Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2008, Article ID 154302, 14 pages
doi:10.1155/2008/154302

Research Article

An Existence Principle for Nonlocal Difference Boundary Value Problems with φ-Laplacian and Its Application to Singular Problems

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Received 15 November 2007; Accepted 22 January 2008

Recommended by Paul Eloe

The paper presents an existence principle for solving a large class of nonlocal regular discrete boundary value problems with the φ-Laplacian. Applications of the existence principle to singular discrete problems are given.

1. Introduction

Let $\mathbb{R}_+ = (0, \infty)$ and let $\mathbb{Z}$ denote the set of all integers. If $a, b \in \mathbb{Z}$, $a < b$, then $\mathbb{T}[a, b]$ denotes the discrete interval $\{a, a+1, \dots, b\}$. Let $\Delta u(k) = u(k+1) - u(k)$ be the forward difference operator.

Let $T, N \in \mathbb{Z}$, $T < N$, and let X stand for the space of functions $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ equipped with the norm $\|u\| = \max\{|u(k)| : k \in \mathbb{T}[T-1, N+1]\}$. Clearly, X is an $(N-T+3)$-dimensional Banach space.

Denote by $\mathcal{A}$ the set of continuous maps $\gamma : X \to \mathbb{R}$. We say that $\alpha, \beta \in \mathcal{A}$ *are compatible* if for each $\mu \in [0,1]$ the problem

$$\Delta(\phi(\Delta u(k-1))) = 0, \quad k \in \mathbb{T}[T, N], \tag{1.1}$$

$$\alpha(u) - \mu\alpha(-u) = 0, \qquad \beta(u) - \mu\beta(-u) = 0 \tag{1.2}$$

has a solution; that is, there exists a function $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ such that equality (1.1) holds for $k \in \mathbb{T}[T, N]$ and u satisfies (1.2). Here ϕ fulfils the following condition:

(H_1) $\phi \in C(\mathbb{R})$ is increasing such that $\phi(0) = 0$, $\phi(\mathbb{R}) = \mathbb{R}$.

Remark 1.1. It is easy to see that $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ is a solution of (1.1) if and only if $\Delta u(k) = B$ for $k \in \mathbb{T}[T-1, N]$, where $B \in \mathbb{R}$. Hence u is a solution of (1.1) if and only if $u(k) = A + Bk$ for $k \in \mathbb{T}[T-1, N-1]$, where $A, B \in \mathbb{R}$. Consequently, problem (1.1)-(1.2) has a solution if and only if the system

$$\begin{gathered} \alpha(A+Bk) - \mu\alpha(-A-Bk) = 0, \\ \beta(A+Bk) - \mu\beta(-A-Bk) = 0 \end{gathered} \tag{1.3}$$

has a solution $(A, B) \in \mathbb{R}^2$. If $\alpha, \beta \in \mathcal{A}$ are linear, then system (1.3) has the form

$$\begin{gathered} A\alpha(1) + B\alpha(k) = 0, \\ A\beta(1) + B\beta(k) = 0 \end{gathered} \tag{1.4}$$

for each $\mu \in [0, 1]$.

Remark 1.2. Due to Remark 1.1, $\alpha, \beta \in \mathcal{A}$ are compatible if system (1.3) has a solution $(A, B) \in \mathbb{R}^2$ for each $\mu \in [0, 1]$. If α, β are linear, then they are compatible. Indeed, system (1.3) has the form of (1.4) for each $\mu \in \mathbb{R}$ and is always solvable in $\mathbb{R}^2$ because $(A, B) = (0, 0)$ is its solution.

Let ϕ satisfy (H_1) and let $h \in C(\mathbb{T}[T, N] \times \mathbb{R}^2)$. We discuss the nonlocal difference problem

$$\Delta(\phi(\Delta u(k-1))) = h(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[T, N], \tag{1.5}$$

$$\alpha(u) = 0, \quad \beta(u) = 0, \quad \alpha, \beta \in \mathcal{A}, \tag{1.6}$$

where α, β are compatible. We say that $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ is *a solution of problem* (1.5)-(1.6) if u fulfils (1.6) and equality (1.5) holds for $k \in \mathbb{T}[T, N]$.

The first aim of this paper is to present an existence principle for solving problem (1.5)-(1.6) and the second aim is to give applications of this principle to singular problems with the ϕ-Laplacian, which include as special cases the Dirichlet problem and the mixed problem.

Singular discrete Dirichlet problems of the type

$$\begin{gathered} -\Delta\left(\phi_p(\Delta u(k-1))\right) = f(k, u(k)), \quad k \in \mathbb{T}[1, T], \\ u(0) = 0, \qquad u(T+1) = 0 \end{gathered} \tag{1.7}$$

were studied with $p = 2$ in [1] and [2–4], where $\phi_p(x) = |x|^{p-2}x$ $(p > 1)$ is the p-Laplacian, $f \in C(\mathbb{T}[1, T] \times (0, \infty))$, and $f(k, x)$ may be singular at $x = 0$. The existence of positive solutions is proved by variational methods [2] and by a combination of the lower and upper solutions method with a nonlinear alternative of Leray-Schauder type [1, 4] and an inequality theory [3]. In [1], the function f is nonnegative, while in [2–4] it may change sign. The paper [2] discusses also multiple positive solutions. The existence of multiple positive solutions is investigated also in [5, 6].

The paper [7] deals with the singular mixed problem

$$\begin{gathered} \Delta\left(\phi_p(\Delta u(k-1))\right) + f(k, u(k), \Delta u(k-1)) = 0, \quad k \in \mathbb{T}[1, T+1], \\ \Delta u(0) = 0, \qquad u(T+2) = 0, \end{gathered} \tag{1.8}$$

where $f \in C(\mathbb{T}[1, T+1] \times (0, \infty) \times \mathbb{R})$ and $f(k, x, y)$ may be singular at $x = 0$. The existence of a positive solution is proved by a combination of the lower and upper functions method with the Brouwer fixed-point theorem.

The rest of the paper is organized as follows. In Section 2, we present an existence principle for solving the discrete problem (1.5)-(1.6) (see Theorem 2.1). This principle is proved using the Brouwer degree and the Borsuk antipodal theorem (see, e.g., [8]). Notice that an analogous principle for continuous regular nonlocal problems with the ϕ-Laplacian was presented in [9, Theorem 2.1]. Section 3 is devoted to applications of the existence principle. We discuss the existence of positive solutions of the difference equation with the ϕ-Laplacian

$$\Delta(\phi(\Delta u(k-1))) = f(k, u(k), \Delta u(k)) \tag{1.9}$$

satisfying two types of nonlocal boundary conditions which include as special cases the Dirichlet problem and the mixed problem. Here f is continuous and $f(k,x,y)$ may be singular at $y=0$. The existence of positive solutions is proved by a combination of regularization and sequential techniques with our existence principle. The results are demonstrated with examples.

