REPRESENTATION OF SOLUTIONS OF LINEAR DISCRETE SYSTEMS WITH CONSTANT COEFFICIENTS AND PURE DELAY

J. DIBLÍK AND D. YA. KHUSAINOV

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The purpose of this contribution is to develop a method for construction of solutions of linear discrete systems with constant coefficients and with pure delay. Solutions are expressed with the aid of a special function called the discrete matrix delayed exponential having between every two adjoining knots the form of a polynomial. These polynomials have increasing degrees in the right direction. Such approach results in a possibility to express initial Cauchy problem in the closed form.

1. Introduction

We use following notation: for integers $s, q, s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \ldots, q\}$, where possibility $s = -\infty$ or $q = \infty$ is admitted too. Throughout this paper, using notation $\mathbb{Z}_s^q$ or another one with a couple of integers s, q, we suppose $s \le q$. In this paper we deal with the discrete system

$$\Delta x(k) = Bx(k-m) + f(k), \tag{1.1}$$

where $m \ge 1$ is a fixed integer, $k \in \mathbb{Z}_0^\infty$, $B = (b_{ij})$ is a constant $n \times n$ matrix, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^n$, $\Delta x(k) = x(k+1) - x(k)$, $x \colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$. Following the terminology (used, e.g., in [1, 3]) we refer to (1.1) as a delayed discrete system if $m \ge 1$ and as a nondelayed discrete system if $m = 0$. Together with (1.1) we consider the initial conditions

$$x(k) = \varphi(k) \tag{1.2}$$

with given $\varphi \colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$.

The *existence* and *uniqueness* of solution of the problem (1.1), (1.2) on $\mathbb{Z}_{-m}^\infty$ is obvious. We recall that *solution* $x \colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$ of the problem (1.1), (1.2) is defined as an

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infinite sequence $\{\varphi(-m),\varphi(-m+1),\ldots,\varphi(0),x(1),x(2),\ldots,x(k),\ldots\}$ such that, for any $k\in\mathbb{Z}_0^\infty$, equality (1.1) holds. Throughout the paper we adopt the customary notations $\sum_{i=k+s}^{k}\circ(i)=0$ and $\prod_{i=k+s}^{k}\circ(i)=1$, where k is an integer, s is a positive integer, and "$\circ$" denotes the function considered irrespective on the fact if it is for indicated arguments defined or not.

1.1. Description of the problem considered. The motivation of our investigation goes back to [10] dealing with the linear system of differential equations with constant coefficients and constant delay. One of the systems considered has the form

$$\dot{x}(t)=Bx(t-\tau), \tag{1.3}$$

where $t\in\mathbb{R}_+=[0,\infty)$, $\tau>0$, $x\colon\mathbb{R}_+\to\mathbb{R}^n$, and B is an $n\times n$ matrix. For a given matrix B we define a matrix function $\exp_\tau(Bt)$, called *delayed exponential of the matrix B*:

$$e_\tau^{Bt}:=\begin{cases}\Theta & \text{if } -\infty<t<-\tau,\\ I & \text{if } -\tau\le t<0,\\ I+\dfrac{1}{1!}Bt & \text{if } 0\le t<\tau,\\ I+\dfrac{1}{1!}Bt+\dfrac{1}{2!}B^2(t-\tau)^2 & \text{if } \tau\le t<2\tau,\\ \cdots & \\ I+\dfrac{1}{1!}Bt+\dfrac{1}{2!}B^2(t-\tau)^2+\cdots+\dfrac{1}{k!}B^k[t-(k-1)\tau]^k & \text{if } (k-1)\tau\le t<k\tau,\\ \cdots & \end{cases} \tag{1.4}$$

with null $n\times n$ matrix Θ and unit $n\times n$ matrix I. We consider initial problem

$$x(t)=\varphi(t),\quad t\in[-\tau,0], \tag{1.5}$$

with continuously differentiable initial function φ on $[-\tau,0]$. In [10], it is proved that the solution of the problem (1.3), (1.5) can be expressed on the interval $[-\tau,\infty)$ in the form

$$x(t)=e_\tau^{Bt}\varphi(-\tau)+\int_{-\tau}^{0}e_\tau^{B(t-\tau-s)}\varphi'(s)ds. \tag{1.6}$$

