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Research Article

Eigenvalue Problems for Systems of Nonlinear Boundary Value Problems on Time Scales

M. Benchohra, J. Henderson, and S. K. Ntouyas

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Values of λ are determined for which there exist positive solutions of the system of dynamic equations, $u^{\Delta\Delta}(t)+\lambda a(t)f(v(\sigma(t)))=0$, $v^{\Delta\Delta}(t)+\lambda b(t)g(u(\sigma(t)))=0$, for $t\in[0,1]_{\mathbf{T}}$, satisfying the boundary conditions, $u(0)=0=u(\sigma^2(1))$, $v(0)=0=v(\sigma^2(1))$, where $\mathbf{T}$ is a time scale. A Guo-Krasnosel'skii fixed point-theorem is applied.

1. Introduction

Let $\mathbf{T}$ be a time scale with $0, \sigma^2(1)\in\mathbf{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbf{T}} := J\cap\mathbf{T}. \tag{1.1}$$

We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions for the system of dynamic equations

$$\begin{aligned} u^{\Delta\Delta}(t)+\lambda a(t)f\big(v(\sigma(t))\big)=0, &\quad t\in[0,1]_{\mathbf{T}},\\ v^{\Delta\Delta}(t)+\lambda b(t)g\big(u(\sigma(t))\big)=0, &\quad t\in[0,1]_{\mathbf{T}}, \end{aligned} \tag{1.2}$$

satisfying the boundary conditions

$$u(0)=0=u\big(\sigma^2(1)\big),\qquad v(0)=0=v\big(\sigma^2(1)\big), \tag{1.3}$$

where

(a) $f,g \in C([0,\infty),[0,\infty))$,

(b) $a,b \in C([0,\sigma(1)]_{\mathbf{T}},[0,\infty))$, and each does not vanish identically on any closed subinterval of $[0,\sigma(1)]_{\mathbf{T}}$,

(c) all of $f_0 := \lim_{x\to 0^+}(f(x)/x)$, $g_0 := \lim_{x\to 0^+}(g(x)/x)$, $f_\infty := \lim_{x\to\infty}(f(x)/x)$, and $g_\infty := \lim_{x\to\infty}(g(x)/x)$ exist as real numbers.

There is an ongoing flurry of research activities devoted to positive solutions of dynamic equations on time scales (see, e.g., [1–7]). This work entails an extension of the paper by Chyan and Henderson [8] to eigenvalue problems for systems of nonlinear boundary value problems on time scales. Also, in that light, this paper is closely related to the works of Li and Sun [9, 10].

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [11–15] and as applications for which only positive solutions are meaningful [16–19]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [20–24].

The main tool in this paper is an application of the Guo-Krasnosel'skii fixed point-theorem for operators leaving a Banach space cone invariant [12]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Some preliminaries

In this section, we state the well-known Guo-Krasnosel'skii fixed point-theorem which we will apply to a completely continuous operator whose kernel, $G(t,s)$, is the Green function for

$$\begin{aligned} -y^{\Delta\Delta} &= 0, \\ y(0) = 0 &= y(\sigma^2(1)). \end{aligned} \tag{2.1}$$

Erbe and Peterson [6] have found that

$$G(t,s) = \frac{1}{\sigma^2(1)} \begin{cases} t(\sigma^2(1)-\sigma(s)), & \text{if } t \le s, \\ \sigma(s)(\sigma^2(1)-t), & \text{if } \sigma(s) \le t, \end{cases} \tag{2.2}$$

from which

$$G(t,s) > 0, \quad (t,s) \in (0,\sigma^2(1))_{\mathbf{T}} \times (0,\sigma(1))_{\mathbf{T}}, \tag{2.3}$$

$$G(t,s) \le G(\sigma(s),s) = \frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)}, \quad t \in [0,\sigma^2(1)]_{\mathbf{T}},\ s \in [0,\sigma(1)]_{\mathbf{T}}, \tag{2.4}$$

and it is also shown in [6] that

$$G(t,s) \ge kG(\sigma(s),s) = k\frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)}, \quad t \in \left[\frac{\sigma^2(1)}{4}, \frac{3\sigma^2(1)}{4}\right]_{\mathbf{T}},\ s \in [0,\sigma(1)]_{\mathbf{T}}, \tag{2.5}$$

where

$$k = \min\left\{\frac{1}{4}, \frac{\sigma^2(1)}{4(\sigma^2(1)-\sigma(0))}\right\}. \tag{2.6}$$

