Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 903982, 5 pages
http://dx.doi.org/10.1155/2013/903982

Research Article

Bounds for the Combinations of Neuman-Sándor, Arithmetic, and Second Seiffert Means in terms of Contraharmonic Mean

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Received 4 January 2013; Accepted 27 February 2013

Academic Editor: Salvatore A. Marano

We give the greatest values r_1, r_2 and the least values s_1, s_2 in $(1/2, 1)$ such that the double inequalities $C(r_1a+(1-r_1)b, r_1b+(1-r_1)a) < \alpha A(a,b) + (1-\alpha)T(a,b) < C(s_1a + (1-s_1)b, s_1b + (1-s_1)a)$ and $C(r_2a + (1-r_2)b, r_2b + (1-r_2)a) < \alpha A(a,b) + (1-\alpha)M(a,b) < C(s_2a + (1-s_2)b, s_2b + (1-s_2)a)$ hold for any $\alpha \in (0,1)$ and all $a, b > 0$ with $a \neq b$, where $A(a,b)$, $M(a,b)$, $C(a,b)$, and $T(a,b)$ are the arithmetic, Neuman-Sándor, contraharmonic, and second Seiffert means of a and b, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a,b)$ [1], second Seiffert mean $T(a,b)$ [2] are defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, \qquad T(a,b) = \frac{a-b}{2\arctan((a-b)/(a+b))}, \tag{1}$$

respectively. Herein, $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a+b)/2$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, quadratic, and contraharmonic means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b) \tag{2}$$

hold for all $a, b > 0$ with $a \neq b$.

Among means of two variables, the Neuman-Sándor, contraharmonic, and second Seiffert means have attracted the attention of several researchers. In particular, many remarkable inequalities and applications for these means can be found in the literature [3–15].

Neuman and Sándor [1, 16] proved that the inequalities

$$A(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})},$$
$$\frac{\pi}{4}T(a,b) < M(a,b) < T(a,b),$$
$$M(a,b) < \frac{2A(a,b) + Q(a,b)}{3},$$
$$P(a,b)M(a,b) < A^2(a,b),$$

$$A(a,b)T(a,b) < M^2(a,b) < \frac{\left(A^2(a,b)+T^2(a,b)\right)}{2} \tag{3}$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$. Then the Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \tag{4}$$

can be found in [1].

Li et al. [17] proved that the double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = I(a,b)$ and $L_{-1}(a,b) = L(a,b)$ is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [18], Neuman proved that the inequalities

$$\begin{aligned} &\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) \\ &\quad < \beta Q(a,b) + (1-\beta)A(a,b), \\ &\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) \\ &\quad < \mu C(a,b) + (1-\mu)A(a,b) \end{aligned} \tag{5}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \le [1 - \log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots$, $\lambda \le [1 - \log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$, $\beta \ge 1/3$ and $\mu \ge 1/6$.

Zhao et al. [19] found the least values $\alpha_1, \alpha_2, \alpha_3$ and the greatest values $\beta_1, \beta_2, \beta_3$ such that the double inequalities

$$\begin{aligned} &\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) \\ &\quad < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ &\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) \\ &\quad < \beta_2 G(a,b) + (1-\beta_2)Q(a,b), \\ &\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) \\ &\quad < \beta_3 H(a,b) + (1-\beta_3)C(a,b) \end{aligned} \tag{6}$$

hold for all $a, b > 0$ with $a \neq b$.

In [20, 21], the authors proved that the double inequalities

$$\begin{aligned} &\alpha_1 T(a,b) + (1-\alpha_1)G(a,b) < A(a,b) \\ &\quad < \beta_1 T(a,b) + (1-\beta_1)G(a,b), \\ &\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < T(a,b) \\ &\quad < \beta_2 Q(a,b) + (1-\beta_2)A(a,b), \\ &Q^{\alpha_3}(a,b)A^{1-\alpha_3}(a,b) < T(a,b) \\ &\quad < Q^{\beta_3}(a,b)A^{1-\beta_3}(a,b) \end{aligned} \tag{7}$$

hold for all $a, b > 0$ with $a \neq b$ if and only of $\alpha_1 \le 3/5$, $\beta_1 \ge 4/\pi$, $\alpha_2 \le (4-\pi)/[(\sqrt{2}-1)\pi]$, $\beta_2 \ge 2/3$, $\alpha_3 \le 2/3$, and $\beta_3 \ge 4 - 2\log\pi/\log 2$.

