ROTATIONALLY INVARIANT PERIODIC SOLUTIONS OF SEMILINEAR WAVE EQUATIONS

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ABSTRACT. Under suitable conditions we are able to solve the semilinear wave equation in any dimension. We are also able to compute the essential spectrum of the linear wave operator for the rotationally invariant periodic case.

1. Introduction

In this paper we continue the work of Smiley [SM] and Ben-Naoum and Mawhin [BNM1] concerning radially symmetric solutions for the problem

(1.1) $$u_{tt} - \Delta u = f(t, x, u), \quad t \in \mathbf{R}, \quad x \in B_R$$

(1.2) $$u(t, x) = 0, \quad t \in \mathbf{R}, \quad x \in \partial B_R$$

(1.3) $$u(t + T, x) = u(t, x), \quad t \in \mathbf{R}, \quad x \in B_R,$$

where

$$B_R = \{x \in \mathbf{R}^n : |x| < R\}.$$

Our basic assumption is that

(1.4) $$8R/T = a/b,$$

where a, b are relatively prime positive integers. We show that

(1.5) $$n \not\equiv 3 \qquad (\mathrm{mod}(4, a))$$

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implies that the linear problem corresponding to (1.1) – (1.3) has no essential spectrum. If

$$n \equiv 3 \qquad (\mathrm{mod}(4, a)), \tag{1.6}$$

then the essential spectrum of the linear operator consists of precisely one point

$$\lambda_0 = -(n-3)(n-1)/4R^2. \tag{1.7}$$

(This shows that the spectrum has at most one limit point.) We can then consider the nonlinear case

$$f(t, r, s) = \mu s + p(t, r, s), \tag{1.8}$$

where μ is a point in the resolvent set, $r = |x|$, and

$$|p(t, r, s)| \le (|s|^\theta + 1), \quad s \in \mathbf{R} \tag{1.9}$$

for some number $\theta < 1$. Our main theorem is

Theorem 1.1. *If (1.5) holds, then (1.1) – (1.3) has a weak rotationally invariant solution. If (1.6) holds and $\lambda_0 < \mu$, assume in addition that $p(t, r, s)$ is nondecreasing in s. If $\mu < \lambda_0$, assume that $p(t, r, s)$ is nonincreasing in s. Then (1.1) – (1.3) has a weak rotationally invariant solution.*

The case T$= 2\pi$, $2R = \pi$ was considered in detail in [BNM1]. They proved the existence of a weak solution for n even and $n = 1, 3$. They consider more general situations than (1.8), (1.9). However, our methods can be adjusted to cover their case as well. Uniqueness theorems were also treated in [BNM1]. They also considered odd $n > 5$ when the spectrum of the linear problem is not dense. However, they do not establish when this is the case.

A main consideration in our approach is the following theorem concerning infinite dimensional linking. It is of interest in its own right and has several other applications.

Theorem 1.2. *Let N be a closed separable subspace of a Hilbert space E. Let G be a continuously differentiable functional on E such that*

$$v_n = Pu_n \to v \text{ weakly in } E, \quad w_n = (I-P)u_n \to w \text{ strongly in } E$$

implies

$$G'(v_n + w_n) \to G'(v + w) \text{ weakly in } E, \tag{1.10}$$

where P is the projection of E onto N. Let Q be a bounded open convex subset of N, and let F be a continuous map of E onto N such that

$$F|_N = I, F(v - w) = v - Fw, \quad v \in N,\ w \in E. \tag{1.11}$$

Assume

$$a_0 := \sup_A G \le b_0 := \inf_B G,\ a_1 := \sup_Q G < \infty, \tag{1.12}$$

where $A = \partial Q, B = F^{-1}(p)$ and p is a point in Q. Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c,\ b_0 \le c \le a_1,\ G'(u_k) \to 0. \tag{1.13}$$

Theorem 1.2 will be proved in Section 4. It generalizes theorems in [KS,S2,3,Wi]. Theorem 1.1 will be proved in Section 3 after the essential spectrum of the linear operator is determined in Section 2.