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.2), (1.3) if and only if

$$\begin{gathered} u(t) = \lambda \int_0^{\sigma(1)} G(t,s)a(s) f\left(\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s, \quad 0 \le t \le \sigma^2(1), \\ v(t) = \lambda \int_0^{\sigma(1)} G(t,s)b(s)g(u(\sigma(s)))\Delta s, \quad 0 \le t \le \sigma^2(1). \end{gathered} \tag{2.7}$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.2), (1.3) will be determined via applications of the following fixed point-theorem [12].

THEOREM 2.1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow \mathcal{P} \tag{2.8}$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Positive solutions in a cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (i.e., positive solutions) of (1.2), (1.3). Assume throughout that $[0, \sigma^2(1)]_{\mathbb{T}}$ is such that

$$\begin{gathered} \xi = \min\left\{t \in T \mid t \ge \frac{\sigma^2(1)}{4}\right\}, \\ \omega = \max\left\{t \in T \mid t \le \frac{3\sigma^2(1)}{4}\right\}; \end{gathered} \tag{3.1}$$

both exist and satisfy

$$\frac{\sigma^2(1)}{4} \le \xi < \omega \le \frac{3\sigma^2(1)}{4}. \tag{3.2}$$

Next, let $\tau \in [\xi, \omega]_{\mathbb{T}}$ be defined by

$$\int_\xi^\omega G(\tau,s)a(s)\Delta s = \max_{t \in [\xi,\omega]_{\mathbb{T}}} \int_\xi^\omega G(t,s)a(s)\Delta s. \tag{3.3}$$

Finally, we define

$$l = \min_{s \in [0,\sigma^2(1)]_{\mathbb{T}}} \frac{G(\sigma(\omega),s)}{G(\sigma(s),s)}, \tag{3.4}$$

and let

$$m = \min\{k, l\}. \tag{3.5}$$

For our construction, let $\mathcal{B} = \{x : [0,\sigma^2(1)]_{\mathbf{T}} \to \mathbb{R}\}$ with supremum norm $\|x\| = \sup\{|x(t)| : t \in [0,\sigma^2(1)]_{\mathbf{T}}\}$ and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [0,\sigma^2(1)]_{\mathbf{T}}, \text{ and } x(t) \geq m\|x\|, \text{ for } t \in [\xi,\sigma(\omega)]_{\mathbf{T}}\right\}. \tag{3.6}$$

For our first result, define positive numbers L_1 and L_2 by

$$\begin{aligned} L_1 &:= \max\left\{\left[m\int_{\xi}^{\omega} G(\tau,s)a(s)\Delta s f_{\infty}\right]^{-1}, \left[m\int_{\xi}^{\omega} G(\tau,s)b(s)\Delta s g_{\infty}\right]^{-1}\right\}, \\ L_2 &:= \min\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a(s)\Delta s f_0\right]^{-1}, \left[\int_0^{\sigma(1)} G(\sigma(s),s)b(s)\Delta s g_0\right]^{-1}\right\}, \end{aligned} \tag{3.7}$$

where we recall that $G(\sigma(s),s) = \sigma(s)(\sigma^2(1) - \sigma(s))/\sigma^2(1)$.

Theorem 3.1. *Assume that conditions (a), (b), and (c) are satisfied. Then, for each λ satisfying*

$$L_1 < \lambda < L_2, \tag{3.8}$$

there exists a pair (u,v) satisfying (1.2), (1.3) such that $u(x) > 0$ and $v(x) > 0$ on $(0,\sigma^2(1))_{\mathbf{T}}$.

Proof. Let λ be as in (3.8). And let $\epsilon > 0$ be chosen such that

$$\begin{aligned} &\max\left\{\left[m\int_{\xi}^{\omega} G(\tau,s)a(s)\Delta s(f_{\infty}-\epsilon)\right]^{-1}, \left[m\int_{\xi}^{\omega} G(\tau,s)b(s)\Delta s(g_{\infty}-\epsilon)\right]^{-1}\right\} \leq \lambda, \\ &\lambda \leq \min\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a(s)\Delta s(f_0+\epsilon)\right]^{-1}, \left[\int_0^{\sigma(1)} G(\sigma(s),s)b(s)\Delta s(g_0+\epsilon)\right]^{-1}\right\}. \end{aligned} \tag{3.9}$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) := \lambda\int_0^{\sigma(1)} G(t,s)a(s)f\left(\lambda\int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s, \quad u \in \mathcal{P}. \tag{3.10}$$

By the remarks in Section 2, we seek suitable fixed points of T in the cone $\mathcal{P}$.