For $\alpha, \beta, \lambda, \mu \in (1/2, 1)$, Chu et al. [22, 23] proved that the inequalities

$$\begin{aligned} &C(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) \\ &\quad < C(\beta a + (1-\beta)b, \beta b + (1-\beta)a), \\ &Q(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < T(a,b) \\ &\quad < Q(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \end{aligned} \tag{8}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \le (1 + \sqrt{4/\pi - 1})/2$, $\beta \ge (3+\sqrt{3})/6$, $\lambda \le (1 + \sqrt{16/\pi^2 - 1})/2$ and $\mu \ge (3+\sqrt{6})/6$.

The aim of this paper is to find the greatest values r_1, r_2 and the least values s_1, s_2 such that the double inequalities

$$\begin{aligned} &C(r_1 a + (1-r_1)b, r_1 b + (1-r_1)a) \\ &\quad < \alpha A(a,b) + (1-\alpha)T(a,b) \\ &\quad < C(s_1 a + (1-s_1)b, s_1 b + (1-s_1)a), \end{aligned} \tag{9}$$

$$\begin{aligned} &C(r_2 a + (1-r_2)b, r_2 b + (1-r_2)a) \\ &\quad < \alpha A(a,b) + (1-\alpha)M(a,b) \\ &\quad < C(s_2 a + (1-s_2)b, s_2 b + (1-s_2)a) \end{aligned} \tag{10}$$

hold for any $\alpha \in (0,1)$ and all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main results, we need three lemmas, which we present in this section.

Lemma 1 (see [24, Theorem 1.25]). *For $-\infty < a < b < +\infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}. \tag{11}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2. *Let $u, \alpha \in (0,1)$ and*

$$f_{u,\alpha}(x) = ux^2 - (1-\alpha)\left(\frac{x}{\arctan x} - 1\right). \tag{12}$$

Then $f_{u,\alpha}(x) > 0$ for all $x \in (0,1)$ if and only if $u \geq (1-\alpha)/3$ and $f_{u,\alpha}(x) < 0$ for all $x \in (0,1)$ if and only if $u \leq (1-\alpha)(4/\pi - 1)$.

Proof. From (12), one has

$$f_{u,\alpha}(0^+) = 0, \tag{13}$$

$$f_{u,\alpha}(1^-) = u - (1-\alpha)\left(\frac{4}{\pi} - 1\right), \tag{14}$$

$$f'_{u,\alpha}(x) = 2x\left[u - \frac{1-\alpha}{2}g(x)\right], \tag{15}$$

where

$$g(x) = \frac{(1+x^2)\arctan x - x}{x(1+x^2)(\arctan x)^2}. \tag{16}$$

Let $g_1(x) = \arctan x - x/(1+x^2)$ and $g_2(x) = x(\arctan x)^2$, then

$$g(x) = \frac{g_1(x)}{g_2(x)}, \qquad g_1(0) = g_2(0) = 0, \tag{17}$$

$$\begin{aligned}\frac{g_1'(x)}{g_2'(x)} &= \frac{2x^2}{2x(1+x^2)\arctan x + (1+x^2)^2(\arctan x)^2} \\ &= \frac{1}{((1+x^2)\arctan x/x) + (1/2)\left[(1+x^2)\arctan x/x\right]^2}.\end{aligned} \tag{18}$$

It is not difficult to verify that the function $(1+x^2)\arctan x/x$ is strictly increasing on $(0,1)$. Then (17) and (18) together with Lemma 1 lead to the conclusion that $g(x)$ is strictly decreasing on $(0,1)$. Moreover, making use of L'Hôpital's rule, we get

$$g(0^+) = \frac{2}{3}, \tag{19}$$

$$g(1^-) = \frac{4(\pi-2)}{\pi^2}. \tag{20}$$

We divide the proof into four cases.

Case 1. $u \geq (1-\alpha)/3$. Then from (15) and (19) together with the monotonicity of $g(x)$, we clearly see that $f_{u,\alpha}(x)$ is strictly increasing on $(0,1)$. Therefore, $f_{u,\alpha}(x) > 0$ for all $x \in (0,1)$ follows from (13) and the monotonicity of $f_{u,\alpha}(x)$.

Case 2. $u \leq 2(1-\alpha)(\pi-2)/\pi^2$. Then from (15) and (20) together with the monotonicity of $g(x)$, we clearly see that $f_{u,\alpha}(x)$ is strictly decreasing on $(0,1)$. Therefore, $f_{u,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (13) and the monotonicity of $f_{u,\alpha}(x)$.

