# MULTIPLE NONNEGATIVE SOLUTIONS FOR BVPs OF FOURTH-ORDER DIFFERENCE EQUATIONS

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First, existence criteria for at least three nonnegative solutions to the following boundary value problem of fourth-order difference equation $\Delta^4 x(t-2) = a(t)f(x(t))$, $t \in [2,T]$, $x(0) = x(T+2) = 0$, $\Delta^2 x(0) = \Delta^2 x(T) = 0$ are established by using the well-known Leggett-Williams fixed point theorem, and then, for arbitrary positive integer $m$, existence results for at least $2m-1$ nonnegative solutions are obtained.



## 1. Introduction

Recently, boundary value problems (BVPs) of difference equations have received considerable attention from many authors, see [1–5, 7–9, 12–19] and the references therein. In particular, Zhang et al. [19] established the existence of positive solution to the fourth-order BVP

$$\begin{gathered} \Delta^4 x(t-2) = \lambda a(t) f(t, x(t)), \quad t \in N,\ 2 \le t \le T, \\ x(0) = x(T+2) = 0, \\ \Delta^2 x(0) = \Delta^2 x(T) = 0 \end{gathered} \tag{1.1}$$

by using the method of upper and lower solutions, and then Sun [15] obtained the existence of one positive solution for the following fourth-order BVP:

$$\begin{gathered} \Delta^4 x(t-2) = a(t) f(x(t)), \quad t \in [2,T], \\ x(0) = x(T+2) = 0, \\ \Delta^2 x(0) = \Delta^2 x(T) = 0 \end{gathered} \tag{1.2}$$

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under the assumption that $f$ is either superlinear or sublinear, where $T > 2$ is a fixed positive integer, $\Delta^m$ denotes the $m$th forward difference operator with stepsize 1, and $[a,b] = \{a, a+1, \dots, b-1, b\} \subset \mathbb{Z}$ the set of all integers. Our main tool was the Guo-Krasnosel'skii fixed point theorem in cone [6, 10].

In this paper we will continue to consider the BVP (1.2). First, existence criteria for at least three nonnegative solutions to the BVP (1.2) are established by using the well-known Leggett-Williams fixed point theorem [11], and then, for arbitrary positive integer $m$, existence results for at least $2m-1$ nonnegative solutions to the BVP (1.2) are obtained.

Throughout this paper, we assume that the following two conditions are satisfied.

(C1) $f : [0,\infty) \to [0,\infty)$ is continuous.

(C2) $a : [2,T] \to [0,\infty)$ is not identical zero.

In order to obtain our main results, we need the following concepts and Leggett-Williams fixed point theorem.

Let $E$ be a real Banach space with cone $P$. A map $\alpha : P \to [0,+\infty)$ is said to be a nonnegative continuous concave functional on $P$ if $\alpha$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y) \tag{1.3}$$

for all $x, y \in P$ and $t \in [0,1]$. Let $a$, $b$ be two numbers such that $0 < a < b$ and let $\alpha$ be a nonnegative continuous concave functional on $P$. We define the following convex sets:

$$\begin{aligned} P_a &= \{x \in P : \|x\| < a\}, \\ P(\alpha, a, b) &= \{x \in P : a \leq \alpha(x),\ \|x\| \leq b\}. \end{aligned} \tag{1.4}$$

Theorem 1.1 (Leggett-Williams fixed point theorem). *Let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and let $\alpha$ be a nonnegative continuous concave functional on $P$ such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < d < a < b \leq c$ such that*

(i) $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \phi$ *and* $\alpha(Ax) > a$ *for* $x \in P(\alpha, a, b)$;

(ii) $\|Ax\| < d$ *for* $\|x\| \leq d$;

(iii) $\alpha(Ax) > a$ *for* $x \in P(\alpha, a, c)$ *with* $\|Ax\| > b$.