2. Existence principle

The following theorem is an existence principle for problem (1.5)-(1.6).

Theorem 2.1. *Let* (H_1) *hold. Let* $h \in C(\mathbb{T}[T,N] \times \mathbb{R}^2)$ *and let* $\alpha, \beta \in \mathcal{A}$ *be compatible. Suppose that there exists a positive constant* S *independent of* λ *such that*

$$\|u\| < S \tag{2.1}$$

for any solution u *of the problem*

$$\begin{gathered}\Delta(\phi(\Delta u(k-1))) = \lambda h(k, u(k), \Delta u(k)), \quad \lambda \in [0,1], \\ \alpha(u) = 0, \qquad \beta(u) = 0.\end{gathered} \tag{2.2}$$

Also assume that there exists a positive constant Λ *such that*

$$\max\{|A|, |B|\} < \Lambda \tag{2.3}$$

for all solutions $(A,B) \in \mathbb{R}^2$ *of system* (1.3) *for each* $\mu \in [0,1]$.

Then problem (1.5)-(1.6) *has a solution.*

Proof. Put $L = (1 + \max\{|T-1|, |N+1|\}\Lambda$ and

$$\Omega = \{u \in X : \|u\| < \max\{S, L\}\}. \tag{2.4}$$

Then Ω is an open, bounded, and symmetric subset of the Banach space X with respect to $0 \in X$. Define an operator $\mathcal{P} : [0,1] \times \overline{\Omega} \to X$ by the formula

$$\mathcal{P}(\lambda, u)(k) = \sum_{j=T}^{k} \phi^{-1}\left(\phi(\Delta u(T-1) + \beta(u)) + \lambda \sum_{s=T}^{j-1} h(s, u(s), \Delta u(s))\right) + u(T-1) + \alpha(u) \tag{2.5}$$

for $k \in \mathbb{T}[T,N]$, where $\sum_{i=T}^{T-1} = 0$. It follows from the continuity of the functions ϕ, ϕ^{-1}, f and the maps α, β that $\mathcal{P}$ is a continuous operator. Suppose that u is a fixed point of $\mathcal{P}(\lambda, \cdot)$ for some $\lambda \in [0,1]$. Then

$$u(k) = \sum_{j=T}^{k} \phi^{-1}\left(\phi(\Delta u(T-1) + \beta(u)) + \lambda \sum_{s=T}^{j-1} h(s, u(s), \Delta u(s))\right) + u(T-1) + \alpha(u) \tag{2.6}$$

for $k \in \mathbb{T}[T,N]$. We set $k = T-1$ and $k = T$ in (2.6), and have $u(T-1) = u(T-1) + \alpha(u)$ and $u(T) = \Delta u(T-1) + \beta(u) + u(T-1) + \alpha(u)$. Hence $\alpha(u) = 0$ and $\beta(u) = 0$, which means that u satisfies the boundary conditions (1.6). In addition,

$$\Delta u(k) = u(k+1) - u(k) = \phi^{-1}\left(\phi(\Delta u(T-1) + \beta(u)) + \lambda\sum_{s=T}^{k} h(s,u(s),\Delta u(s))\right), \tag{2.7}$$

and consequently

$$\Delta(\phi(\Delta u(k-1))) = \phi(\Delta u(k)) - \phi(\Delta u(k-1)) = \lambda h(k,u(k),\Delta u(k)) \tag{2.8}$$

for $k \in \mathbb{T}[T,N]$. Hence u is a solution of the equation in (2.2). We have proved that for each $\lambda \in [0,1]$ any fixed point of the operator $\mathcal{P}(\lambda,\cdot)$ is a solution of problem (2.2). In particular, any fixed point of $\mathcal{P}(1,\cdot)$ is a solution of problem (1.5)-(1.6). In order to prove the solvability of problem (1.5)-(1.6), it suffices to show, by the Brouwer degree theory, that

$$d(\mathcal{I} - \mathcal{P}(1,\cdot),\Omega,0) \neq 0, \tag{2.9}$$

where "d" stands for the Brouwer degree and $\mathcal{I}$ is the identical operator on X. We know that $\mathcal{P}$ is a continuous operator and, by the assumptions of our theorem, for each $\lambda \in [0,1]$ and any fixed point u of $\mathcal{P}(\lambda,\cdot)$ the estimate (2.1) is true with a positive constant S. Hence for each $\lambda \in [0,1]$, the operator $\mathcal{P}(\lambda,\cdot)$ is fixed point free on the boundary $\partial\Omega$ of Ω. Consequently, by the homotopy property,

$$d(\mathcal{I} - \mathcal{P}(1,\cdot),\Omega,0) = d(\mathcal{I} - \mathcal{P}(0,\cdot),\Omega,0). \tag{2.10}$$

We now define an operator $\mathcal{L} : [0,1] \times \overline{\Omega} \to X$ by the formula

$$\mathcal{L}(\mu,u)(k) = \begin{cases} u(T-1) + \alpha(u) - \mu\alpha(-u) \\ \quad +(k+1-T)[\Delta u(T-1) + \beta(u) - \mu(\beta(-u))] \\ \qquad\qquad\qquad\qquad \text{for } k \in \mathbb{T}[T-1,N+1]. \end{cases} \tag{2.11}$$

The operator $\mathcal{L}$ is continuous because of the continuity of α, β. In addition, $\mathcal{L}(0,\cdot) = \mathcal{P}(0,\cdot)$ and $\mathcal{L}(1,\cdot)$ is an odd operator, that is, $\mathcal{L}(1,-u) = -\mathcal{L}(1,u)$ for $u \in \overline{\Omega}$. Suppose that u_0 is a fixed point of $\mathcal{L}(\mu,\cdot)$ for some $\mu \in [0,1]$. Then

$$u_0(k) = \begin{cases} u_0(T-1) + \alpha(u_0) - \mu\alpha(-u_0) \\ \quad +(k+1-T)[\Delta u_0(T-1) + \beta(u_0) - \mu(\beta(-u_0))] \\ \qquad\qquad\qquad\qquad \text{for } k \in \mathbb{T}[T-1,N+1]. \end{cases} \tag{2.12}$$

Therefore

$$u_0(T-1) = u_0(T-1) + \alpha(u_0) - \mu\alpha(-u_0), \tag{2.13}$$

$$u_0(T) = u_0(T-1) + \alpha(u_0) - \mu\alpha(-u_0) + \Delta u_0(T-1) + \beta(u_0) - \mu\beta(-u_0), \tag{2.14}$$

$$u_0(k+1) - u_0(k) = \Delta u_0(T-1) + \beta(u_0) - \mu\beta(-u_0), \quad k \in \mathbb{T}[T,N]. \tag{2.15}$$