2. The Spectrum of the linear operator

In proving Theorem 1.1 we shall need to calculate the spectrum of the linear operator $\Box$ applied to periodic rotationally symmetric functions. Specifically, we shall need

Theorem 2.1. *Let L_0 be the operator*

$$L_0 u = u_{tt} - u_{rr} - r^{-1}(n-1)u_r \tag{2.1}$$

applied to functions $u(t,r)$ in $C^\infty(\bar{\Omega})$ satisfying

$$u(T,r) = u(0,r),\ u_t(T,r) = u_t(0,r), \quad 0 \le r \le R \tag{2.2}$$

$$u(t,R) = u_R(t,0) = 0, \quad t \in \mathbf{R}, \tag{2.3}$$

where $\Omega = [0,T] \times [0,R]$. Then L_0 is symmetric on $L^2(\Omega,\rho)$, where $\rho = r^{n-1}$. Assume that $8R/T = a/b$, where a,b are relatively prime integers (i.e., $(a,b) = 1$). Then L_0 has a selfadjoint extension L having no essential spectrum other than the point $\lambda_0 = -(n-3)(n-1)/4R^2$. If $n \not\equiv 3(mod(4,a))$, then L has no essential spectrum. If $n \equiv 3(mod(4,a))$, then the essential spectrum of L is precisely the point λ_0.

Proof. Let $\nu = (n-2)/2$, and let γ be a positive root of $J_\nu(x) = 0$, where J_ν is the Bessel function of the first kind. Set

$$\varphi(r) = J_\nu(\gamma r/R)/r^\nu. \tag{2.4}$$

Then

$$\varphi'' + (n-1)\varphi'/r = (x^2 J_\nu'' + xJ_\nu' - \nu^2 J_\nu)/r^{\nu+2} = -\gamma^2 J_\nu/R^2.$$

If

$$\psi(t,r) = \varphi(r)e^{2\pi ikt/T},$$

then

$$L_0\psi = [(\gamma/R)^2 - (2\pi k/T)^2]\psi. \tag{2.5}$$

Let γ_j be the j-th positive root of $J_\nu(x) = 0$, and set

$$\psi_{jk}(t,r) = r^{-\nu} J_\nu(\gamma_j r/R)e^{2\pi ikt/T}. \tag{2.6}$$

Then $\psi_{jk}(t,r)$ is an eigenfunction of L_0 with eigenvalue

$$\lambda_{jk} = (\gamma_j/R)^2 - (2\pi k/T)^2. \tag{2.7}$$

It is easily checked that the functions ψ_{jk}, when normalized, form a complete orthonormal sequence in $L^2(\Omega,\rho)$. We shall show that the corresponding eigenvalues (2.7) are not dense in $\mathbf{R}$. It will then follow that L_0 has a selfadjoint extension L with spectrum equal to the closure of the set $\{\lambda_{jk}\}$. Now

$$\gamma_j = \beta_j - (\mu-1)/8\beta_j + O(\beta_j^{-3}) \text{ as } \beta_j \to \infty, \tag{2.8}$$

where

$$\beta_j = \pi(j + \frac{1}{2}\,\nu - \frac{1}{4}), \quad \mu = 4\nu^2 \tag{2.9}$$

(cf., e.g., [WA]). Thus

$$\begin{aligned} \lambda_{jk}R^2 &= [\beta_j - \tau_k - (\mu-1)/8\beta_j + O(\beta_j^{-3})] \\ &\quad \cdot[\beta_j + \tau_k - (\mu-1)/8\beta_j + O(\beta_j^{-3})] \\ &= \beta_j^2 - \tau_k^2 - (\mu-1)/4 + O(\beta_j^{-2}), \end{aligned}$$

where $\tau_k = 2k\pi R/T$. (We may assume $k \geq 0$.) Now

$$\beta_j - \tau_k = \pi(j + \frac{1}{2}\,\nu - \frac{1}{4} - ak/4b) = \pi[(4j+n-3)b - ak]/4b. \tag{2.10}$$

Since the expression in the brackets is an integer, we see that either $\beta_j = \tau_k$ or

$$|\beta_j - \tau_k| \geq \pi/4b. \tag{2.11}$$

Thus

$$\lim_{\substack{j,|k|\to\infty \\ \beta_j=\tau_k}} \lambda_{jk} = -(\mu-1)/4R^2 = \lambda_0 \tag{2.12}$$

and

$$\lim_{\substack{j,|k|\to\infty \\ \beta_j\neq\tau_k}} |\lambda_{jk}| = \infty. \tag{2.13}$$