Case 3. $2(1-\alpha)(\pi-2)/\pi^2 < u \leq (1-\alpha)(4/\pi - 1)$. Then (14) leads to

$$f_{u,\alpha}(1^-) \leq 0. \tag{21}$$

From (15), (19), and (20) together with the monotonicity of $g(x)$, we clearly see that there exists unique $x_0 \in (0,1)$ such that $f_{u,\alpha}(x)$ is strictly decreasing on $(0, x_0]$ and strictly increasing on $[x_0, 1)$. Therefore, $f_{u,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (13) and (21) together with the piecewise monotonicity of $f_{u,\alpha}(x)$.

Case 4. $(1-\alpha)(4/\pi - 1) < u \leq (1-\alpha)/3$. Then (14) leads to

$$f_{u,\alpha}(1^-) > 0. \tag{22}$$

It follows from (15), (19), and (20) together with the monotonicity of $g(x)$, there exists unique $x_1 \in (0,1)$ such that $f_{u,\alpha}(x)$ is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1)$. Equation (13) and inequality (22) together with the piecewise monotonicity of $f_{u,\alpha}(x)$ lead to the conclusion that there exists $x_2 \in (x_1, 1)$ such that $f_{u,\alpha}(x) < 0$ for $x \in (0, x_2)$ and $f_{u,\alpha}(x) > 0$ for $x \in (x_2, 1)$. □

Lemma 3. *Let $\lambda, \alpha \in (0,1)$ and*

$$\varphi_{\lambda,\alpha}(x) = \lambda x^2 - (1-\alpha)\left(\frac{x}{\sinh^{-1}(x)} - 1\right). \tag{23}$$

Then $\varphi_{\lambda,\alpha}(x) > 0$ for all $x \in (0,1)$ if and only if $\lambda \geq (1-\alpha)/6$ and $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0,1)$ if and only if $\lambda \leq (1-\alpha)(1 - \log(1+\sqrt{2}))/\log(1+\sqrt{2})$.

Proof. From (23) we get

$$\varphi_{\lambda,\alpha}(0^+) = 0, \tag{24}$$

$$\varphi_{\lambda,\alpha}(1^-) = \lambda - \frac{(1-\alpha)\left[1 - \log\left(1+\sqrt{2}\right)\right]}{\log\left(1+\sqrt{2}\right)}, \tag{25}$$

$$\varphi'_{\lambda,\alpha}(x) = 2x\left[\lambda - \frac{1-\alpha}{2}\psi(x)\right], \tag{26}$$

where

$$\psi(x) = \frac{\sinh^{-1}(x) - x/\sqrt{1+x^2}}{x\left(\sinh^{-1}(x)\right)^2}. \tag{27}$$

Let $\psi_1(x) = \sinh^{-1}(x) - x/\sqrt{1+x^2}$ and $\psi_2(x) = x(\sinh^{-1}(x))^2$, then

$$\psi(x) = \frac{\psi_1(x)}{\psi_2(x)}, \qquad \psi_1(0) = \psi_2(0) = 0,$$

$$\frac{\psi_1'(x)}{\psi_2'(x)} = x^2 \times \left(\left(1+x^2\right)^{3/2} \left(\sinh^{-1}(x)\right)^2 + 2x\left(1+x^2\right)\sinh^{-1}(x) \right)^{-1} = \left(\left(\left(1+x^2\right)^{3/4} \sinh^{-1}(x)/x \right)^2 + 2\left(1+x^2\right)^{1/4} \left(\left(1+x^2\right)^{3/4} \sinh^{-1}(x)/x \right) \right)^{-1}. \tag{28}$$

It is not difficult to verify that the function $(1+x^2)^{3/4}\sinh^{-1}(x)/x$ is strictly increasing on $(0,1)$. Then (28) together with Lemma 1 leads to the conclusion that $\psi(x)$ is strictly decreasing on $(0,1)$. Moreover, making use of L'Hôpital's rule, we have

$$\psi(0^+) = \frac{1}{3}, \tag{29}$$

$$\psi(1^-) = \frac{\sqrt{2}\log\left(1+\sqrt{2}\right)-1}{\sqrt{2}\log^2\left(1+\sqrt{2}\right)}. \tag{30}$$

We divide the proof into four cases.

Case 1. $\lambda \geq (1-\alpha)/6$. Then from (26) and (29) together with the monotonicity of $\psi(x)$, we clearly see that $\varphi_{\lambda,\alpha}(x)$ is strictly increasing on $(0,1)$. Therefore, $\varphi_{\lambda,\alpha}(x) > 0$ for all $x \in (0,1)$ follows from (24) and the monotonicity of $\varphi_{\lambda,\alpha}(x)$.