Then, by (2.13) and (2.14),

$$\alpha(u_0) - \mu\alpha(-u_0) = 0, \qquad \beta(u_0) - \mu\beta(-u_0) = 0, \tag{2.16}$$

which combined with (2.15) yield $\Delta u_0(k) = \Delta u_0(T-1)$ for $k \in \mathbb{T}[T, N]$. Hence

$$u_0(k) = A + kB \quad \text{for } k \in \mathbb{T}[T-1, N+1], \tag{2.17}$$

where $A = u_0(T-1) + (1-T)\Delta u_0(T-1)$ and $B = \Delta u_0(T-1)$. It follows from (2.16) and (2.17) that (A, B) is a solution of system (1.3) and therefore $\max\{|A|, |B|\} < \Lambda$ by the assumptions of our theorem. From this we conclude that $\|u_0\| < (1 + \max\{|T-1|, |N+1|\}\Lambda$. As a result for each $\mu \in [0,1]$ and any fixed point u of $\mathcal{L}(\mu, \cdot)$, we have $u \notin \partial\Omega$. Hence, by the Borsuk antipodal theorem and the homotopy property,

$$d(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0) \neq 0, \qquad d(\mathcal{I} - \mathcal{L}(0,\cdot), \Omega, 0) = d(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0). \tag{2.18}$$

Relation (2.9) follows from $\mathcal{L}(0,\cdot) = \mathcal{P}(0,\cdot)$ and from (2.10) and (2.18). □

3. Applications of the existence principle

Theorem 2.1 presents an existence principle which can be used for a large class of nonlocal boundary value problems. In this section, we apply Theorem 2.1 to prove the existence of positive solutions of a generalized singular Dirichlet problem and a generalized singular mixed problem. Both of these problems are called "generalized" since by the special choice of their boundary conditions we obtain the Dirichlet conditions $u(-N-1) = C$, $u(N+1) = C$ and the mixed conditions $\Delta u(0) = 0$, $u(N+1) = C$.

3.1. Generalized singular Dirichlet problem

Denote by $\mathcal{C}_1$ the set of functions $q \in C(\mathbb{R}^2)$ such that

(i) $q(x, y)$ is increasing in x and nondecreasing in y,

(ii) $q(x, y) = -q(-x, -y)$ for $(x, y) \in \mathbb{R}^2$,

(iii) $\lim_{x\to\infty} q(x, 0) = \infty$.

It is obvious that for each $q \in \mathcal{C}_1$ we have $q(0,0) = 0$ and $q(x, y) > 0$ for $(x, y) \in \mathbb{R}^2_+$.

Let $N \geq 1$ be a positive integer. We discuss the singular boundary value problem

$$\Delta(\phi(\Delta u(k-1))) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[-N, N], \tag{3.1}$$

$$\begin{gathered} q(u(-N-1), -\Delta u(-N-1)) = C, \\ q(u(N+1), \Delta u(N)) = C, \quad q \in \mathcal{C}_1, \ C > 0, \end{gathered} \tag{3.2}$$

where ϕ satisfies (H_1) and f satisfies the condition

(H_2) $f \in C(\mathbb{T}[-N, N] \times \mathfrak{D})$, $\mathfrak{D} = [0, \infty) \times (\mathbb{R} \setminus \{0\})$, $f(k, x, y) > 0$ for $k \in \mathbb{T}[-N, N]$, $(x, y) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, $f(k, 0, y) = 0$ for $k \in \mathbb{T}[-N, N]$, $y \in \mathbb{R} \setminus \{0\}$, and for each $k \in \mathbb{T}[-N, N]$, $\lim_{y\to 0} f(k, x, y) = \infty$ locally uniformly on $\mathbb{R}_+$.

We say that $u \in \mathbb{T}[-N-1, N+1] \to \mathbb{R}$ is *a solution of problem* (3.1)-(3.2) if u satisfies the boundary conditions (3.2) and fulfils equality (3.1) for $k \in \mathbb{T}[-N, N]$.

Notice that a special case of the boundary conditions (3.2) is the Dirichlet conditions $u(-N-1) = C$, $u(N+1) = C$ which we get by setting $q(x, y) = x$.

We apply sequential and regularization methods to show the existence of a solution of problem (3.1)-(3.2). To this end, for each $n \in \mathbb{N}$ define $f_n \in C(\mathbb{T}[-N, N] \times \mathbb{R}^2)$ by the formula

$$f_n(k,x,y) = \begin{cases} f_*(k,x,y) & \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in \mathbb{R} \times \left(\mathbb{R} \setminus \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]\right), \\ \dfrac{n}{2}\left[f_*\left(k,x,\dfrac{1}{n}\right)\left(y+\dfrac{1}{n}\right) - f_*\left(k,x,-\dfrac{1}{n}\right)\left(y-\dfrac{1}{n}\right)\right] & \\ \qquad \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in \mathbb{R} \times \left[-\dfrac{1}{n}, \dfrac{1}{n}\right], \end{cases} \tag{3.3}$$

where

$$f_*(k,x,y) = \begin{cases} f(k,x,y) & \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in \mathfrak{D}, \\ 0 & \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in (-\infty,0) \times (\mathbb{R} \setminus \{0\}). \end{cases} \tag{3.4}$$

If condition (H_2) holds, then

$$f_n(k,x,y) > 0 \quad \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.5}$$

$$f_n(k,x,y) = 0 \quad \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in (-\infty,0] \times \mathbb{R}, \tag{3.6}$$

$$\lim_{n\to\infty} f_n(k,x,y) = f(k,x,y) \quad \text{for } k \in \mathbb{T}[-N,N],\ (x,y) \in [0,\infty) \times (\mathbb{R} \setminus \{0\}). \tag{3.7}$$

Throughout this section, X denotes the Banach space of functions $u : \mathbb{T}[-N-1, N+1] \to \mathbb{R}$ with the norm $\|u\| = \max\{|u(k)| : k \in \mathbb{T}[-N-1, N+1]\}$.

Keeping in mind the boundary conditions (3.2), put

$$\begin{gathered} \alpha(u) = q(u(-N-1), -\Delta u(-N-1)) - C, \\ \beta(u) = q(u(N+1), \Delta u(N)) - C, \quad q \in \mathcal{C}_1,\ C > 0, \end{gathered} \tag{3.8}$$

for $u \in X$. Then $\alpha, \beta \in \mathcal{A}$ and we can write the boundary conditions (3.2) in the form of (1.6).