If $n-3$ is not a multiple of $(4,a)$, then

$$\beta_j - \tau_k = \pi(4j + n - 3 - ak/b)/4$$

can never vanish. To see this, note that if $(b,k) \neq b$, then ak/b is not an integer. Hence $\beta_j \neq \tau_k$. If $b = (b,k)$, then

$$(n-3) \neq ak' - 4j \quad \forall j,\ k' = k/b.$$

Thus in this case we always have $\beta_j \neq \tau_k$ and $|\lambda_{jk}| \to \infty$ as $j,k \to \infty$. On the other hand, if $n \equiv 3\ (mod(4,a))$, then there is an infinite number of positive integers j, k' such that

$$n - 3 = ak' - 4j.$$

Hence, the point λ_0 is a limit point of eigenvalues. Consequently, it is in $\sigma_e(L)$. This completes the proof. ∎

3. The Nonlinear Case

We now turn to the problem of solving

$$Lu = f(t,r,u), \quad u \in D(L), \tag{3.1}$$

where L is the selfadjoint extension of the operator L_0 given in Theorem 2.1. Under the hypotheses of that theorem the spectrum of L is discrete. We assume that

$$f(t,r,s) = \mu s + p(t,r,s), \tag{3.2}$$

where μ is a point in the resolvent set of L and $p(t,r,s)$ is a Carathéodory function on $\Omega \times \mathbf{R}$ such that

$$|p(t,r,s)| \leq c(|s|^\theta + 1), \quad s \in \mathbf{R} \tag{3.3}$$

for some number $\theta < 1$. We have

Theorem 3.1. *Let $f(t,r,s)$ satisfy (3.2) and (3.3), and assume the hypotheses of Theorem 2.1. If*

$$n \not\equiv 3(mod(4,a)), \tag{3.4}$$

make no further assumptions. If

$$n \equiv 3(mod(4,a)) \tag{3.5}$$

and $\lambda_0 < \mu$, assume that $p(t,r,s)$ is nondecreasing in s. If (3.5) holds and $\mu < \lambda_0$, assume that $p(t,r,s)$ is nonincreasing in s. Then (3.1) has at least one weak solution.

Proof. Since μ is in the resolvent set of L, there is a $\delta > 0$ such that

$$|\lambda_{jk} - \mu| \geq \delta \quad \forall\, j,k, \tag{3.6}$$

where the λ_{jk} are given by (2.7). Each $u \in L^2(\Omega,\rho)$ can be expanded in the form

$$u = \sum \alpha_{jk}\psi_{jk}(t,r), \tag{3.7}$$

where the ψ_{jk} are given by (2.6). Let N_0 be the subspace of those $u \in L^2(\Omega,\rho)$ for which $\alpha_{jk} = 0$ if $\beta_j \neq \tau_k$ (cf. the proof of Theorem 2.1). For $u \in N_0$

$$u = \sum_{[0]} \alpha_{jk}\psi_{jk}(t,r), \tag{3.8}$$

where summation is taken over those j,k for which $\beta_j = \tau_k$. Let E be the subspace of $L^2(\Omega,\rho)$ consisting of those u for which

$$\|u\|_E^2 = \sum |\lambda_{jk} - \mu||\alpha_{jk}|^2 \tag{3.9}$$

is finite. With this norm, E becomes a separable Hilbert space. Note that $E \subset D(|L|^{1/2})$, and the embedding of $E \ominus N_0$ into $L^2(\Omega,\rho)$ is compact (we use (2.13) for this purpose). Let

$$G(u) = ([L-\mu]u, w) - 2\int\int_\Omega P(t,r,u)\rho\, dt dr, \quad u \in E, \tag{3.10}$$

where

$$P(t,r,s) = \int_0^s p(t,r,\sigma)d\sigma, \tag{3.11}$$

and the scalar product is that of $L^2(\Omega,\rho)$. One checks readily that G is a C^1 functional on E with

$$(G'(u),v)/2 = ([L-\mu]u,v) - (p(u),v), \quad u,v \in E, \tag{3.12}$$

where we write $p(u)$ in place of $p(t,r,u)$. This shows that u is a weak solution of (3.1) iff $G'(u) = 0$.