It is easy to deduce that the delayed exponential is a useful tool for the formalizing of computation of initial problems for systems of the form (1.3), since the usually used method of steps (being nevertheless hidden in the notion of delayed exponential) gives unwieldy formulas. Discrete systems of the form (1.1) containing only one delay are often called *systems with pure delay*. The main goal of the present paper is to extend the notion of the delayed exponential of a matrix relative to discrete delayed equations and give an analogue of formula (1.6) for homogeneous and nonhomogeneous problems (1.1), (1.2) with pure delay.

2. Discrete matrix delayed exponential

Now we give the notion of the so-called discrete matrix delayed exponential as well as of its main property. Before we consider an example, we make possible understanding better the ensuing definition of discrete matrix delayed exponential.

2.1. An example. We consider a scalar discrete equation together with an initial problem

$$\Delta x(k) = bx(k-3), \tag{2.1}$$

$$x(-3) = x(-2) = x(-1) = x(0) = 1, \tag{2.2}$$

where $b \in \mathbb{R}$, $b \neq 0$. Rewriting (2.1) as

$$x(k+1) = x(k) + bx(k-3) \tag{2.3}$$

and solving it by the method of steps, we conclude that the solution of the problem (2.1), (2.2) can be written in the form

$$x(k) = \begin{cases} 1 \quad \text{if } k \in \mathbb{Z}_{-3}^{0}, \\ 1 + b \cdot \binom{k}{1} \quad \text{if } k \in \mathbb{Z}_1^4, \\ 1 + b \cdot \binom{k}{1} + b^2 \cdot \binom{k-3}{2} \quad \text{if } k \in \mathbb{Z}_5^8, \\ 1 + b \cdot \binom{k}{1} + b^2 \cdot \binom{k-3}{2} + b^3 \cdot \binom{k-6}{3} \quad \text{if } k \in \mathbb{Z}_9^{12}, \\ \cdots \\ 1 + b \cdot \binom{k}{1} + b^2 \cdot \binom{k-3}{2} + \cdots \\ \quad + b^{\ell} \cdot \binom{k-(\ell-1)\cdot 3}{\ell} \quad \text{if } k \in \mathbb{Z}_{(\ell-1)4+1}^{(\ell-1)4+4},\ \ell = 1,2,\dots. \end{cases} \tag{2.4}$$

Such expression of x serves as a motivation for the definition of discrete matrix delayed exponential.

2.2. Definition of a discrete matrix delayed exponential. We define a discrete matrix function $\exp_m(Bk)$ called the discrete matrix delayed exponential of an $n \times n$ constant matrix B:

$$
e_m^{Bk} := \begin{cases} \Theta \quad \text{if } k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I \quad \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ I + B \cdot \binom{k}{1} \quad \text{if } k \in \mathbb{Z}_1^{m+1}, \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} \quad \text{if } k \in \mathbb{Z}_{(m+1)+1}^{2(m+1)}, \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} + B^3 \cdot \binom{k-2m}{3} \quad \text{if } k \in \mathbb{Z}_{2(m+1)+1}^{3(m+1)}, \\ \cdots \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} + \cdots \\ \quad + B^\ell \cdot \binom{k-(\ell-1)m}{\ell} \quad \text{if } k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}, \ \ell = 0,1,2,\ldots. \end{cases} \tag{2.5}
$$

