = 1, 2, 4, 5$. By Theorem 1.1, we know that the unique positive equilibrium $\overline{x} = 1$ of (1.2) is globally asymptotically stable.

Remark 3.4. Let $f_1(u_0, u_1, \ldots, u_k) = 1/u_0$, $f_2(u_0, u_1, \ldots, u_k) = u_1$, $f_3(u_0, u_1, \ldots, u_k) = u_2 + \cdots + u_{k-1}$, $f_4(u_0, u_1, \ldots, u_k) = u_k$, and $f_5(u_0, u_1, \ldots, u_k) = 1$. Then, (1.4) is (1.3), since $[f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0^*, u_1^*, \cdots, u_k^*\}$, for $i = 1, 2, 4, 5$. By Theorem 1.1, we know that the unique positive equilibrium $\overline{x} = 1$ of (1.3) is globally asymptotically stable.

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