Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 512383, 5 pages
http://dx.doi.org/10.1155/2013/512383

Research Article

Rotationally Symmetric Harmonic Diffeomorphisms between Surfaces

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Received 12 February 2013; Revised 22 April 2013; Accepted 22 April 2013

Academic Editor: Yuriy Rogovchenko

We show the nonexistence of rotationally symmetric harmonic diffeomorphism between the unit disk without the origin and a punctured disc with hyperbolic metric on the target.

1. Introduction

The existence of harmonic diffeomorphisms between complete Riemannian manifolds has been extensively studied, please see, for example, [1–34]. In particular, Heinz [17] proved that there is no harmonic diffeomorphism from the unit disc onto $\mathbb{C}$ with its flat metric. On the other hand, Schoen [25] mentioned a question about the existence, or nonexistence, of a harmonic diffeomorphism from the complex plane onto the hyperbolic 2-space. At the present time, many beautiful results about the asymptotic behavior of harmonic embedding from $\mathbb{C}$ into the hyperbolic plane have been obtained, please see, for example, [4, 5, 14, 32] or the review [33] by Wan and the references therein. In 2010, Collin and Rosenberg [10] constructed harmonic diffeomorphisms from $\mathbb{C}$ onto the hyperbolic plane. In [7, 24, 28, 29], the authors therein studied the rotational symmetry case. One of their results is the nonexistence of rotationally symmetric harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic plane.

In this paper, we will study the existence, or nonexistence, of rotationally symmetric harmonic diffeomorphisms from the unit disk without the origin onto a punctured disc. For simplicity, let us denote

$$\mathbb{D}^* = \mathbb{D} \setminus \{0\}, \quad P(a) = \mathbb{D} \setminus \{|z| \le e^{-a}\} \quad \text{for } a > 0, \tag{1}$$

where $\mathbb{D}$ is the unit disc and z is the complex coordinate of $\mathbb{C}$. We will prove the following results.

Theorem 1. *For any $a > 0$, there is no rotationally symmetric harmonic diffeomorphism from $\mathbb{D}^*$ onto $P(a)$ with its hyperbolic metric.*

And vice versa as shown below.

Theorem 2. *For any $a > 0$, there is no rotationally symmetric harmonic diffeomorphism from $P(a)$ onto $\mathbb{D}^*$ with its hyperbolic metric.*

We will also consider the Euclidean case and will prove the following theorem.

Theorem 3. *For any $a > 0$, there is no rotationally symmetric harmonic diffeomorphism from $\mathbb{D}^*$ onto $P(a)$ with its Euclidean metric; but on the other hand, there are rotationally symmetric harmonic diffeomorphisms from $P(a)$ onto $\mathbb{D}^*$ with its Euclidean metric.*

This paper is organized as follows. In Section 2, we will prove Theorems 1 and 2. Theorem 3 will be proved in Section 3. At the last section, we will give another proof for the nonexistence of rotationally symmetric harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic disc.

2. Harmonic Maps from $\mathbb{D}^*$ to $P(a)$ with Its Hyperbolic Metric and Vice Versa

For convenience, let us recall the definition about the harmonic maps between surfaces. Let M and N be two oriented surfaces with metrics $\tau^2|dz|^2$ and $\sigma^2|du|^2$, respectively, where z and u are local complex coordinates of M and N, respectively. A C^2 map u from M to N is harmonic if and only if u satisfies

$$u_{z\bar{z}} + \frac{2\sigma_u}{\sigma} u_z u_{\bar{z}} = 0. \tag{2}$$

Now let us prove Theorem 1.

Proof of Theorem 1. First of all, let us denote (r, θ) as the polar coordinates of $\mathbb{D}^*$ and u as the complex coordinates of $P(a)$ in $\mathbb{C}$; then the hyperbolic metric $\sigma_1 d|u|$ on $P(a)$ can be written as

$$\frac{-\pi\,|du|}{a\,|u|\sin\left((\pi/a)\ln|u|\right)}. \tag{3}$$

Here $|u|$ is the norm of u with respect to the Euclidean metric.