We underline a parallelism between the delayed exponential $\exp_\tau(Bt)$ of the matrix B and its discrete analogy $\exp_m(Bk)$. Discrete matrix delayed exponential $\exp_m(Bk)$ is a matrix function having the form of a matrix polynomial. Similarly as values of $\exp_\tau(Bt)$ are pasted at the boundary points $t = k\tau$, $k = 0,1,\ldots$, values of $\exp_m(Bk)$ are in a sense "pasted" at the boundary knots $k = \ell(m+1)$, $\ell = 0,1,\ldots$. It becomes clear if we put, by definition,

$$
\frac{s!}{(-1)!} := 0 \tag{2.6}
$$

for any nonnegative integer s. The definition of the discrete matrix delayed exponential can be shortened as

$$
e_m^{Bk} := I + \sum_{j=1}^{\ell} B^j \cdot \binom{k-(j-1)m}{j} \tag{2.7}
$$

for $k = (\ell-1)(m+1)+1,\ldots,\ell(m+1)$ and $\ell = 0,1,\ldots$.

2.3. Basic property of the discrete matrix delayed exponential. Main property of $\exp_m(Bk)$ is given in the following theorem.

Theorem 2.1. *Let B be a constant $n \times n$ matrix. Then for $k \in \mathbb{Z}_{-m}^{\infty}$,*

$$
\Delta e_m^{Bk} = B e_m^{B(k-m)}. \tag{2.8}
$$

Proof. Let a matrix B and a positive integer m be fixed. Then for integer k satisfying

$$(\ell-1)(m+1)+1 \le k \le \ell(m+1), \tag{2.9}$$

in accordance with the definition of e_m^{Bk} relation,

$$\Delta e_m^{Bk} = \Delta\left[I + \sum_{j=1}^{\ell} B^j \cdot \binom{k-(j-1)m}{j}\right] \tag{2.10}$$

holds. Since $\Delta I = \Theta$ we have

$$\Delta e_m^{Bk} = \Delta\left[\sum_{j=1}^{\ell} B^j \cdot \binom{k-(j-1)m}{j}\right]. \tag{2.11}$$

Considering the increment by its definition, for example,

$$\Delta e_m^{Bk} = e_m^{B(k+1)} - e_m^{Bk}, \tag{2.12}$$

we conclude that it is reasonable to divide the proof into two parts with respect to the value of the integer k. One case is represented with k such that

$$(\ell-1)(m+1)+1 \le k < k+1 \le \ell(m+1), \tag{2.13}$$

the second one with $k = \ell(m+1)$.

The case $(\ell-1)(m+1)+1 \le k < k+1 \le \ell(m+1)$. In this case

$$k-m \in [(\ell-2)(m+1)+1, (\ell-1)(m+1)] \tag{2.14}$$

and, by definition,

$$e_m^{B(k-m)} = I + \sum_{j=1}^{\ell-1} B^j \cdot \binom{k-m-(j-1)m}{j} = I + \sum_{j=1}^{\ell-1} B^j \cdot \binom{k-jm}{j}. \tag{2.15}$$

We prove that

$$\Delta e_m^{Bk} = B e_m^{B(k-m)} = B\left[I + \sum_{j=1}^{\ell-1} B^j \cdot \binom{k-jm}{j}\right]. \tag{2.16}$$

With the aid of (2.11) and (2.12) we get

$$\begin{aligned}
\Delta \mathrm{e}_m^{Bk} &= \mathrm{e}_m^{B(k+1)} - \mathrm{e}_m^{Bk} \\
&= \sum_{j=1}^{\ell} B^j \cdot \binom{k+1-(j-1)m}{j} - \sum_{j=1}^{\ell} B^j \cdot \binom{k-(j-1)m}{j} \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\left[\frac{(k+1-(j-1)m)!}{(k+1-(j-1)m-j)!} - \frac{(k-(j-1)m)!}{(k-(j-1)m-j)!}\right] \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\frac{(k-(j-1)m)!}{(k+1-(j-1)m-j)!}\left[(k+1-(j-1)m)-(k+1-(j-1)m-j)\right] \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\frac{(k-(j-1)m)!\cdot j}{(k+1-(j-1)m-j)!} \\
&= B\sum_{j=1}^{\ell} \frac{B^{j-1}}{(j-1)!}\frac{(k-(j-1)m)!}{(k-(j-1)m-(j-1))!} \\
&= B\left[I + \sum_{j=2}^{\ell} B^{j-1} \cdot \binom{k-(j-1)m}{j-1}\right].
\end{aligned} \tag{2.17}$$