*Then $A$ has at least three fixed points $x_1$, $x_2$, $x_3$ in $\overline{P_c}$ satisfying*

$$\|x_1\| < d, \quad a < \alpha(x_2), \quad \|x_3\| > d, \quad \alpha(x_3) < a. \tag{1.5}$$

## 2. Main results

For convenience, we denote

$$G_1(t,s) = \frac{1}{T}\begin{cases} (t-1)(T+1-s), & 1 \leq t \leq s \leq T, \\ (s-1)(T+1-t), & 2 \leq s \leq t \leq T+1, \end{cases}$$

$$G_2(t,s) = \frac{1}{T+2}\begin{cases} t(T+2-s), & 0 \leq t \leq s \leq T+1, \\ s(T+2-t), & 1 \leq s \leq t \leq T+2, \end{cases}$$

$$D = \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v),$$
$$C = \min_{t\in[2,T]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v). \tag{2.1}$$

It is easily seen from the expression of $G_2(t,s)$ that

$$G_2(t,s) \le G_2(s,s), \quad (t,s) \in [0,T+2]\times[1,T+1],$$
$$G_2(t,s) \ge \frac{1}{T+1}G_2(s,s), \quad (t,s) \in [1,T+1]\times[1,T+1]. \tag{2.2}$$

Our main result is the following theorem.

Theorem 2.1. *Assume that there exist numbers $d$, $a$, and $c$ with $0 < d < a < (T+1)a < c$ such that*

$$f(x) < \frac{d}{D}, \quad x \in [0,d], \tag{2.3}$$
$$f(x) > \frac{a}{C}, \quad x \in [a,(T+1)a], \tag{2.4}$$
$$f(x) < \frac{c}{D}, \quad x \in [0,c]. \tag{2.5}$$

*Then the BVP (1.2) has at least three nonnegative solutions.*

*Proof.* Let the Banach space $E = \{x : [0,T+2] \to R\}$ be equipped with the norm

$$\|x\| = \max_{t\in[0,T+2]} |x(t)|. \tag{2.6}$$

We define

$$P = \{x \in E : x(t) \ge 0,\ t \in [0,T+2]\}, \tag{2.7}$$

then it is obvious that $P$ is a cone in $E$.

For $x \in P$, we define

$$\alpha(x) = \min_{t\in[2,T]} x(t),$$
$$(Ax)(t) = \sum_{s=1}^{T+1} G_2(t,s)\sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)), \quad t \in [0,T+2]. \tag{2.8}$$

It is easy to check that $\alpha$ is a nonnegative continuous concave functional on $P$ with $\alpha(x) \le \|x\|$ for $x \in P$ and that $A : P \to P$ is completely continuous and fixed points of $A$ are solutions of the BVP (1.2).

We first assert that if there exists a positive number $r$ such that $f(x) < r/D$ for $x \in [0,r]$, then $A : \overline{P_r} \to P_r$.

Indeed, if $x \in \overline{P_r}$, then for $t \in [0, T+2]$,

$$
\begin{aligned}
(Ax)(t) &= \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)) \\
&< \frac{r}{D} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v) \\
&\leq \frac{r}{D} \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v) = r.
\end{aligned} \tag{2.9}
$$

Thus, $\|Ax\| < r$, that is, $Ax \in P_r$.

Hence, we have shown that if (2.3) and (2.5) hold, then $A$ maps $\overline{P}_d$ into $P_d$ and $\overline{P}_c$ into $P_c$.

Next, we assert that $\{x \in P(\alpha, a, (T+1)a) : \alpha(x) > a\} \neq \phi$ and $\alpha(Ax) > a$ for all $x \in P(\alpha, a, (T+1)a)$.

In fact, the constant function

$$
\frac{(T+2)a}{2} \in \{x \in P(\alpha, a, (T+1)a) : \alpha(x) > a\}. \tag{2.10}
$$

Moreover, for $x \in P(\alpha, a, (T+1)a)$, we have

$$
(T+1)a \geq \|x\| \geq x(t) \geq \min_{t\in[2,T]} x(t) = \alpha(x) \geq a \tag{2.11}
$$

for all $t \in [2, T]$. Thus, in view of (2.4), we see that

$$
\begin{aligned}
\alpha(Ax) &= \min_{t\in[2,T]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)) \\
&> \frac{a}{C} \min_{t\in[2,T]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v) = a
\end{aligned} \tag{2.12}
$$

as required.

Finally, we assert that if $x \in P(\alpha, a, c)$ and $\|Ax\| > (T+1)a$, then $\alpha(Ax) > a$.