We will prove this theorem by contradiction. Suppose u is a rotationally symmetric harmonic diffeomorphism from $\mathbb{D}^*$ onto $P(a)$, with the metric $\sigma_1 d|u|$. Because $\mathbb{D}^*$, $P(a)$ and the metric $\sigma_1 d|u|$ are rotationally symmetric, we can assume that such a map u has the form $u = f(r)e^{i\theta}$. Substituting $u,\ \sigma_1$ to (2), we can get

$$\begin{aligned} & f'' + \frac{f'}{r} - \frac{f}{r^2} \\ & \quad - \frac{\sin\left((\pi/a)\ln f\right) + (\pi/a)\cos\left((\pi/a)\ln f\right)}{f\sin\left((\pi/a)\ln f\right)} \\ & \quad \times \left(\left(f'\right)^2 - \frac{f^2}{r^2}\right) = 0 \end{aligned} \tag{4}$$

for $1 > r > 0$. Since u is a harmonic diffeomorphism from $\mathbb{D}^*$ onto $P(a)$, we have

$$\begin{gathered} f(0) = e^{-a}, \quad f(1) = 1, \\ f'(r) > 0 \quad \text{for } 1 > r > 0, \end{gathered} \tag{5}$$

or

$$f(0) = 1, \quad f(1) = e^{-a}, \tag{6}$$

$$f'(r) < 0 \quad \text{for } 1 > r > 0. \tag{7}$$

We will just deal with the case that (5) is satisfied; the rest case is similar. Let $F = \ln f \in (-a, 0)$, then we have

$$f' = \frac{f'}{f} > 0, \quad F'' = \frac{f''}{f} - \left(\frac{f'}{f}\right)^2. \tag{8}$$

Using this fact, we can get from (4) the following equation:

$$\begin{aligned} & F'' + \frac{1}{r}F' - \frac{\pi}{a}\operatorname{ctg}\left(\frac{\pi}{a}F\right)\left(F'\right)^2 + \frac{1}{r^2}\frac{\pi}{a}\operatorname{ctg}\left(\frac{\pi}{a}F\right) \\ & \quad = 0 \quad \text{for } 1 > r > 0, \end{aligned} \tag{9}$$

with $F(0) = -a$, $F(1) = 0$, and $F'(r) > 0$ for $1 > r > 0$.

Regarding r as a function of F, we have the following relations:

$$F_r = r_F^{-1}, \qquad F_{rr} = -r_F^{-3} r_{FF}. \tag{10}$$

Using these facts, we can get from (9) the following equation:

$$\begin{aligned} & \frac{r''}{r} - \left(\frac{r'}{r}\right)^2 + \frac{\pi}{a}\operatorname{ctg}\left(\frac{\pi}{a}F\right)\frac{r'}{r} \\ & \quad - \left(\frac{r'}{r}\right)^3 \frac{\pi}{a}\operatorname{ctg}\left(\frac{\pi}{a}F\right) = 0 \end{aligned} \tag{11}$$

for $0 > F > -a$. Let $x = (\ln r)'(F)$; from (11) we can get the following equation:

$$x' + \frac{\pi}{a}\operatorname{ctg}\left(\frac{\pi}{a}F\right)\cdot x - \frac{\pi}{a}\operatorname{ctg}\left(\frac{\pi}{a}F\right)\cdot x^3 = 0. \tag{12}$$

One can solve this Bernoulli equation to obtain

$$x^{-2} = 1 + c_0\left(\sin\left(\frac{\pi}{a}F\right)\right)^2. \tag{13}$$

Here c_0 is a constant depending on the choice of the function f. So

$$x = \frac{1}{\sqrt{1 + c_0\left(\sin\left((\pi/a)F\right)\right)^2}}. \tag{14}$$

Since $x = (\ln r)'(F)$, we can get

$$(\ln r)(F) = \int_0^F x(t)\,dt = \int_0^F \frac{1}{\sqrt{1 + c_0\left(\sin\left((\pi/a)t\right)\right)^2}}\,dt. \tag{15}$$

Noting that $x(F)$ is continuous in $(-a, 0)$ and is equal to 1 as $F = -a$, or 0, one can get x is uniformly bounded for $F \in [-a, 0]$. So the right-hand side of (15) is uniformly bounded, but the left-hand side will tend to $-\infty$ as $F \to -a$. Hence, we get a contradiction. Therefore, such f does not exist, Theorem 1 has been proved. □

We are going to prove Theorem 2.

Proof of Theorem 2. First of all, let us denote (r, θ) as the polar coordinates of $P(a)$ and u as the complex coordinates of $\mathbb{D}^*$ in $\mathbb{C}$; then the hyperbolic metric $\sigma_2 d|u|$ on $\mathbb{D}^*$ can be written as

$$\frac{|du|}{|u|\ln\left(1/|u|\right)}. \tag{16}$$

Here $|u|$ is the norm with respect to the Euclidean metric.