Notice from (a), (b), and (2.3) that, for $u \in \mathcal{P}$, $Tu(t) \geq 0$ on $[0,\sigma^2(1)]_{\mathbf{T}}$. Also, for $u \in \mathcal{P}$, we have from (2.4) that

$$\begin{aligned} Tu(t) &= \lambda\int_0^{\sigma(1)} G(t,s)a(s)f\left(\lambda\int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\ &\leq \lambda\int_0^{\sigma(1)} G(\sigma(s),s)a(s)f\left(\lambda\int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \end{aligned} \tag{3.11}$$

so that

$$\|Tu\| \leq \lambda \int_0^{\sigma(1)} G(\sigma(s),s)a(s)f\left(\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s. \tag{3.12}$$

Next, if $u \in \mathcal{P}$, we have from (2.5), (3.5), and (3.10) that

$$\begin{aligned}\min_{t\in[\xi,\omega]_{\mathbb{T}}} Tu(t) &= \min_{t\in[\xi,\omega]_{\mathbb{T}}} \lambda \int_0^{\sigma(1)} G(t,s)a(s)f\left(\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\ &\geq \lambda m \int_0^{\sigma(1)} G(\sigma(s),s)a(s)f\left(\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\ &\geq m\|Tu\|.\end{aligned} \tag{3.13}$$

Consequently, $T:\mathcal{P}\to\mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \leq (f_0+\epsilon)x, \quad g(x) \leq (g_0+\epsilon)x, \quad 0 < x \leq H_1. \tag{3.14}$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.4) and choice of ϵ, for $0 \leq s \leq \sigma(1)$, that

$$\begin{aligned}\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r &\leq \lambda \int_0^{\sigma(1)} G(\sigma(r),r)b(r)g(u(\sigma(r)))\Delta r \\ &\leq \lambda \int_0^{\sigma(1)} G(\sigma(r),r)b(r)(g_0+\epsilon)u(r)\Delta r \\ &\leq \lambda \int_0^{\sigma(1)} G(\sigma(r),r)b(r)\Delta r(g_0+\epsilon)\|u\| \\ &\leq \|u\| = H_1.\end{aligned} \tag{3.15}$$

As a consequence, we next have from (2.4) and choice of ϵ, for $0 \leq t \leq \sigma^2(1)$, that

$$\begin{aligned}Tu(t) &= \lambda \int_0^{\sigma(1)} G(t,s)a(s)f\left(\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\ &\leq \lambda \int_0^{\sigma(1)} G(\sigma(s),s)a(s)(f_0+\epsilon)\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\Delta s \\ &\leq \lambda \int_0^{\sigma(1)} G(\sigma(s),s)a(s)(f_0+\epsilon)H_1\Delta s \\ &\leq H_1 = \|u\|.\end{aligned} \tag{3.16}$$

So, $\|Tu\| \leq \|u\|$. If we set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\}, \tag{3.17}$$

then

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.18}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2 > 0$ such that

$$f(x) \geq (f_\infty - \epsilon)x, \quad g(x) \geq (g_\infty - \epsilon)x, \quad x \geq \overline{H}_2. \tag{3.19}$$

Let

$$H_2 = \max\left\{2H_1, \frac{\overline{H}_2}{m}\right\}. \tag{3.20}$$

Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,

$$\min_{t\in[\xi,\omega]_{\mathbb{T}}} u(t) \geq m\|u\| \geq \overline{H}_2. \tag{3.21}$$