Let N be the subspace of E spanned by the ψ_{jk} corresponding to those $\lambda_{jk} < \mu$ and let M denote the subspace of E spanned by the rest. Thus $M = N^{\perp}$ in E. Assume first that $N \cap N_0 = \{0\}$. Then

$$\|u\|_E^2 = \sum (\mu - \lambda_{jk})|\alpha_{jk}|^2, \quad u \in N. \tag{3.13}$$

Thus

$$\begin{aligned} G(v) &= -\|v\|_E^2 - 2\int\int_\Omega P(t,r,v)\rho dtdr \\ &\leq -\|v\|_E^2 + C\int\int_\Omega (|v|^{1+\theta} + |v|)\rho dtdr \\ &\leq -\|v\|_E^2 + C'(\|v\|^{1+\theta} + \|v\|) \to -\infty, \quad \|v\|_E \to \infty, \quad v \in N. \end{aligned}$$

If $w \in M$,

$$G(w) \geq \delta\|w\|^2 - C(\|w\|^{1+\theta} + \|w\|) \geq -K, \quad w \in M.$$

We can now make use of Theorem 1.2. If Q is a large ball in N, then

$$\sup_{\partial Q} G \leq \inf_M G.$$

Moreover, if $\{u_k\} \subset E$ is a sequence such that $v_k = Pu_k \to v = Pu$ weakly on N and $w_k = (I-P)u_k \to w = (I-P)u$ strongly in M, where P is the projection of E onto N, then $\{u_k\}$ has a renamed subsequence which converges strongly in $L^2(\Omega, \rho)$. The reason is that $\{v_k\}$ has such a subsequences because the embedding of $E \ominus N_0$ in $L^2(\Omega, \rho)$ is compact. Thus $G'(u_n) \to G'(u)$ weakly in E. Hence all of the hypotheses of Theorem 1.2 are satisfied, and we can conclude that there is a sequence $\{u_k\}$ satisfying (1.13). Write $u_k = v_k + w_k + y_k$, where $v_k \in N$, $w_k \in M \ominus N_0$, $y_k \in N_0$. Then

$$(G'(u_k), v_k)/2 = ([L-\mu]u_k, v_k) - (p(u_k), v_k),$$

and consequently

$$\|v_k\|_E^2 \leq \|G'(u_k)\|\|v_k\|_E/2 + C\|v_k\|(\|u_k\|^\theta + 1) \tag{3.14}$$

in view of (3.3) and (3.9). Similarly

$$\|w_k\|_E^2 \leq \|G'(u_k)\|\|w_k\|_E/2 + C\|w_k\|(\|u_k\|^\theta + 1). \tag{3.15}$$

If $N_0 = \{0\}$, then if follows from (3.14) and (3.15) that $\|u_k\|_E$ is bounded, and consequently there is a renamed subsequence which converges weakly in E and strongly in $L^2(\Omega, \rho)$ to a function u. Thus $G'(u_k) \to G'(u)$ weakly. But $G'(u_k) \to 0$. Consequently $G'(u) = 0$, and the proof for this case is complete. If $N_0 \neq \{0\}$, we note that

$$\|y_k\|_E^2 \leq \|G'(u_k)\|\|y_k\|_E/2 + C\|y_k\|(\|u_k\|^\theta + 1) \tag{3.16}$$

as well. Again this together with (3.14) and (3.15) implies that $\|u_k\|_E$ is bounded and has a renamed subsequence which converges weakly in E and such that $u_k' = v_k + w_k$ converges strongly in $L^2(\Omega, \rho)$. Now

$$(3.17)\qquad \begin{aligned}(G'(u_k), y_k - y)/2 = &([L-\mu](y_k - y), y_k - y) \\ &- (p(u_k) - p(u_k' + y), y_k - y) \\ &+ (p(u_k' + y) - p(u), y_k - y) \\ &+ ([L-\mu]y, y_k - y),\end{aligned}$$

where $y_k \to y$ weakly in E and $L^2(\Omega, \rho)$ and $u_k' \to u'$ weakly in E and strongly in $L^2(\Omega, \rho)$. By hypothesis

$$(3.18)\qquad (p(u_k) - p(u_k' + y), y_k - y) \geq 0$$

since $\mu < \lambda_0$. Moreover,

$$\begin{aligned}(G'(u_k), y_k - y) &\to 0 \\ (p(u_k' + y) - p(u), y_k - y) &\to 0\end{aligned}$$

and

$$([L-\mu]y, y_k - y) \to 0.$$

Hence

$$\|y_k - y\|_E^2 \leq o(1), \quad k \to \infty.$$

This shows that $y_k \to y$ in E, and the proof proceeds as before. If $\lambda_0 < \mu$, we apply Theorem 1.2 to $-G(u)$ and come to the same conclusion. In this case, the inequality in (3.17) is reversed. This completes the proof. ∎