We will prove this theorem by contradiction. The idea is similar to the proof of Theorem 1. Suppose ψ is a rotationally symmetric harmonic diffeomorphism from $P(a)$ onto $\mathbb{D}^*$ with the metric $\sigma_2 d|u|$, with the form $\psi = g(r)e^{i\theta}$, then substituting ψ, σ_2 to u, σ in (2), respectively, we can get

$$g'' + \frac{g'}{r} - \frac{g}{r^2} - \frac{1+\ln g}{g \ln g}\left(\left(g'\right)^2 - \frac{g^2}{r^2}\right) = 0 \tag{17}$$

for $1 > r > e^{-a}$. Since v is a harmonic diffeomorphism from $P(a)$ onto $\mathbb{D}^*$, we have

$$\begin{gathered} g\left(e^{-a}\right) = 0, \quad g(1) = 1, \\ g'(r) > 0 \quad \text{for } 1 > r > e^{-a}, \end{gathered} \tag{18}$$

or

$$g\left(e^{-a}\right) = 1, \; g(1) = 0, \tag{19}$$

$$g'(r) < 0 \quad \text{for } 1 > r > e^{-a}. \tag{20}$$

We will only deal with the case that (18) is satisfied; the rest case is similar. Let $G = \ln g$, then (17) can be rewritten as

$$G'' + \frac{1}{r}G' - \frac{1}{G}\left(G'\right)^2 + \frac{1}{r^2 G} = 0 \tag{21}$$

for $1 > r > e^{-a}$, with $G(1) = 0$ and $\lim_{r \to e^{-a}} G(r) = -\infty$.

Regarding r as a function of G, using a similar formula of (10), from (21) we can get

$$\frac{r''}{r} - \left(\frac{r'}{r}\right)^2 + \frac{r'}{rG} - \frac{1}{G}\left(\frac{r'}{r}\right)^3 = 0, \quad G \in (-\infty, 0). \tag{22}$$

Similar to solving (11), we can get the solution to (22) as follows:

$$(\ln r)'(G) = \frac{1}{\sqrt{1 + c_1 G^2}}, \quad G \in (-\infty, 0) \tag{23}$$

for some nonnegative constant c_1 depending on the choice of g.

If c_1 is equal to 0, then $g = r$; this is in contradiction to (18).

If c_1 is positive, then taking integration on both sides of (23), we can get

$$\begin{aligned} (\ln r)(G) &= \int_0^G \frac{1}{\sqrt{1 + c_1 t^2}} dt \\ &= \frac{1}{\sqrt{c_1}} \ln\left(\sqrt{c_1}G + \sqrt{1 + c_1 G^2}\right). \end{aligned} \tag{24}$$

So

$$r = \left(\sqrt{c_1} \ln g + \sqrt{1 + c_1 \ln^2 g}\right)^{1/\sqrt{c_1}}, \tag{25}$$

with $\lim_{g \to 0+} r(g) = 0$. On the other hand, from (18), we have $r(0) = e^{-a}$. Hence, we get a contradiction. Therefore, such g does not exist, Theorem 2 has been proved. □

3. Harmonic Maps from $\mathbb{D}^*$ to $P(a)$ with Its Euclidean Metric and Vice Versa

Now let us consider the case of that the target has the Euclidean metric.

Proof of Theorem 3. Let us prove the first part of this theorem, that is, show the nonexistence of rotationally symmetric harmonic diffeomorphism from $\mathbb{D}^*$ onto $P(a)$ with its Euclidean metric. The idea is similar to the proof of Theorem 1, so we just sketch the proof here. Suppose there is such a harmonic diffeomorphism φ from $\mathbb{D}^*$ onto $P(a)$ with its Euclidean metric with the form $\varphi = h(r)e^{i\theta}$, and then we can get

$$h'' + \frac{1}{r}h' - \frac{1}{r^2}h = 0 \quad \text{for } 1 > r > 0, \tag{26}$$

with

$$\begin{gathered} h(0) = e^{-a}, \quad h(1) = 1, \\ h'(r) > 0 \quad \text{for } 1 > r > 0, \end{gathered} \tag{27}$$

or

$$\begin{gathered} h(0) = 1, \quad h(1) = e^{-a}, \\ h'(r) < 0 \quad \text{for } 1 > r > 0. \end{gathered} \tag{28}$$