Now we change the index of summation j by $j+1$. Then

$$\Delta \mathrm{e}_m^{Bk} = B\left[I + \sum_{j=1}^{\ell-1} B^j \cdot \binom{k-jm}{j}\right] \tag{2.18}$$

and due to (2.15) we conclude that formula (2.16) is valid.

The case $k = \ell(m+1)$. In this case we have by definition

$$\begin{aligned}
\mathrm{e}_m^{Bk} = \mathrm{e}_m^{B\ell(m+1)} &= I + \sum_{j=1}^{\ell} B^j \cdot \binom{\ell(m+1)-(j-1)m}{j}, \\
\mathrm{e}_m^{B(k+1)} = \mathrm{e}_m^{B(\ell(m+1)+1)} &= I + \sum_{j=1}^{\ell+1} B^j \cdot \binom{\ell(m+1)+1-(j-1)m}{j}.
\end{aligned} \tag{2.19}$$

Since

$$k - m = \ell(m+1) - m \in \left[(\ell-1)(m+1)+1, \ell(m+1)\right] \tag{2.20}$$

the discrete matrix delayed exponential $e_m^{B(k-m)}$ is expressed by

$$e_m^{B(k-m)} = I + \sum_{j=1}^{\ell} B^j \cdot \binom{k-m-(j-1)m}{j}. \tag{2.21}$$

Therefore

$$\begin{aligned}
\Delta e_m^{Bk} &= e_m^{B(k+1)} - e_m^{Bk} = e_m^{B(\ell(m+1)+1)} - e_m^{B\ell(m+1)} \\
&= \sum_{j=1}^{\ell+1} B^j \cdot \binom{\ell(m+1)+1-(j-1)m}{j} - \sum_{j=1}^{\ell} B^j \cdot \binom{\ell(m+1)-(j-1)m}{j} \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\left[\frac{(\ell(m+1)+1-(j-1)m)!}{(\ell(m+1)+1-(j-1)m-j)!} - \frac{(\ell(m+1)-(j-1)m)!}{(\ell(m+1)-(j-1)m-j)!}\right] \\
&\quad + \frac{B^{\ell+1}}{(\ell+1)!}\frac{(\ell(m+1)+1-(\ell+1-1)m)!}{(\ell(m+1)+1-(\ell+1-1)m-(\ell+1))!} \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\left[\frac{(\ell(m+1)+1-(j-1)m)!}{(\ell(m+1)+1-(j-1)m-j)!} - \frac{(\ell(m+1)-(j-1)m)!}{(\ell(m+1)-(j-1)m-j)!}\right] \\
&\quad + \frac{B^{\ell+1}}{(\ell+1)!}\frac{(\ell(m+1)+1-\ell m)!}{(\ell(m+1)+1-\ell m-(\ell+1))!} \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\frac{(\ell(m+1)-(j-1)m)!}{(\ell(m+1)+1-(j-1)m-j)!} \\
&\quad \times \left[(\ell(m+1)+1-(j-1)m) - (\ell(m+1)+1-(j-1)m-j)\right] + B^{\ell+1} \\
&= \sum_{j=1}^{\ell} \frac{B^j}{j!}\frac{(\ell(m+1)-(j-1)m)!}{(\ell(m+1)+1-(j-1)m-j)!} \cdot j + B^{\ell+1} \\
&= B\sum_{j=1}^{\ell} \frac{B^{j-1}}{(j-1)!}\frac{(\ell(m+1)-(j-1)m)!}{(\ell(m+1)+1-(j-1)m-j)!} + B^{\ell+1} \\
&= B + B\sum_{j=2}^{\ell} B^{j-1} \cdot \binom{\ell(m+1)-(j-1)m}{j-1} + B^{\ell+1}.
\end{aligned} \tag{2.22}$$