Lemma 3.1. *Let $\alpha, \beta \in \mathcal{A}$ be defined in* (3.8). *Then for each $\mu \in [0,1]$ system* (1.3) *has a unique solution $(A, B) \in \mathbb{R}^2$ and there exists a positive constant Λ independent of μ such that*

$$\max\{|A|, |B|\} < \Lambda. \tag{3.9}$$

Proof. Using property (ii) of $q \in \mathcal{C}_1$ we can write system (1.3) in the form

$$\begin{gathered} q(A - (N+1)B, -B) = \frac{(1-\mu)C}{1+\mu}, \\ q(A + (N+1)B, B) = \frac{(1-\mu)C}{1+\mu}. \end{gathered} \tag{3.10}$$

Suppose that some $(A,B) \in \mathbb{R}^2$ is a solution of (3.10). If $B \neq 0$, then $q(A-(N+1)B,-B) \neq q(A+(N+1)B,B)$ due to property (i) of functions belonging to the set $\mathcal{C}_1$, which is impossible. Hence $B=0$ and $q(A,0)=(1-\mu)C/(1+\mu)$. Put

$$p(x)=q(x,0) \quad \text{for } x \in \mathbb{R}. \tag{3.11}$$

Then $p \in C(\mathbb{R})$ is increasing and odd on $\mathbb{R}$ and $\lim_{x\to\infty} p(x)=\infty$. Therefore $A=p^{-1}((1-\mu)C/(1+\mu))$ is the unique solution of the equation $q(A,0)=(1-\mu)C/(1+\mu)$. It is easy to check that $(A,B)=(p^{-1}((1-\mu)C/(1+\mu)),0)$ is a solution of system (1.3) for each $\mu \in [0,1]$. This proves that system (1.3) has the unique solution $(A,B)=(p^{-1}((1-\mu)C/(1+\mu)),0)$ for each $\mu \in [0,1]$. It follows from the inequality $0 \le p^{-1}((1-\mu)C/(1+\mu)) \le P^{-1}(C)$ that (A,B) fulfils the estimate (3.9) with $\Lambda = p^{-1}(C)+1$. □

Remark 3.2. Due to Lemma 3.1 and Remark 1.2 the boundary conditions (3.2) are compatible.

The following result gives the properties of solutions to a regular problem depending on a parameter λ.

Lemma 3.3. *Let (H_1) and (H_2) hold. Let u be a solution of the equation*

$$\Delta(\phi(\Delta u(k-1))) = \lambda f_n(k,u(k),\Delta u(k)), \quad k \in \mathbb{T}[-N,N],\ \lambda \in (0,1], \tag{3.12}$$

fulfilling the boundary conditions (3.2)*. Then there exists a positive constant S independent of n and λ such that*

$$0 < u(k) < S \quad \textit{for } k \in \mathbb{T}[-N-1,N+1], \tag{3.13}$$

$$\Delta u(k-1) < \Delta u(k) \quad \textit{for } k \in \mathbb{T}[-N,N], \tag{3.14}$$

$$\Delta u(-N-1) < 0, \qquad \Delta u(N) > 0. \tag{3.15}$$

Proof. Suppose that $u(N+1) \le 0$. If $\Delta u(N) \le 0$, then $q(u(N+1),\Delta u(N)) \le q(0,0)=0$, contrary to $q(u(N+1),\Delta u(N)) = C > 0$. Hence $\Delta u(N) > 0$ and therefore $u(N) < u(N+1) \le 0$, which gives $\Delta(\phi(\Delta u(N-1))) = 0$ because $f_n(N,u(N),\Delta u(N)) = 0$ by (3.6). It follows from $\Delta(\phi(\Delta u(N-1))) = 0$, $\Delta u(N) > 0$, and from condition (H_1) that $\Delta u(N-1) = \Delta u(N) > 0$, and consequently $u(N-1) < u(N) < 0$. Applying the above arguments repeatedly, we get $\Delta u(j) = \Delta u(N)$ for $j \in \mathbb{T}[-N-1,N]$. Then $\Delta u(-N-1) > 0$ and $u(-N-1) < u(N) < 0$, which yields $q(u(-N-1),-\Delta u(-N-1)) < 0$, contrary to $q(u(-N-1),-\Delta u(-N-1)) = C > 0$ by (3.2). Hence $u(N+1) > 0$. Suppose that there exists $j \in \mathbb{T}[-N-1,N]$ such that $u(j) \le 0$ and $u(j+1) > 0$. If $j > -N-1$, then $\Delta(\phi(\Delta u(j-1))) = \lambda f_n(j,u(j),\Delta u(j)) = 0$ and therefore $\Delta u(j-1) = \Delta u(j)$, which gives $u(j-1) < u(j)$ because $\Delta u(j) > 0$. Essentially, the same reasoning as in the above part of the proof yields $\Delta u(k) = \Delta u(j) > 0$ for $k \in \mathbb{T}[-N-1,j]$. In particular, $u(-N-1) < u(j) \le 0$ and $\Delta u(-N-1) > 0$. Consequently, $q(u(-N-1),-\Delta u(-N-1)) < 0$, which is impossible by (3.2). If $j=-N-1$, then $u(-N-1) \le 0$ and $\Delta(-N-1) > 0$, which gives $q(u(-N-1),-\Delta u(-N-1)) \le 0$, contrary to (3.2). We have

$$u(k) > 0 \quad \text{for } k \in \mathbb{T}[-N-1,N+1]. \tag{3.16}$$

Then $f_n(k,u(k),\Delta u(k)) > 0$ for $k \in \mathbb{T}[-N,N]$ by (3.5) and so $\Delta(\phi(\Delta u(k-1))) > 0$ for these k, which means that inequality (3.14) is true.

We now prove that inequality (3.15) holds. Suppose that $\Delta u(-N-1) \geq 0$. Then $\Delta u(k) > \Delta u(-N-1) \geq 0$ for $k \in \mathbb{T}[-N, N]$ by (3.14) and $u(N+1) - u(-N-1) = \sum_{k=-N}^{N} \Delta u(k) > 0$. In particular, $\Delta u(N) > 0$ and

$$u(N+1) > u(-N-1). \tag{3.17}$$

Hence $C = q(u(-N-1), -\Delta u(-N-1)) \leq q(u(-N-1), 0)$, $C = q(u(N+1), \Delta u(N)) \geq q(u(N+1), 0)$. Therefore $q(u(-N-1), 0) \geq q(u(N+1), 0)$, which contradicts (3.17), because the function $q(\cdot, 0)$ is increasing on $\mathbb{R}$. We have shown that the first inequality in (3.15) holds. In order to prove that the second inequality in (3.15) is true we assume, on the contrary, that $\Delta u(N) \leq 0$. By (3.14), $\Delta u(k) < \Delta u(N) \leq 0$ for $k \in \mathbb{T}[-N-1, N-1]$ and so $u(N+1) - u(-N-1) = \sum_{k=-N}^{N} \Delta u(k) < 0$. It follows from $C = q(u(-N-1), -\Delta u(-N-1)) \geq q(u(-N-1), 0)$ and $C = q(u(N+1), \Delta u(N)) \leq q(u(N+1), 0)$ that $q(u(-N-1), 0) \leq q(u(N+1), 0)$, which contradicts $u(N+1) < u(-N-1)$, because $q(\cdot, 0)$ is increasing on $\mathbb{R}$.