Case 2. $\lambda \leq (1-\alpha)[\sqrt{2}\log(1+\sqrt{2})-1]/[2\sqrt{2}\log^2(1+\sqrt{2})]$. Then from (26) and (30) together with the monotonicity of $\psi(x)$, we clearly see that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0,1)$. Therefore, $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (24) and the monotonicity of $\varphi_{\lambda,\alpha}(x)$.

Case 3. $((1-\alpha)[\sqrt{2}\log(1+\sqrt{2})-1]/2\sqrt{2}\log^2(1+\sqrt{2})) < \lambda \leq ((1-\alpha)[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}))$. Then (25) leads to

$$\varphi_{\lambda,\alpha}(1^-) \leq 0. \tag{31}$$

From (26), (29), and (30) together with the monotonicity of $\psi(x)$, we clearly see that there exists $x_3 \in (0,1)$ such that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0,x_3]$ and strictly increasing on $[x_3,1)$. Therefore, $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0,1)$ follows from (24) and (31) together with the piecewise monotonicity of $\varphi_{\lambda,\alpha}(x)$.

Case 4. $((1-\alpha)[1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})) < \lambda < ((1-\alpha)/6)$. Then (25) leads to

$$\varphi_{\lambda,\alpha}(1^-) > 0. \tag{32}$$

It follows from (26), (29), and (30) together with the monotonicity of $\psi(x)$, there exists $x_4 \in (0,1)$ such that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0,x_4]$ and strictly increasing on $[x_4,1)$. Equation (24) and inequality (32) together with the piecewise monotonicity of $\varphi_{\lambda,\alpha}(x)$ lead to the conclusion that there exists $x_5 \in (x_4,1)$ such that $\varphi_{\lambda,\alpha}(x) < 0$ for $x \in (0,x_5)$ and $\varphi_{\lambda,\alpha}(x) > 0$ for $x \in (x_5,1)$. □

3. Main Results

Theorem 4. *If $\alpha \in (0,1)$ and $r_1, s_1 \in (1/2,1)$, then the double inequality*

$$C(r_1a+(1-r_1)b, r_1b+(1-r_1)a) < \alpha A(a,b)+(1-\alpha)T(a,b) < C(s_1a+(1-s_1)b, s_1b+(1-s_1)a) \tag{33}$$

holds for all $a,b>0$ with $a \neq b$ if and only if $r_1 \leq [1+\sqrt{(1-\alpha)(4-\pi)/\pi}]/2$ and $s_1 \geq [1+\sqrt{(1-\alpha)/3}]/2$.

Proof. Since $A(a,b)$, $T(a,b)$, and $C(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a>b$. Let $p \in (1/2,1)$ and $x=(a-b)/(a+b)$, then $x \in (0,1)$ and

$$C(pa+(1-p)b, pb+(1-p)a) - [\alpha A(a,b)+(1-\alpha)T(a,b)] = A(a,b)\left[(2p-1)^2x^2-(1-\alpha)\left(\frac{x}{\arctan x}-1\right)\right]. \tag{34}$$

Therefore, Theorem 4 follows easily from Lemma 2 and (34). □

Theorem 5. *If $\alpha \in (0,1)$ and $r_2, s_2 \in (1/2,1)$, then the double inequality*

$$C(r_2a+(1-r_2)b, r_2b+(1-r_2)a) < \alpha A(a,b)+(1-\alpha)M(a,b) < C(s_2a+(1-s_2)b, s_2b+(1-s_2)a) \tag{35}$$

holds for all $a,b>0$ with $a \neq b$ if and only if $s_2 \geq [1+\sqrt{(1-\alpha)/6}]/2$ and $r_2 \leq [1+\sqrt{(1-\alpha)(1-\log(1+\sqrt{2}))/\log(1+\sqrt{2})}]/2$.

Proof. Since $A(a,b)$, $M(a,b)$, and $C(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a>b$. Let $q \in (1/2,1)$ and $x=(a-b)/(a+b)$, then $x \in (0,1)$ and

$$C(qa+(1-q)b, qb+(1-q)a) - [\alpha A(a,b)+(1-\alpha)M(a,b)] = A(a,b)\left[(2q-1)^2x^2-(1-\alpha)\left(\frac{x}{\sinh^{-1}(x)}-1\right)\right]. \tag{36}$$

Therefore, Theorem 5 follows easily from Lemma 3 and (36). □

Remark 6. If $\alpha = 0$, then Theorem 4 reduces to the first double inequality in (8).