Now we change the index of summation j by $j+1$. Then

$$\Delta e_m^{Bk} = B\left[I + \sum_{j=1}^{\ell-1} B^j \cdot \binom{\ell(m+1)-jm}{j} + B^{\ell}\right]. \tag{2.23}$$

With the aid of the relation $k = \ell(m+1)$ we get

$$\Delta e_m^{Bk} = B\left[I + \sum_{j=1}^{\ell-1} B^j \cdot \binom{k-m-(j-1)m}{j} + B^\ell \cdot \binom{k-m-(\ell-1)m}{\ell}\right]. \tag{2.24}$$

Finally due to (2.21),

$$\Delta e_m^{Bk} = B\left[I + \sum_{j=1}^{\ell} B^j \cdot \binom{k-m-(j-1)m}{j}\right] = Be_m^{B(k-m)} \tag{2.25}$$

and formula (2.16) is proved. □

Remark 2.2. Analyzing the formula (2.8) we conclude that the discrete matrix delayed exponential is the matrix solution of the initial Cauchy problem

$$\begin{aligned} \Delta X(k) &= BX(k-m), \quad k \in \mathbb{Z}_0^\infty, \\ X(k) &= I, \quad k \in \mathbb{Z}_{-m}^0. \end{aligned} \tag{2.26}$$

So we have $X(k) = \exp_m(Bk)$, $k \in \mathbb{Z}_{-m}^\infty$.

3. Representation of the solution of initial problem via discrete matrix delayed exponential

In this section we prove the main results of the paper. With the aid of discrete matrix delayed exponential we give formulas for the solution of the homogeneous and nonhomogeneous problems (1.1), (1.2).

3.1. Representation of the solution of homogeneous initial problem. Consider at first homogeneous problem (1.1), (1.2)

$$\Delta x(k) = Bx(k-m), \quad k \in \mathbb{Z}_0^\infty, \tag{3.1}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0. \tag{3.2}$$

Theorem 3.1. *Let B be a constant $n \times n$ matrix. Then the solution of the problem (3.1), (3.2) can be expressed as*

$$x(k) = e_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)}\Delta\varphi(j-1), \tag{3.3}$$

where $k \in \mathbb{Z}_{-m}^\infty$.

Proof. We are going to find the solution of the problem (3.1), (3.2) in the form

$$x(k) = e_m^{Bk}C + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)}\Delta\psi(j-1), \quad k \in \mathbb{Z}_{-m}^\infty, \tag{3.4}$$

with an unknown constant vector C and a discrete function $\psi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$. Due to linearity (taking into account that k varies), we have

$$\begin{aligned}
\Delta x(k) &= \Delta\left[e_m^{Bk} C + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)} \Delta\psi(j-1)\right] \\
&= \Delta e_m^{Bk} C + \sum_{j=-m+1}^{0} \Delta\left[e_m^{B(k-m-j)} \Delta\psi(j-1)\right] \\
&= \Delta\left[e_m^{Bk}\right] C + \sum_{j=-m+1}^{0} \Delta\left[e_m^{B(k-m-j)}\right] \Delta\psi(j-1).
\end{aligned} \tag{3.5}$$

We use formula (2.8):

$$\begin{aligned}
\Delta x(k) &= B e_m^{B(k-m)} C + \sum_{j=-m+1}^{0} B e_m^{B(k-2m-j)} \Delta\psi(j-1) \\
&= B\left[e_m^{B(k-m)} C + \sum_{j=-m+1}^{0} e_m^{B(k-2m-j)} \Delta\psi(j-1)\right].
\end{aligned} \tag{3.6}$$