4. Weak Linking

We now give a proof of Theorem 1.2. It is similar to those of [KS,S2,3,Wi]. Assume that there is no sequence satisfying (1.13). Then there is a positive number δ such that

$$(4.1)\qquad \|G'(u)\| \geq 2\delta$$

whenever u belongs to the set

$$(4.2)\qquad E_1 = \{u \in E : b_0 - 2\delta \leq G(u) \leq a_1 + 2\delta\}.$$

Since N is separable, we can norm it with a norm $|v|_w$ satisfying

$$(4.3)\qquad |v|_w \leq \|v\|, \quad v \in N$$

and such that the topology induced by this norm is equivalent to the weak topology of N on bounded subsets of N (cf., e.g., [DS,p.426]). For $u \in E$, we write $u = v + w$, where $v \in N, w \in M = N^{\perp}$, and take

$$(4,4)\qquad |u|_w^2 = |v|_w^2 + \|w\|^2.$$

Then clearly

$$(4.5)\qquad |u|_w \leq \|u\|, \quad u \in E,$$

and convergence of a bounded sequence $u_n = v_n + w_n$ with respect to this norm means that v_n converges weakly in N and w_n converges strongly in

M. We denote E equipped with this norm by E_w. For $u \in E_1$, let $q(u) = G'(u)/\|G'(u)\|$. Then by (4.1)

$$(G'(u), q(u)) \geq 2\delta, \quad u \in E_1. \tag{4.6}$$

Let $T = (a_1 - b_0 + 4\delta)/\delta$, $R = T + \sup_Q \|u\|$, and $B = \bar{B}_R \cap E_1$, where $B_R = \{u \in E : \|u\| < R\}$. For each $u \in B$ there is a E_w neighborhood $W(u)$ of u such that

$$(G'(h), q(u)) > \delta, \quad h \in W(u) \cap B. \tag{4.7}$$

For otherwise there would be a sequence $\{h_k\} \subset B$ such that

$$|h_k - u|_w \to 0 \text{ and } (G'(h_k), q(u)) \leq \delta. \tag{4.8}$$

Since B is bounded in $E, Ph_k \to Pu$ weakly in N and $(I-P)h_k \to (I-P)u$ strongly in M. Hence, by hypothesis,

$$(G'(h_k), q(u)) \to (G'(u), q(u)) \geq 2\delta$$

in view of (4.6). This contradicts (4.8). Let B_w be the set B with the inherited topology of E_w. It is a metric space, and $W(u) \cap B$ is an open set in this space. Thus $\{W(u) \cap B\}, u \in B$, is an open covering of the paracompact space B_w. Consequently, there is a locally finite refinement $\{W_\tau\}$ of this cover. For each τ there is an element u_τ such that $W_\tau \subset W(u_\tau)$. Let $\{\psi_\tau\}$ be a partition of unity subordinate to this covering. Each ψ_τ is locally Lipschitz continuous with respect to the norm $|u|_w$ and consequently with respect to the norm of E. Let

$$Y(u) = \sum \psi_\tau(u) q(u_\tau), \quad u \in B. \tag{4.9}$$

Then $Y(u)$ is locally Lipschitz continuous with respect to both norms. Moreover,

$$\|Y(u)\| \leq \sum \psi_\tau(u) \|q(u_\tau)\| \leq 1 \tag{4.10}$$

and

$$(G'(u), Y(u)) = \sum \psi_\tau(u)(G'(u), h(u_\tau)) \geq \delta, \quad u \in B. \tag{4.11}$$

For $u \in \bar{Q} \cap E_1$, let $\sigma(t)u$ be the solution of

$$\sigma'(t) = -Y(\sigma(t)), \quad t \geq 0, \ \sigma(0) = u. \tag{4.12}$$

Note that $\sigma(t)u$ will exist as long as $\sigma(t)u$ is in B. Moreover, it is continuous in (u,t) with respect to both topologies.

Next we note that if $u \in \bar{Q} \cap E_1$ and $\sigma(t)u \in B$, then

$$\begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma), \sigma') \\ &= -(G'(\sigma), Y(\sigma)) \leq -\delta. \end{aligned}$$