To see this, suppose $x \in P(\alpha, a, c)$ and $\|Ax\| > (T+1)a$, then in view of (2.2), we have

$$
\begin{aligned}
\alpha(Ax) &= \min_{t\in[2,T]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)) \\
&\geq \frac{1}{T+1} \sum_{s=1}^{T+1} G_2(s,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)) \\
&\geq \frac{1}{T+1} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v))
\end{aligned} \tag{2.13}
$$

for $t \in [0, T+2]$. Thus

$$\begin{aligned}\alpha(Ax) &\geq \frac{1}{T+1} \max_{t\in[0,T+2]} \sum_{s=1}^{T+1} G_2(t,s) \sum_{v=2}^{T} G_1(s,v)a(v)f(x(v)) \\ &= \frac{1}{T+1}\|Ax\| > \frac{1}{T+1}(T+1)a = a.\end{aligned} \tag{2.14}$$

To sum up, all the hypotheses of the Leggett-Williams theorem are satisfied. Hence $A$ has at least three fixed points, that is, the BVP (1.2) has at least three nonnegative solutions $u$, $v$, and $w$ such that

$$\begin{gathered}\|u\| < d, \quad a < \min_{t\in[2,T]} v(t), \quad \|w\| > d, \\ \min_{t\in[2,T]} w(t) < a.\end{gathered} \tag{2.15}$$

The proof is complete. □

Corollary 2.2. *Let $m$ be an arbitrary positive integer. Assume that there exist numbers $d_j$ $(1 \leq j \leq m)$ and $a_h$ $(1 \leq h \leq m-1)$ with $0 < d_1 < a_1 < (T+1)a_1 < d_2 < a_2 < (T+1)a_2 < \cdots < d_{m-1} < a_{m-1} < (T+1)a_{m-1} < d_m$ such that*

$$f(x) < \frac{d_j}{D}, \quad x \in [0, d_j],\ 1 \leq j \leq m, \tag{2.16}$$

$$f(x) > \frac{a_h}{C}, \quad x \in [a_h, (T+1)a_h],\ 1 \leq h \leq m-1. \tag{2.17}$$

*Then, the BVP (1.2) has at least $2m-1$ nonnegative solutions in $\overline{P_{d_m}}$.*

*Proof.* We prove this conclusion by induction.

First, for $m = 1$, we know from (2.16) that $A : \overline{P_{d_1}} \to P_{d_1} \subset \overline{P_{d_1}}$, then, it follows from Schauder fixed point theorem that the BVP (1.2) has at least one nonnegative solution in $\overline{P_{d_1}}$.

Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion also holds for $m = k+1$, we suppose that there exist numbers $d_j$ $(1 \leq j \leq k+1)$ and $a_h$ $(1 \leq h \leq k)$ with $0 < d_1 < a_1 < (T+1)a_1 < d_2 < a_2 < (T+1)a_2 < \cdots < d_k < a_k < (T+1)a_k < d_{k+1}$ such that

$$\begin{gathered}f(x) < \frac{d_j}{D}, \quad x \in [0, d_j],\ 1 \leq j \leq k+1, \\ f(x) > \frac{a_h}{C}, \quad x \in [a_h, (T+1)a_h],\ 1 \leq h \leq k.\end{gathered} \tag{2.18}$$

By the assumption, (2.18), we know that the BVP (1.2) has at least $2k-1$ nonnegative solutions $x_i$ $(i = 1, 2, \ldots, 2k-1)$ in $\overline{P_{d_k}}$. At the same time, it follows from Theorem 2.1 and (2.18) that the BVP (1.2) has at least three nonnegative solutions $u$, $v$, and $w$ in $\overline{P_{d_{k+1}}}$

such that

$$\|u\| < d_k, \quad a_k < \min_{t\in[2,T]} v(t), \quad \|w\| > d_k,$$
$$\min_{t\in[2,T]} w(t) < a_k. \tag{2.19}$$

Obviously, $v$ and $w$ are different from $x_i$ $(i = 1,2,\dots,2k-1)$. Therefore, the BVP (1.2) has at least $2k+1$ nonnegative solutions in $\overline{P_{d_{k+1}}}$, which shows that this conclusion also holds for $m = k+1$. The proof is complete. □

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