Now we conclude that for any C and ψ the relation $\Delta x(k) = Bx(k-m)$ holds. We will try to satisfy the initial conditions (3.2). Due to (3.1), we have

$$e_m^{B(k-m)} C + \sum_{j=-m+1}^{0} e_m^{B(k-2m-j)} \Delta\psi(j-1) = x(k-m). \tag{3.7}$$

We consider values k such that $k-m \in \mathbb{Z}_{-m}^{0}$. Simultaneously we change the argument k by $k+m$. We get

$$e_m^{Bk} C + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)} \Delta\psi(j-1) = \varphi(k) \tag{3.8}$$

for $k \in \mathbb{Z}_{-m}^{0}$. We rewrite the last formula as

$$e_m^{Bk} C + \sum_{j=-m+1}^{k} e_m^{B(k-m-j)} \Delta\psi(j-1) + \sum_{j=k+1}^{0} e_m^{B(k-m-j)} \Delta\psi(j-1) = \varphi(k). \tag{3.9}$$

Due to the definition of the discrete matrix delayed exponential, the first sum becomes

$$\sum_{j=-m+1}^{k} e_m^{B(k-m-j)} \Delta\psi(j-1) = \sum_{j=-m+1}^{k} \Delta\psi(j-1) = \psi(k) - \psi(-m) \tag{3.10}$$

and the second one turns into zero vector. Finally, since

$$e_m^{Bk} \equiv I, \quad k \in \mathbb{Z}_{-m}^{0}, \tag{3.11}$$

relation (3.9) becomes

$$C+\psi(k)-\psi(-m)=\varphi(k) \tag{3.12}$$

and one can define

$$\psi(k):=\varphi(k), \quad k\in\mathbb{Z}_{-m}^{0}; \quad C:=\psi(-m)=\varphi(-m). \tag{3.13}$$

In order to get formula (3.3) it remains to put C and ψ into (3.4). □

Example 3.2. Let us represent the solution of the problem (2.1), (2.2) with the aid of formula (3.3). In this case $m=3$, $n=1$, $B=b$, $\varphi(-3)=1$, $\Delta\varphi(-3)=\Delta\varphi(-2)=\Delta\varphi(-1)=0$, and for $k\in\mathbb{Z}_{-3}^{\infty}$, we get

$$\begin{aligned} x(k)&=e_m^{Bk}\varphi(-m)+\sum_{j=-m+1}^{0}e_m^{B(k-m-j)}\Delta\varphi(j-1)\\ &=e_3^{bk}\varphi(-3)+\sum_{j=-2}^{0}e_3^{b(k-3-j)}\Delta\varphi(j-1)=e_3^{bk}. \end{aligned} \tag{3.14}$$

This formula coincides with corresponding formula given in Section 2.1.

3.2. Representation of the solution of nonhomogeneous initial problem. We consider the nonhomogeneous problem (1.1), (1.2)

$$\Delta x(k)=Bx(k-m)+f(k), \quad k\in\mathbb{Z}_0^{\infty}, \tag{3.15}$$

$$x(k)=\varphi(k), \quad k\in\mathbb{Z}_{-m}^{0}. \tag{3.16}$$

We get this solution, in accordance with the theory of linear equations, as the sum of the solution of adjoint homogeneous problem (3.1), (3.2) (satisfying the same initial data) and a particular solution of (3.15) being zero on initial interval. Therefore we are going to find such a particular solution. We give some auxiliary material.

Definition 3.3. Let a function $F(k,n)$ of two discrete variables be given. The operator Δ_k acting by the formula

$$\Delta_k F(k,n):=F(k+1,n)-F(k,n) \tag{3.17}$$

is said to be a partial difference operator, provided that the right-hand side exists.

In the following formula (which proof is omitted) we suppose that all used expressions are well defined.