Consequently, from (2.5) and choice of ϵ, for $0 \leq s \leq \sigma(1)$, we have that

$$\begin{aligned}
\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r &\geq \lambda \int_\xi^\omega G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r \\
&\geq \lambda \int_\xi^\omega G(\tau,r)b(r)g(u(\sigma(r)))\Delta r \\
&\geq \lambda \int_\xi^\omega G(\tau,r)b(r)(g_\infty - \epsilon)u(r)\Delta r \\
&\geq m\lambda \int_\xi^\omega G(\tau,r)b(r)(g_\infty - \epsilon)\Delta r\|u\| \\
&\geq \|u\| = H_2.
\end{aligned} \tag{3.22}$$

And so, we have from (2.5) and choice of ϵ that

$$\begin{aligned}
Tu(\tau) &= \lambda \int_0^{\sigma(1)} G(\tau,s)a(s)f\left(\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\
&\geq \lambda \int_0^{\sigma(1)} G(\tau,s)a(s)(f_\infty - \epsilon)\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\geq \lambda \int_0^{\sigma(1)} G(\tau,s)a(s)(f_\infty - \epsilon)H_2\Delta s \\
&\geq mH_2 > H_2 = \|u\|.
\end{aligned} \tag{3.23}$$

Hence, $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\}, \tag{3.24}$$

then

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.25}$$

Applying Theorem 2.1 to (3.18) and (3.25), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, and with v being defined by

$$v(t) = \lambda \int_0^{\sigma(1)} G(t,s)b(s)g(u(\sigma(s)))\Delta s, \tag{3.26}$$

the pair (u,v) is a desired solution of (1.2), (1.3) for the given λ. The proof is complete. □

Prior to our next result, we introduce another hypothesis.

(d) $g(0) = 0$, and f is an increasing function.

We now define positive numbers L_3 and L_4 by

$$\begin{gathered} L_3 := \max\left\{\left[m\int_\xi^\omega G(\tau,s)a(s)\Delta s f_0\right]^{-1}, \left[m\int_\xi^\omega G(\tau,s)b(s)\Delta s g_0\right]^{-1}\right\}, \\ L_4 := \min\left\{\left[\int_0^{\sigma(1)} G(\sigma(s(s)))a(s)\Delta s f_\infty\right]^{-1}, \left[\int_0^{\sigma(1)} G(\sigma(s(s)))b(s)\Delta s g_\infty\right]^{-1}\right\}. \end{gathered} \tag{3.27}$$

Theorem 3.2. *Assume that conditions (a)–(d) are satisfied. Then, for each λ satisfying*

$$L_3 < \lambda < L_4, \tag{3.28}$$

there exists a pair (u,v) satisfying (1.2), (1.3) such that $u(x) > 0$ and $v(x) > 0$ on $(0,\sigma^2(1))_{\mathbf{T}}$.

Proof. Let λ be as in (3.28). And let $\epsilon > 0$ be chosen such that

$$\begin{gathered} \max\left\{\left[m\int_\xi^\omega G(\tau,s)a(s)\Delta s(f_0-\epsilon)\right]^{-1}, \left[m\int_\xi^\omega G(\tau,s)b(s)\Delta s(g_0-\epsilon)\right]^{-1}\right\} \le \lambda, \\ \lambda \le \min\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a(s)\Delta s(f_\infty+\epsilon)\right]^{-1}, \left[\int_0^{\sigma(1)} G(\sigma(s),s)b(s)\Delta s(g_\infty+\epsilon)\right]^{-1}\right\}. \end{gathered} \tag{3.29}$$

Let T be the cone preserving, completely continuous operator that was defined by (3.10).

From the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \ge (f_0-\epsilon)x, \quad g(x) \ge (g_0-\epsilon)x, \quad 0 < x \le H_1. \tag{3.30}$$

Now, $g(0) = 0$, and so there exists $0 < H_2 < H_1$ such that

$$\lambda g(x) \le \frac{H_1}{\int_0^{\sigma(1)} G(\sigma(s),s)b(s)\Delta s}, \qquad 0 \le x \le H_2. \tag{3.31}$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_2$. Then, for $0 \le s \le \sigma(1)$, we have

$$\lambda \int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r \le \frac{\int_0^{\sigma(1)} G(\sigma(s),r)b(r)H_1\Delta r}{\int_0^{\sigma(1)} G(\sigma(s),s)b(s)\Delta s} \le H_1. \tag{3.32}$$