Corollary 7. *If $\lambda, \mu \in (1/2, 1)$, then the double inequality*

$$C(\lambda a + (1-\lambda) b, \lambda b + (1-\lambda) a) < M(a,b) < C(\mu a + (1-\mu) b, \mu b + (1-\mu) a) \tag{37}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq [1 + \sqrt{1/\log(1+\sqrt{2}) - 1}]/2$ and $\mu \geq (6+\sqrt{6})/12$.

Proof. Corollary 7 follows easily from Theorem 5 with $\alpha = 0$. □

Acknowledgments

This research was supported by the Natural Science Foundation of China (Grants no.11171307, 61173123), the Natural Science Foundation of Zhejiang Province (Grants no. Z1110551, LY12F02012), and the Natural science Foundation of Huzhou City (Grant no. 2012YZ06).

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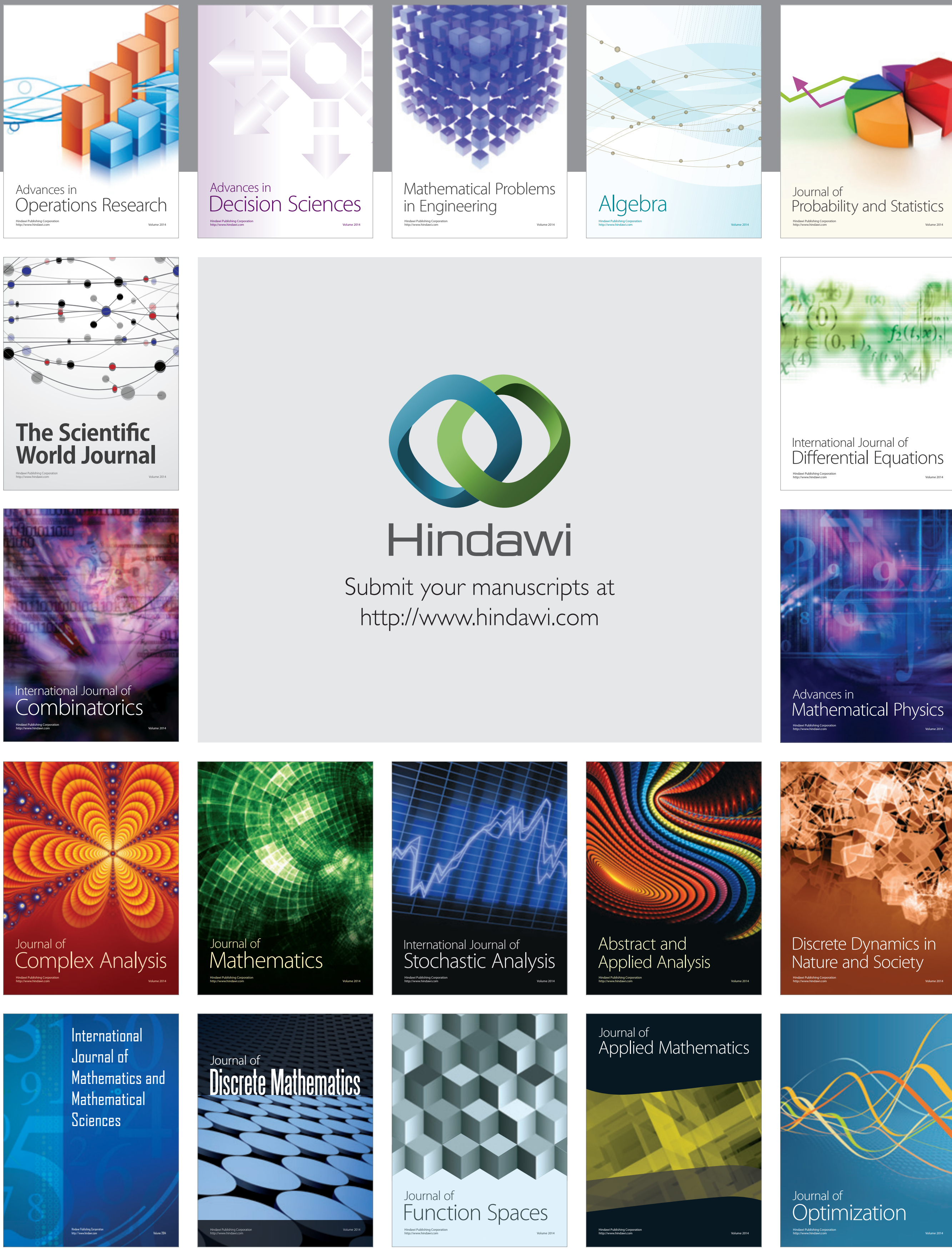
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