It remains to prove that $u(k) < S$ for $k \in \mathbb{T}[-N-1, N+1]$, where S is a positive constant independent of n and λ. We see from (3.14) and (3.15) that there exists $j \in \mathbb{T}[-N, N-1]$ such that

$$\Delta u(k) < 0 \quad \text{for } k \in \mathbb{T}[-N-1, j-1], \qquad \Delta u(k) > 0 \quad \text{for } k \in \mathbb{T}[j+1, N]. \tag{3.18}$$

Hence $u(k) \leq \max\{u(-N-1), u(N+1)\}$ for $k \in \mathbb{T}[-N-1, N+1]$. We conclude from $C = q(u(-N-1), -\Delta u(-N-1)) \geq q(u(-N-1), 0)$, $C = q(u(N+1), \Delta u(N)) \geq q(u(N+1), 0)$ that $q(u(-N-1), 0) \leq C$, $q(u(N+1), 0) \leq C$, and consequently $\max\{u(-N-1), u(N+1)\} \leq p^{-1}(C)$, where p^{-1} is the inverse function to p given in (3.11). Therefore estimate (3.13) holds with $S = p^{-1}(C) + 1$. □

Remark 3.4. Problem (3.12)–(3.2) with $\lambda = 0$ has the unique solution u, $u(k) = p^{-1}(C)$, for $k \in \mathbb{T}[-N-1, N+1]$, where p is given in (3.11). This fact follows from Remark 1.1 and from the proof of Lemma 3.1 with $\mu = 0$.

The next lemma gives an existence result for problem (3.19)–(3.2), where

$$\Delta(\phi(\Delta u(k-1))) = f_n(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[-N, N]. \tag{3.19}$$

Lemma 3.5. *Let (H_1) and (H_2) hold. Then for each $n \in \mathbb{N}$ there exists a solution of problem* (3.19)–(3.2) *and any of its solutions u_n fulfils the inequalities*

$$0 < u_n(k) < S \quad \text{for } k \in \mathbb{T}[-N-1,\ N+1], \tag{3.20}$$

where S is a positive constant independent of n, and

$$\Delta u_n(k-1) < \Delta u_n(k) \quad \text{for } k \in \mathbb{T}[-N, N], \tag{3.21}$$

$$\Delta u_n(-N-1) < 0, \qquad \Delta u_n(N) > 0. \tag{3.22}$$

Proof. Let us choose $n \in \mathbb{N}$. Put $h(k, x, y) = f_n(k, x, y)$ for $k \in \mathbb{T}[-N, N]$, $(x, y) \in \mathbb{R}^2$ and let $\alpha, \beta \in \mathcal{A}$ be given in (3.8). By Remark 3.2, the boundary conditions (3.2) are compatible. Due to Lemma 3.3 and Remark 3.4 there exists a positive constant S such that $\|u\| < S$ for all solutions u of problem (2.2). By Lemma 3.1, there exists a positive constant Λ such that estimate (3.9) is true for any solutions $(A, B) \in \mathbb{R}^2$ of problem (1.3) for each $\mu \in [0, 1]$. Hence the conditions of Theorem 2.1 are satisfied and therefore problem (3.19)–(3.2) has a solution. In addition, any of its solutions u_n fulfils inequalities (3.20)–(3.22) by Lemma 3.3. □

The main existence result for problem (3.1)-(3.2) is given in the following theorem.

Theorem 3.6. *Let* (H_1) *and* (H_2) *hold. The problem* (3.1)-(3.2) *has a solution* u *and* $u(k) > 0$ *for* $k \in \mathbb{T}[-N-1, N+1]$.

Proof. By Lemma 3.5, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (3.19)–(3.2) satisfying inequalities (3.20)–(3.22). As a result, the sequence $\{u_n(k)\}$ is bounded for $k \in \mathbb{T}[-N-1, N+1]$, and therefore by the Bolzano-Weierstrass compactness theorem, there exist a subsequence $\{\ell_n\}$ of $\{n\}$ and some $u \in X$ such that $\lim_{n\to\infty} u_{\ell_n} = u$. Letting $n \to \infty$ in (3.20)–(3.22) (with ℓ_n instead of n) and in the boundary conditions $q(u_{\ell_n}(-N-1), -\Delta u_{\ell_n}(-N-1)) = C$, $q(u_{\ell_n}(N+1), -\Delta u_{\ell_n}(N1)) = C$, we obtain

$$0 \le u(k) \le S \quad \text{for } k \in \mathbb{T}[-N-1, N+1], \tag{3.23}$$

$$\Delta u(k-1) \le \Delta u(k) \quad \text{for } k \in \mathbb{T}[-N, N], \tag{3.24}$$

$$\Delta u(-N-1) \le 0, \qquad \Delta u(N) \ge 0, \tag{3.25}$$

and u satisfies the boundary conditions (3.2).

If $u(N+1) = 0$, then $u(N) = -\Delta u(N)$, and since $u(N) \ge 0$ by (3.23) and $\Delta u(N) \ge 0$ by (3.25), we have $\Delta u(N) = 0$. Hence $q(u(N+1), \Delta u(N)) = q(0,0) = 0$, contrary to (3.2). We have $u(N+1) > 0$. In order to prove that $u(k) > 0$ for $k \in \mathbb{T}[-N-1, N]$ we first assume that there exists $j \in \mathbb{T}[-N, N]$ such that $u(j) = 0$ and $u(k) > 0$ for $k \in \mathbb{T}[j+1, N+1]$. Then $\Delta u(j) > 0$ and therefore

$$\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(j-1)\big)\big) = \lim_{n\to\infty} f_{\ell_n}\big(j, u_{\ell_n}(j), \Delta u_{\ell_n}(j)\big) = f(j, 0, \Delta u(j)) = 0, \tag{3.26}$$

by (3.7) and (H_2). Since $\lim_{n\to\infty}\Delta(\phi(\Delta u_{\ell_n}(j-1))) = \Delta(\phi(\Delta u(j-1)))$, we have $\Delta(\phi(\Delta u(j-1))) = 0$. Consequently, $\Delta u(j-1) = \Delta u(j) > 0$, which contradicts $u(j-1) = -\Delta u(j-1) < 0$ and (3.23). We have proved that $u(k) > 0$ for $k \in \mathbb{T}[-N, N+1]$. If $u(-N-1) = 0$, then it follows from $u(-N) \ge 0$, and $\Delta u(-N-1) \le 0$ by (3.23) and (3.25) that $u(-N) = 0$, $\Delta u(-N-1) = 0$, and consequently $q(u(-N-1), \Delta u(-N-1)) = q(0,0) = 0$, contrary to (3.2). Hence $u(-N-1) > 0$. To summarize, we have

$$u(k) > 0 \quad \text{for } k \in [-N-1,\ N+1]. \tag{3.27}$$

We now prove that

$$\Delta u(k) \ne 0 \quad \text{for } k \in [-N, N]. \tag{3.28}$$

On the contrary, suppose that $\Delta u(j) = 0$ for some $j \in \mathbb{T}[-N, N]$. Then $\lim_{n\to\infty} f_{\ell_n}(j, u_{\ell_n}(j), \Delta u_{\ell_n}(j)) = \infty$ by (H_2) since $\lim_{n\to\infty} u_{\ell_n}(j) = u(j) > 0$ and $(\ell_n/2)\max\{\Delta u_{\ell_n}(j) + 1/\ell_n, -\Delta u_{\ell_n}(j) + 1/\ell_n\} \ge 1/2$ for each n such that $|\Delta u_{\ell_n}(j)| \le 1/\ell_n$. Therefore $\lim_{n\to\infty}\Delta(\phi(\Delta u_{\ell_n}(j-1))) = \lim_{n\to\infty} f_{\ell_n}(j, u_{\ell_n}(j), \Delta u_{\ell_n}(j)) = \infty$, which contradicts $\lim_{n\to\infty}\Delta(\phi(\Delta u_{\ell_n}(j-1))) = \Delta(\phi(\Delta u(j-1))) \in \mathbb{R}$.