Then,

$$\begin{aligned}
Tu(\tau) &= \lambda\int_0^{\sigma(1)} G(\tau,s)a(s)f\left(\lambda\int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\
&\geq \lambda\int_\xi^{\omega} G(\tau,s)a(s)(f_0-\epsilon)\lambda\int_0^{\sigma(1)} G(\sigma(s),r)b(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\geq \lambda\int_\xi^{\omega} G(\tau,s)a(s)(f_0-\epsilon)\lambda\int_\xi^{\omega} G(\tau,r)b(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\geq \lambda\int_\xi^{\omega} G(\tau,s)a(s)(f_0-\epsilon)\lambda m\int_\xi^{\omega} G(\tau,r)b(r)(g_0-\epsilon)\|u\|\Delta r\Delta s \\
&\geq \lambda\int_\xi^{\omega} G(\tau,s)a(s)(f_0-\epsilon)\|u\|\Delta s \\
&\geq \lambda m\int_\xi^{\omega} G(\tau,s)a(s)(f_0-\epsilon)\|u\|\Delta s \geq \|u\|.
\end{aligned} \tag{3.33}$$

So, $\|Tu\| \geq \|u\|$. If we put

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_2\}, \tag{3.34}$$

then

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathcal{P}\cap\partial\Omega_1. \tag{3.35}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_1$ such that

$$f(x) \leq (f_0-\epsilon)x, \quad g(x) \leq (g_0-\epsilon)x, \quad x \geq \overline{H}_1. \tag{3.36}$$

There are two cases: (a) g is bounded, and (b) g is unbounded.

For case (a), suppose $N > 0$ is such that $g(x) \leq N$ for all $0 < x < \infty$. Then, for $0 \leq s \leq \sigma(1)$ and $u \in \mathcal{P}$,

$$\lambda\int_0^{\sigma(1)} G(\sigma(s(r)))b(r)g(u(\sigma(r)))\Delta r \leq N\lambda\int_0^{\sigma(1)} G(\sigma(r),r)b(r)\Delta r. \tag{3.37}$$

Let

$$M = \max\left\{f(x) \mid 0 \leq x \leq N\lambda\int_0^{\sigma(1)} G(\sigma(r),r)b(r)\Delta r\right\}, \tag{3.38}$$

and let

$$H_3 > \max\left\{2H_2, M\lambda\int_0^{\sigma(1)} G(\sigma(s),s)a(s)\Delta s\right\}. \tag{3.39}$$

Then, for $u \in \mathcal{P}$ with $\|u\| = H_3$,

$$\begin{aligned}
Tu(t) &\leq \lambda\int_0^{\sigma(1)} G(\sigma(s),s)a(s)M\Delta s \\
&\leq H_3 = \|u\|
\end{aligned} \tag{3.40}$$

so that $\|Tu\| \le \|u\|$. If

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_3\}, \tag{3.41}$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.42}$$

For case (b), there exists $H_3 > \max\{2H_2, \overline{H}_1\}$ such that $g(x) \le g(H_3)$, for $0 < x \le H_3$. Similarly, there exists $H_4 > \max\{H_3, \lambda\int_0^{\sigma(1)} G(\sigma(r),r)b(r)g(H_3)\Delta r)\}$ such that $f(x) \le f(H_4)$, for $0 < x \le H_4$. Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$ we have by (d) that

$$\begin{aligned} Tu(t) &\le \lambda \int_0^{\sigma(1)} G(t,s)a(s)f\left(\lambda \int_0^{\sigma(1)} G(\sigma(r),r)b(r)g(H_3)\Delta r\right)\Delta s \\ &\le \lambda \int_0^{\sigma(1)} G(t,s)a(s)f(H_4)\Delta s \\ &\le \lambda \int_0^{\sigma(1)} G(\sigma(s),s)a(s)\Delta s(f_\infty + \epsilon)H_4 \\ &\le H_4 = \|u\|, \end{aligned} \tag{3.43}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_4\}, \tag{3.44}$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.45}$$

In either cases, application of part (ii) of Theorem 2.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which in turn yields a pair (u,v) satisfying (1.2), (1.3) for the chosen value of λ. The proof is complete. □

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M. Benchohra: Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algeria
Email address: benchohra@univ-sba.dz

J. Henderson: Department of Mathematics, Baylor University, Waco, TX 76798, USA
Email address: johnny_henderson@baylor.edu

S. K. Ntouyas: Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
Email address: sntouyas@uoi.gr