Lemma 3.4. *Let a function $F(k,n)$ of two discrete variables be given. Then*

$$\Delta_k\left[\sum_{j=1}^{k}F(k,j)\right]=F(k+1,k+1)+\sum_{j=1}^{k}\Delta_k F(k,j). \tag{3.18}$$

Now we are ready to find a particular solution $x_p(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$, of the initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{3.19}$$

$$x(k) = 0, \quad k \in \mathbb{Z}_{-m}^{0}. \tag{3.20}$$

Theorem 3.5. *Solution $x = x_p(k)$ of the initial Cauchy problem (3.19), (3.20) can be represented on $\mathbb{Z}_{-m}^{\infty}$ in the form*

$$x_p(k) = \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1). \tag{3.21}$$

Proof. We are going to find particular solution $x_p(k)$ of the problem (3.19), (3.20) following the idea of the method of variation of arbitrary constants (see, e.g., [1]) in the form

$$x_p(k) = \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} \omega(j), \tag{3.22}$$

where $\omega : \mathbb{Z}_1^{\infty} \to \mathbb{R}^n$ is a discrete function. We put (3.22) into (3.19). Then

$$\Delta\left[\sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} \omega(j)\right] = B\left[\sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-2m-j)} \omega(j)\right] + f(k). \tag{3.23}$$

With the aid of (3.18) we obtain

$$\mathrm{e}_m^{B((k+1)-m-(k+1))} \omega(k+1) + \sum_{j=1}^{k} \Delta\left[\mathrm{e}_m^{B(k-m-j)} \omega(j)\right] = B\left[\sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-2m-j)} \omega(j)\right] + f(k). \tag{3.24}$$

Using formula (2.8) we get

$$\Delta \mathrm{e}_m^{B(k-m-j)} = B\mathrm{e}_m^{B(k-2m-j)}, \tag{3.25}$$

and the last relation becomes

$$\mathrm{e}_m^{B(-m)} \omega(k+1) + B\sum_{j=1}^{k} \left[\mathrm{e}_m^{B(k-2m-j)} \omega(j)\right] = B\left[\sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-2m-j)} \omega(j)\right] + f(k). \tag{3.26}$$

Since $\mathrm{e}_m^{B(-m)} \equiv I$ and

$$\sum_{j=1}^{k} \left[\mathrm{e}_m^{B(k-2m-j)} \omega(j)\right] = \sum_{j=1}^{k-m} \left[\mathrm{e}_m^{B(k-2m-j)} \omega(j)\right] + \sum_{j=k-m+1}^{k} \left[\mathrm{e}_m^{B(k-2m-j)} \omega(j)\right], \tag{3.27}$$

where due to the definition of the discrete matrix delayed exponential

$$e_m^{B(k-2m-j)} \equiv \Theta, \quad j \in \mathbb{Z}_{k-m+1}^{k}, \tag{3.28}$$

the relation (3.26) turns into

$$\omega(k+1) + B\sum_{j=1}^{k-m}\left[e_m^{B(k-2m-j)}\omega(j)\right] = B\left[\sum_{j=1}^{k-m} e_m^{B(k-2m-j)}\omega(j)\right] + f(k). \tag{3.29}$$

We define

$$\omega(k) := f(k-1), \quad k \in \mathbb{Z}_0^{\infty}, \tag{3.30}$$

and we put this function into (3.22). This ends the proof. □

Collecting the results of Theorems 3.1 and 3.5 we get immediately the following.