Keeping in mind (3.27) and (3.28), we have

$$\begin{aligned}\Delta(\phi(\Delta u(k-1))) &= \lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(k-1)\big)\big) \\ &= \lim_{n\to\infty} f_{\ell_n}\big(k, u_{\ell_n}(k), \Delta u_{\ell_n}(k)\big) \\ &= f(k, u(k), \Delta u(k))\end{aligned} \tag{3.29}$$

for $k \in \mathbb{T}[-N, N]$, which means that u is a solution of (3.1). Hence u is a positive solution of problem (3.1)-(3.2). □

Example 3.7. Let $a, b, c \in \mathbb{R}_+$, $\mu \geq 0$, and $n \in \mathbb{N}$. Then $f(k,x,y) = e^k \arctan x + x^a + x^b/|y|^c$, $k \in \mathbb{T}[-N,N]$, $(x,y) \in [0,\infty)\times(\mathbb{R}\setminus\{0\})$, satisfies condition (H$_2$) and $q(x,y) = x^{2n-1} + \mu(e^y - e^{-y})$, $(x,y) \in \mathbb{R}^2$, belongs to the set $\mathcal{C}_1$. If ϕ fulfils (H$_1$) then, by Theorem 3.6, the singular equation

$$\Delta(\phi(\Delta u(k-1))) = e^k \arctan(u(k)) + (u(k))^a + \frac{(u(k))^b}{|\Delta u(k)|^c}, \quad k \in \mathbb{T}[-N,N], \tag{3.30}$$

has a positive solution fulfilling the boundary conditions

$$\begin{gathered}(u(-N-1))^{2n-1} + \mu\left(e^{-\Delta u(-N-1)} - e^{\Delta u(-N-1)}\right) = C,\\ (u(N+1))^{2n-1} + \mu\left(e^{\Delta u(N)} - e^{-\Delta u(N)}\right) = C, \quad C > 0.\end{gathered} \tag{3.31}$$

3.2. Generalized singular mixed problem

In this section, $N \in \mathbb{N}$, $N > 1$. Denote by $\mathcal{C}_2$ the set of functions $Q \in C(\mathbb{R}^{N+1})$ such that

(i) $Q(x_1,\ldots,x_{N+1})$ is nondecreasing in its arguments $x_1,\ldots,x_N$ and increasing in x_{N+1},

(ii) $Q(x_1,\ldots,x_{N+1}) = -Q(-x_1,\ldots,-x_{N+1})$ for $(x_1,\ldots,x_{N+1}) \in \mathbb{R}^{N+1}$,

(iii) $\lim_{x_{N+1}\to\infty} Q(0,\ldots,0,x_{N+1}) = \infty$.

It is clear that for each $Q \in \mathcal{C}_2$ we have $Q(0,\ldots,0) = 0$ and $Q(x_1,\ldots,x_{N+1}) > 0$ for $(x_1,\ldots,x_{N+1}) \in \mathbb{R}_+^{N+1}$.

Consider the nonlocal singular boundary value problem

$$\Delta(\phi(\Delta(u(k-1))) = f(k,u(k),\Delta u(k)), \quad k \in \mathbb{T}[1,N], \tag{3.32}$$

$$\Delta u(0) = 0, \qquad Q(u(1),\ldots,u(N+1)) = C, \quad Q \in \mathcal{C}_2,\ C > 0, \tag{3.33}$$

where ϕ satisfies (H$_1$) and f fulfils the condition

(H$_3$) $f \in C(\mathbb{T}[1,N] \times \mathfrak{D})$, $\mathfrak{D} = [0,\infty) \times \mathbb{R}_+$, $f(k,x,y) > 0$ for $k \in \mathbb{T}[1,N]$, $(x,y) \in \mathbb{R}_+^2$, $f(k,0,y) = 0$ for $k \in \mathbb{T}[1,N]$, $y \in \mathbb{R}_+$, and $\lim_{y\to 0^+} f(1,x,y) = \infty$ locally uniformly on $\mathbb{R}_+$.

We say that $u \in \mathbb{T}[0,N+1] \to \mathbb{R}$ is a solution of problem (3.32)-(3.33) if u satisfies (3.33) and fulfils equality (3.32) for $k \in \mathbb{T}[1,N]$.

Notice that a special case of the boundary conditions (3.33) is the mixed conditions $\Delta u(0) = 0$, $u(N+1) = C$ which we get by setting $Q(x_1,\ldots,x_{N+1}) = x_{N+1}$.

The existence of a solution to problem (3.32)-(3.33) is proved by regularization and sequential techniques. To this end, for each $n \in \mathbb{N}$ define $f_n \in C(\mathbb{T}[1,N] \times \mathbb{R}^2)$ by the formula

$$f_n(k,x,y) = f^*\left(k, x, \max\left\{\frac{1}{n}, y\right\}\right), \quad k \in \mathbb{T}[1,N],\ (x,y) \in \mathbb{R}^2, \tag{3.34}$$

where

$$f^*(k,x,y) = \begin{cases} f(k,x,y) & \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in [0,\infty) \times \mathbb{R}_+, \\ 0 & \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in (-\infty,0) \times \mathbb{R}_+. \end{cases} \tag{3.35}$$

Under condition (H_3), we have

$$f_n(k,x,y) > 0 \quad \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in (0,\infty) \times \mathbb{R}, \tag{3.36}$$

$$f_n(k,x,y) = 0 \quad \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in (-\infty,0] \times \mathbb{R}, \tag{3.37}$$

$$\lim_{n\to\infty} f_n(k,x,y) = f(k,x,y) \quad \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in [0,\infty) \times \mathbb{R}_+. \tag{3.38}$$

Throughout this section, X denotes the Banach space of functions $u : \mathbb{T}[0,N+1] \to \mathbb{R}$ equipped with the norm $\|u\| = \max\{|u(k)| : k \in \mathbb{T}[0,N+1]\}$.