Hence if $\sigma(t)u \in B$ for $0 \leq t \leq T$, then

$$G(\sigma(T)u) \leq G(u) - \delta T \leq b_0 - 4\delta. \tag{4.14}$$

But if $\sigma(s)u$ exists for $0 \leq s \leq t < T$, then $\sigma(t)u \in B$. To see this note that

$$u - \sigma(t)u = z_t(u) := \int_0^t Y(\sigma(s)u) ds. \tag{4.15}$$

By (4.10)

$$\|z_t(u)\| \le t.$$

Consequently

$$\|\sigma(t)u\| \le \|u\| + t < R. \tag{4.16}$$

Thus $\sigma(t)u \in B$. We can now conclude that for each $u \in \bar{Q} \cap E_1$ there is a $t < T$ such that $\sigma(s)u$ exists for $0 \le s \le t$ and $G(\sigma(t)u) \le b_0 - \delta$. Let

$$T_u := \inf\{t \ge 0 : G(\sigma(t)u) \le b_0 - \delta\}, u \in \bar{Q} \cap E_1. \tag{4.17}$$

Then $\sigma(t)u$ exists for $0 \le t \le T_u < T$. Moreover, T_u is continuous in u. Define

$$\begin{aligned} \sigma_1(t)u &= \sigma(t)u, \quad 0 \le t \le T_u \\ &= \sigma(T_u)u, \quad T_u \le t \le T \end{aligned} \tag{4.18}$$

for $u \in \bar{Q} \cap E_1$. For $u \in \bar{Q} \setminus E_1$, define $\sigma_1(t)u = u, \quad 0 \le t \le T$. Then $\sigma_1(t)u$ is continuous in (u, t), and

$$G(\sigma_1(T)u) \le b_0 - \delta, \quad u \in \bar{Q}. \tag{4.19}$$

Let

$$\varphi(v, t) = F\sigma_1(t)v, \quad v \in \bar{Q},\ 0 \le t \le T. \tag{4.20}$$

Then φ is a continuous map of $K := \bar{Q} \times [0, T]$ to N. Moreover, K is a compact subset of $N_w \times [0, T]$. For if $(v_k, t_k) \in K$, there is a renamed subsequence such that $v_k \to v_0$ weakly in N and $t_k \to t_0$ in $[0, T]$. Since $\bar{Q}$ is convex, v_0 is in $\bar{Q}$. Since $\bar{Q}$ is bounded, $|v_k - v_0|_w \to 0$. Each $u_0 \in B$ has a neighborhood $W(u_0)$ in E_w and a finite dimensional subspace $S(u_0)$ such that $Y(u) \subset S(u_0)$ for $u \in W(u_0) \cap B$. Since $\sigma_1(t)v$ is continuous is (v, t), for each (v_0, t_0) in K there are a neighborhood $W(v_0, t_0) \subset N \times [0, T]$ and a finite dimensional subspace $S(v_0, t_0) \subset N$ such that $Fz_{1t}(v) \subset S(v_0, t_0)$ for $(v, t) \in W(v_0, t_0)$, where

$$z_{1t}(v) := \int_0^t Y(\sigma_1(s)v)ds.$$

Since K is compact, there is a finite number of points $(v_j, t_j) \subset K$ such that $K \subset W = \cup W(v_j, t_j)$. Let S be a finite dimensional subspace of N containing p and all the $S(v_j, t_j)$. We note that $\varphi(v, t)$ maps $\bar{Q} \cap S \times [0, T]$ into S since $F\sigma_1(t)v = v - Fz_{1t}(v)$, and $Fz_{1t}(v)$ is in S when v is in $\bar{Q}$. Let $\varphi_t(v) = \varphi(v, t)$, $(v, t) \in K$. Then

$$\varphi_t(v) \ne p, \quad v \in \partial(Q \cap S) = \partial Q \cap S,\ 0 \le t \le T. \tag{4.21}$$

For if $\varphi(v, t) = p$, then $\sigma_1(t)v \in F^{-1}(p) = B$. This implies $G(\sigma_1(t)v) \ge b_0$ by (1.12). But (4.13) and (1.12) imply that $G(\sigma_1(t)v) < b_0$ for $t > 0$. Since $p \notin \partial Q$ by hypothesis, $\varphi_0(v) = v \ne p$. Thus (4.21) holds. Consequently the Brouwer degree $d(\varphi_t, Q \cap S, p)$ can be defined. Since $\varphi_t(v)$ is continuous, we have

$$d(\varphi_T, Q \cap S, p) = d(\varphi_0, Q \cap S, p) = d(I, Q \cap S, p) = 1.$$

Hence there is a $v \in Q$ such that $F\sigma_1(T)v = p$. Consequently, $\sigma_1(T)v \in F^{-1}(p) = B$. In view of (1.12), this implies

$$G(\sigma_1(T)u) \geq b_0,$$

contradicting (4.19). This completes the proof. ∎

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