Theorem 3.6. *Solution $x = x(k)$ of the problem (1.1), (1.2) can be on $\mathbb{Z}_{-m}^{\infty}$ represented in the form*

$$x(k) = e_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} e_m^{B(k-m-j)} f(j-1). \tag{3.31}$$

Example 3.7. Let us represent the solution of the problem

$$\begin{gathered} \Delta x(k) = bx(k-3) + k + 1, \\ x(-3) = x(-2) = x(-1) = x(0) = 1, \end{gathered} \tag{3.32}$$

$b \neq 0$, $b \in \mathbb{R}$, by formula (3.31). Taking into account the representation of the solution of the problem (2.1), (2.2) given in Example 3.2, we get (in our case $f(k) := k+1$)

$$\begin{aligned} x(k) &= e_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} e_m^{B(k-m-j)} f(j-1) \\ &= e_3^{bk}\varphi(-3) + \sum_{j=-2}^{0} e_3^{b(k-3-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} e_3^{b(k-3-j)} f(j-1) = e_3^{bk} + \sum_{j=1}^{k} j e_3^{b(k-3-j)}. \end{aligned} \tag{3.33}$$

4. Concluding remarks

Method of representation of solutions developed in the paper can be used to the investigation of some boundary value problems for linear discrete systems with constant coefficients on finite intervals. Moreover results obtained can be useful in investigation of such asymptotic problems as describing the asymptotic behavior of solutions and the investigation concerning boundedness, convergence, or stability of solutions. With the aid of different methods (Liapunov type technique and retract principle), some of these problems have been investigated, for example, in the recent papers [2–9, 11–13].

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References

[1] R. P. Agarwal, *Differential Equations and Inequalities*, 2nd ed., Marcel Dekker, New York, 2000.

[2] J. Baštinec and J. Diblík, *Asymptotic formulae for a particular solution of linear nonhomogeneous discrete equations*, Computers & Mathematics with Applications **45** (2003), no. 6–9, 1163–1169.

[3] ______, *Subdominant positive solutions of the discrete equation* $\Delta u(k+n) = -p(k)u(k)$, Abstract and Applied Analysis **2004** (2004), no. 6, 461–470.

[4] A. Boichuk and M. Růžičková, *Solutions of nonlinear difference equations bounded on the whole line*, Colloquium on Differential and Difference Equations, CDDE 2002 (Brno), Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math., vol. 13, Masaryk University, Brno, 2003, pp. 45–60.

[5] ______, *Solutions bounded on the whole line for perturbed difference systems*, Proceedings of the Eighth International Conference on Difference Equations and Applications, Chapman & Hall/CRC, Florida, 2005, pp. 51–59.

[6] J. Čermák, *On the related asymptotics of delay differential and difference equations*, Dynamic Systems and Applications **14** (2005), no. 3-4, 419–429.

[7] J. Diblík, *Discrete retract principle for systems of discrete equations*, Computers & Mathematics with Applications **42** (2001), no. 3–5, 515–528.

[8] ______, *Asymptotic behaviour of solutions of systems of discrete equations via Liapunov type technique*, Computers & Mathematics with Applications **45** (2003), no. 6–9, 1041–1057.

[9] ______, *Anti-Lyapunov method for systems of discrete equations*, Nonlinear Analysis **57** (2004), no. 7-8, 1043–1057.

[10] D. Ya. Khusainov and G. V. Shuklin, *Linear autonomous time-delay system with permutation matrices solving*, Studies of the University of Žilina. Mathematical Series **17** (2003), no. 1, 101–108.

[11] E. Liz and M. Pituk, *Asymptotic estimates and exponential stability for higher-order monotone difference equations*, Advances in Difference Equations **2005** (2005), no. 1, 41–55.

[12] M. Migda, A. Musielak, and E. Schmeidel, *On a class of fourth-order nonlinear difference equations*, Advances in Difference Equations **2004** (2004), no. 1, 23–36.

[13] Ch. G. Philos and I. K. Purnaras, *An asymptotic result for some delay difference equations with continuous variable*, Advances in Difference Equations **2004** (2004), no. 1, 1–10.

J. Diblík: Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 602 00 Brno, Czech Republic
E-mail address: diblik@feec.vutbr.cz

D. Ya. Khusainov: Faculty of Cybernetics, Kiev National Taras Shevchenko University, 01033 Kiev, Ukraine
E-mail address: khusainov@unicyb.kiev.ua