Finally, let $\alpha, \beta \in \mathcal{A}$ be defined on X by

$$\alpha(u) = \Delta u(0), \qquad \beta(u) = Q(u(1),\dots,u(N+1)) - C, \quad Q \in \mathcal{C}_2,\ C > 0. \tag{3.39}$$

Then we can write the boundary conditions (3.33) in the form of (1.6).

Lemma 3.8. *Let $\alpha, \beta \in \mathcal{A}$ be defined in* (3.39). *Then for each $\mu \in [0,1]$ system* (1.3) *has a unique solution $(A,B) \in \mathbb{R}^2$ and there exists a positive constant Λ independent of μ such that*

$$\max\{|A|,|B|\} < \Lambda. \tag{3.40}$$

Proof. Since α is a linear map and Q is an odd function, we can write system (1.3) in the form

$$\begin{gathered}(1+\mu)B = 0,\\ (1+\mu)Q(A+B,\dots,A+(N+1)B) = (1-\mu)C.\end{gathered} \tag{3.41}$$

In particular, $B = 0$ and A is a solution of the equation

$$Q(A,\dots,A) = \frac{(1-\mu)C}{1+\mu}. \tag{3.42}$$

Put $p(x) = Q(x,\dots,x)$ for $x \in \mathbb{R}$. Then $p \in C(\mathbb{R})$ is increasing on $\mathbb{R}$, $p(0) = 0$ and $\lim_{x\to\infty} p(x) = \infty$. Hence $A = p^{-1}((1-\mu)C/(1+\mu))$ is the unique solution of (3.42), and for each $\mu \in [0,1]$ we have $0 < A \le p^{-1}(C)$. To summarize, for each $\mu \in [0,1]$ system (1.3) has a unique solution $(A,B) = (p^{-1}((1-\mu)C/(1+\mu)),0)$ and the estimate (3.40) is true with $\Lambda = p^{-1}(C) + 1$. □

Remark 3.9. By Lemma 3.8 and Remark 1.2, the boundary conditions (3.33) are compatible.

Lemma 3.10. *Let (H_1) and (H_3) hold. Let $u : \mathbb{T}[1,N] \to \mathbb{R}$ be a solution of the equation*

$$\Delta(\phi(\Delta u(k-1))) = \lambda f_n(k,u(k),\Delta u(k)), \quad k \in \mathbb{T}[1,N],\ \lambda \in (0,1], \tag{3.43}$$

fulfilling the boundary conditions (3.33). *Then there exists a positive constant S independent of n and λ such that*

$$0 < u(k) < S \quad \textit{for } k \in \mathbb{T}[0,N+1], \tag{3.44}$$

$$\Delta u(k-1) < \Delta u(k) \quad \textit{for } k \in \mathbb{T}[1,N]. \tag{3.45}$$

Proof. Suppose that $u(0) \le 0$. Then $u(1) = u(0) \le 0$ and, by equality (3.37), $\Delta(\phi(\Delta u(0))) = \lambda f_n(1, u(1), \Delta u(1)) = 0$. Hence $\Delta u(1) = \Delta u(0) = 0$ and so $u(2) = u(0) \le 0$. Applying the above arguments repeatedly, we have $\Delta u(j-1) = \Delta u(0) = 0$ and $u(j) = u(0) \le 0$ for $j \in \mathbb{T}[2, N+1]$. Therefore $Q(u(1), \dots, u(N+1)) \le Q(0, \dots, 0) = 0$, which contradicts the fact that $Q(u(1), \dots, u(N+1)) = C > 0$ by (3.33). Consequently, $u(0) = u(1) > 0$. By (3.36) and (3.37), $f_n(k, u(k), \Delta u(k)) \ge 0$ for $k \in \mathbb{T}[1, N]$, which gives $\Delta(\phi(\Delta u(k-1))) \ge 0$ for these k. Therefore $\Delta u(k) \ge \Delta u(k-1)$ for $k \in \mathbb{T}[1, N]$. This and $\Delta u(0) = 0$ and $u(1) > 0$ yield

$$u(k) > 0 \quad \text{for } k \in [0, N+1]. \tag{3.46}$$

Then $\Delta(\phi(\Delta u(k-1))) = \lambda f_n(k, u(k), \Delta u(k)) > 0$ by (3.36), and consequently inequality (3.45) is true, which means that the sequence $\{u(k)\}_{k=1}^{N+1}$ is increasing and $\max\{u(k) : k \in \mathbb{T}[0, N+1]\} = u(N+1)$. It remains to prove that $u(N+1) < S$, where S is a positive constant independent of n and λ. To this end, put $r(x) = Q(0, \dots, 0, x)$ for $x \in \mathbb{R}$. Then $C = Q(u(1), \dots, u(N), u(N+1)) \ge Q(0, \dots, 0, u(N+1)) = r(u(N+1))$. Since $r \in C(\mathbb{R})$ is increasing on $\mathbb{R}$ and $\lim_{x \to \infty} r(x) = \infty$, it follows from the inequality $C \ge r(u(N+1))$ that $u(N+1) \le r^{-1}(C)$. Hence $u(N+1) < S$, where $S = r^{-1}(C) + 1$. Clearly, S is independent of n and λ. □

Remark 3.11. Let $\lambda = 0$ in (3.43). Then problem (3.43)–(3.33) has a unique solution u, $u(k) = p^{-1}(C)$, for $k \in \mathbb{T}[0, N+1]$, where p^{-1} is the inverse function to p defined by $p(x) = Q(x, \dots, x)$ for $x \in \mathbb{R}$. This fact follows from Remark 1.1 and the proof of Lemma 3.8 with $\mu = 0$. Since $p(x) \ge r(x)$ for $x \in \mathbb{R}_+$, we have $p^{-1}(C) \le r^{-1}(C)$. Here $r(x) = Q(0, \dots, 0, x)$ for $x \in \mathbb{R}$. Hence $0 < u(k) < S$ for $k \in \mathbb{T}[0, N+1]$, where $S = r^{-1}(C) + 1$.

The next lemma gives an existence result for problem (3.47)–(3.33), where

$$\Delta(\phi(\Delta u(k-1))) = f_n(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[1, N]. \tag{3.47}$$

Lemma 3.12. *Let (H_1) and (H_3) hold. Then for each $n \in \mathbb{N}$ there exists a solution of problem* (3.47)–(3.33) *and any of its solutions u_n satisfies the estimate*

$$0 < u_n(k) < S \quad \text{for } k \in \mathbb{T}[0, N+1], \tag{3.48}$$

where S is a positive constant independent of n and

$$\Delta u_n(k-1) < \Delta u_n(k) \quad \text{for } k \in \mathbb{T}[1, N]. \tag{3.49}$$

Proof. Let us choose $n \in \mathbb{N}$. Put $h(k, x, y) = f_n(k, x, y)$ for $k \in \mathbb{T}[1, N]$, $(x, y) \in \mathbb{R}^2$, and let $\alpha, \beta \in \mathcal{A}$ be given in (3.39). By Remark 3.9, the boundary conditions (3.33) are compatible, and it follows from Lemma 3.10 and Remark 3.11 that there exists a positive constant S independent of n such that $\|u\| < S$ for any solution u of problem (3.43)–(3.33), where $\lambda \in [0, 1]$. Besides, by Lemma 3.8, there exists a positive constant Λ such that estimate (3.40) holds for all solutions $(A, B) \in \mathbb{R}^2$ of problem (1.3) for each $\mu \in [0, 1]$. Therefore the conditions of Theorem 2.1 are fulfilled, and consequently problem (3.47)–(3.33) has a solution. In addition, any of its solutions u_n satisfies inequalities (3.48) and (3.49) by Lemma 3.10. □

We are now in a position to give our result for the solvability of problem (3.32)-(3.33).