We will just deal with the case that (27) is satisfied; the rest case is similar. Let $H = (\ln h)'$; then we can get

$$H' + H^2 + \frac{1}{r}H = 0 \quad \text{for } 1 > r > 0. \tag{29}$$

Solving this equation, we can get

$$H = \frac{1}{r} + \frac{1}{c_3 r^3 - r^2} = \frac{1}{r} + \frac{c_3^2}{c_3 r - 1} - \frac{c_3}{r} - \frac{1}{r^2}. \tag{30}$$

Here c_3 is a constant depending on the choice of h. So

$$\ln h = \ln r + c_3 \ln\left(c_3 r - 1\right) - c_3 \ln r + \frac{1}{r} + c_4. \tag{31}$$

Here c_4 is a constant depending on the choice of h. Hence

$$h = r\left(c_3 r - 1\right)^{c_3} r^{-c_3} e^{1/r} e^{c_4}. \tag{32}$$

From (32), we can get $\lim_{r \to 0} h(r) = \infty$. On the other hand, from (27), $h(0) = e^{-a}$. We get a contradiction. Hence such a function h does not exist; the first part of Theorem 3 holds.

Now let us prove the second part of this theorem, that is, show the existence of rotationally symmetric harmonic diffeomorphisms from $P(a)$ onto $\mathbb{D}^*$ with its Euclidean metric. It suffices to find a map from $P(a)$ onto $\mathbb{D}^*$ with the form $q(r)e^{i\theta}$ such that

$$q'' + \frac{1}{r}q' - \frac{1}{r^2}q = 0 \quad \text{for } 1 > r > e^{-a} \tag{33}$$

with $q(e^{-a}) = 0$, $q(1) = 1$, and $q' > 0$ for $1 > r > e^{-a}$. Using the boundary condition and (32), we can get that

$$q = e^{-1}\left(e^a - 1\right)^{-e^a} r\left(e^a r - 1\right)^{e^a} r^{-e^a} e^{1/r} \tag{34}$$

is a solution to (33).

Therefore, we finished the proof of Theorem 3. □

4. Harmonic Maps from $\mathbb{C}$ to the Hyperbolic Disc

In this section, we will give another proof of the following result.

Proposition 4. *There is no rotationally symmetric harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic disc.*

Proof. It is well-known that the hyperbolic metric on the unit disc is $(2/(1-z|^2))|dz|$. We will also use the idea of the proof of Theorem 1. Suppose there is such a harmonic diffeomorphism ϕ from $\mathbb{C}$ onto $\mathbb{D}$ with its hyperbolic metric with the form $\phi = k(r)e^{i\theta}$, and then we can get

$$k'' + \frac{1}{r}k' - \frac{1}{r^2}k + \frac{2k}{1-k^2}\left[\left(k'\right)^2 - \frac{k^2}{r^2}\right] = 0 \quad \text{for } r > 0 \tag{35}$$

with

$$k(0) = 0, \quad k'(r) > 0 \quad \text{for } r > 0. \tag{36}$$

Regarding r as a function of k, setting $v = (\ln r)'(k)$, (35) can be rewritten as

$$\left(1-k^2\right)v' - 2kv + v^3\left(k+k^3\right) = 0. \tag{37}$$

That is,

$$\left(v^{-2}\right)' + \frac{4k}{1-k^2}v^{-2} = \frac{2\left(k+k^3\right)}{1-k^2}. \tag{38}$$

One can solve this equation to obtain

$$v^{-2} = k^2 + c_5\left(1-k^2\right)^2 \tag{39}$$

for some nonnegative constant c_5 depending on the choice of the function k.

If $c_5 = 0$, then we can get $r = c_6 k$ for some constant c_6. On the other hand, ϕ is a diffeomorphism, so $k \to 1$ as $r \to \infty$. This is a contradiction.

If $c_5 > 0$, then $b_1 \geq k^2 + c_5(1-k^2)^2 \geq b_2$ for some positive constants b_1 and b_2. So

$$\left|(\ln r)'(k)\right| \leq \frac{1}{\sqrt{b_2}}. \tag{40}$$

This is in contradiction to the assumption that $r \to \infty$ as $k \to 1$.

Therefore, Proposition 4 holds. □

Acknowledgments

The author (Xu-Qian Fan) would like to thank Professor Luen-fai Tam for his very worthy advice. The first author is partially supported by the National Natural Science Foundation of China (11201131); the second author is partially supported by the National Natural Science Foundation of China (11101106).

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