Theorem 3.13. *Let* (H_1) *and* (H_3) *hold. Then problem* (3.32)-(3.33) *has a positive solution.*

Proof. Due to Lemma 3.12, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (3.47)–(3.33) satisfying inequalities (3.48) and (3.49). Hence the sequence $\{u_n(k)\}$ is bounded for each $k \in \mathbb{T}[0, N+1]$, and consequently by the Bolzano-Weierstrass compactness theorem, there exists a subsequence $\{\ell_n\}$ of $\{n\}$ and $u \in X$ such that $\lim_{n\to\infty} u_{\ell_n} = u$. Letting $n \to \infty$ in (3.48) and (3.49) (with ℓ_n instead of n) and in the boundary conditions $\Delta u_{\ell_n}(0) = 0$, $Q(u_{\ell_n}(1), \dots, u_{\ell_n}(N+1)) = C$, we have

$$0 \le u(k) \le S \quad \text{for } k \in \mathbb{T}[0, N+1], \tag{3.50}$$

$$\Delta u(k-1) \le \Delta u(k) \quad \text{for } k \in \mathbb{T}[1, N], \tag{3.51}$$

and u satisfies the boundary conditions (3.33). It follows from $\Delta u(0) = 0$ and inequalities (3.50)-(3.51) that

$$0 \le u(0) = u(1) \le u(2) \le \cdots \le u(N+1) \le S. \tag{3.52}$$

If $u(N+1) = 0$, then $u(k) = 0$ for $k \in \mathbb{T}[0, N+1]$. Therefore $Q(u(1), \dots, u(N+1)) = Q(0, \dots, 0) = 0$, contrary to (3.33). We have $u(N+1) > 0$. Suppose now that $u(N) = 0$. Then $\Delta u(N) = u(N+1) > 0$ and

$$\begin{aligned}
\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(N-1)\big)\big) &= \lim_{n\to\infty} f_{\ell_n}\big(N, u_{\ell_n}(N), \Delta u_{\ell_n}(N)\big) \\
&= \lim_{n\to\infty} f\left(N, u_{\ell_n}(N), \max\left\{\frac{1}{\ell_n}, \Delta u_{\ell_n}(N)\right\}\right) \\
&= f(N, 0, \Delta u(N)) \\
&= 0.
\end{aligned} \tag{3.53}$$

Since $\lim_{n\to\infty} \Delta(\phi(\Delta u_{\ell_n}(N-1))) = \Delta(\phi(\Delta u(N-1)))$, we have $\Delta(\phi(\Delta u(N-1))) = 0$. This gives $\Delta u(N-1) = \Delta u(N) > 0$ and therefore $u(N-1) = -\Delta u(N-1) < 0$, which is impossible. Hence $u(N) > 0$. Repeated application of the above arguments yields $u(k) > 0$ for $k \in \mathbb{T}[0, N-1]$. Hence

$$u(k) > 0 \quad \text{for } k \in \mathbb{T}[0, N+1]. \tag{3.54}$$

We proceed to show that

$$\Delta u(k) > 0 \quad \text{for } k \in \mathbb{T}[1, N]. \tag{3.55}$$

Suppose that $0 = \Delta u(0) = \Delta u(1)$. Then

$$\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(0)\big)\big) = \Delta(\phi(\Delta u(0))) = 0. \tag{3.56}$$

Since $\lim_{n\to\infty} u_{\ell_n}(1) = u(1) > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(0)\big)\big) &= \lim_{n\to\infty} f_{\ell_n}\big(1, u_{\ell_n}(1), \Delta u_{\ell_n}(1)\big) \\
&= \lim_{n\to\infty} f\left(1, u_{\ell_n}(1), \max\left\{\frac{1}{\ell_n}, \Delta u_{\ell_n}(1)\right\}\right) \\
&= \infty
\end{aligned} \tag{3.57}$$

by (H_3), contrary to (3.56). Hence $\Delta u(1) > 0$. From this and from (3.51), it follows that inequality (3.55) is true. Having in mind (3.55), we get

$$\begin{aligned}\Delta(\phi(\Delta u(k-1))) &= \lim_{n\to\infty}\Delta\left(\phi\left(\Delta u_{\ell_n}(k-1)\right)\right)\\ &= \lim_{n\to\infty} f\left(k, u_{\ell_n}(k), \max\left\{\frac{1}{\ell_n}, \Delta u_{\ell_n}(k)\right\}\right)\\ &= f(k, u(k), \Delta u(k))\end{aligned} \tag{3.58}$$

for $k \in \mathbb{T}[1,N]$. In particular, u is a solution of (3.32). Since u satisfies (3.33) and (3.41), it follows that u is a positive solution of problem (3.32)-(3.33). □

Example 3.14. Let $a, b, a_{N+1} \in \mathbb{R}_+$ and $a_j \in [0,\infty)$ for $j \in \mathbb{T}[1,N]$. Then $f(k,x,y) = (e^x - 1)(\ln k + x^a + 1/y^b)$, $k \in \mathbb{T}[1,N]$, $(x,y) \in [0,\infty) \times \mathbb{R}_+$, satisfies condition ($H_3$), and the function $Q(x_1,\dots,x_{N+1}) = \sum_{j=1}^{N+1} a_j x_j^{2j-1}$ belongs to the set $\mathcal{C}_2$. If ϕ fulfils (H_1) then, by Theorem 3.13, the singular problem

$$\begin{gathered}\Delta(\phi(\Delta u(k-1))) = \left(e^{u(k)} - 1\right)\left(\ln k + (u(k))^a + \frac{1}{(\Delta u(k))^b}\right), \quad k \in \mathbb{T}[1,N],\\ \Delta u(0) = 0, \qquad \sum_{j=1}^{N+1} a_j (u(j))^{2j-1} = C, \quad C > 0,\end{gathered} \tag{3.59}$$

has a positive solution.

Acknowledgments

This work is supported by Grant no. A100190703 of the Grant Agency of the Academy of Science of the Czech Republic and by the Council of Czech Government, MSM 6198959214.

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