Research Article

Common Fixed Point Theorems for Nonlinear Contractive Mappings in Dislocated Metric Spaces

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In 1986, Matthews generalized Banach contraction mapping theorem in dislocated metric space that is a wider space than metric space. In this paper, we established common fixed point theorems for a class of contractive mappings. Our results extend the corresponding ones of other authors in dislocated metric spaces.

1. Introduction

Fixed point theory is an important branch of nonlinear analysis and can be used to many discipline branches, such as control theory, convex optimization, differential inclusion, and economics. Banach proved a celebrated fixed point theorem for contraction mappings in complete metric space which is one of the pivotal results of analysis. Dass and Gupta [1] generalized Banach contraction mapping in metric space. Also Rhoades [2] established a partial ordering for various definitions of contractive mapping. The concept of metric spaces, as an environmental space in fixed point theory, has been generalized in several directions. Some of such generalizations are dislocated spaces, quasimetric spaces, dislocated quasimetric spaces, and generalized quasimetric spaces. The concept of dislocated spaces is treated differently by different authors. Matthews [3] generalized Banach contraction mapping theorem in dislocated metric space. Hitzler [4] presented variants of Banach contraction principle for various modified forms of a metric space including dislocated metric space and applied them to semantic analysis of logic programs. In this context, Hitzler and Seda [5] raised some related questions on the topological aspects of dislocated metrics. In 2005, Zeyada et al. [6] generalized a fixed point theorem in dislocated quasimetric spaces. In 2008, Aage and Salunke [7] proved some results on fixed points in dislocated quasimetric space. Recently, Isufati [8] proved fixed point theorem for contractive type condition with rational expression in dislocated quasimetric space. In this paper, we study the mapping refereed by Xia Dafeng and obtained fixed point theorems in dislocated metric space. For fixed point theorems, see [9, 10]. The following definition is introduced by Xia et al. [11].

Definition 1 (see [11]). Let $R_+ = [0, \infty)$. Let $G_1 : R_+^2 \to R_+$, $G_2 : R_+^3 \to R_+$, satisfying the following:

(1) if $w \le G_1(u, v)$, then there exists $c \in (0, 1)$, such that $w \le c \max\{u, v\}$;

(2) if $w \le G_2(u, v, r)$, then there exists $c \in (0, 1)$, such that $w \le c \max\{u, v, r\}$.

Theorem 2 (see [11]). *Let (X, d) be complete metric spaces, let $f, g : X \to X$ be continuous mappings, and for all $x, y \in X$, such that*

$$d(f(x), g(y)) \le G_1(d(x, f(x)), d(y, g(y))) \quad (1)$$

or

$$d(f(x), g(y)) \le G_2(d(x, y), d(x, f(x)), d(y, g(y))) \quad (2)$$

then f, g have a unique common fixed point.

2. Preliminaries

Definition 3 (see [6]). Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function called a distance function. If for all $x, y, z \in X$,

(1) nonnegativity: $d(x, y) \geq 0$;

(2) faithful: $d(x, y) = 0 \Leftrightarrow x = y$;

(3) the triangle inequality: $d(x, y) \leq d(x, z) + d(z, x)$,

So, here d is a quasimetric on X, and (X, d) is called a quasimetric space.

Definition 4 (see [6]). Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function called a distance function. If for all $x, y, z \in X$,

(1) nonnegativity: $d(x, y) \geq 0$;

(2) indistancy implies equality: $d(x, y) = 0 = d(y, x)$ implies $x = y$;

(3) the triangle inequality: $d(x, y) \leq d(x, z) + d(z, x)$,

so, here d is called a dislocated quasimetric or d_q-metric on X, and (X, d) is called a dislocated quasimetric space.

Definition 5 (see [6]). Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function called a distance function. If for $x, y, z \in X$,

(1) nonnegativity: $d(x, y) \geq 0$;

(2) indistancy implies equality: $d(x, y) = 0 = d(y, x)$ implies $x = y$;

(3) symmetry: $d(x, y) = d(y, x)$;

(4) the triangle inequality: $d(x, y) \leq d(x, z) + d(z, x)$,

so, here d is called a dislocated metric or d-metric on X and the pair (X, d) is called a dislocated metric space.

Definition 6 (see [6]). A sequence $\{x_n\}$ in d_q-metric space (dislocated quasimetric space) (X, d) is called a Cauchy sequence, if for a given $\epsilon > 0$, there exists $n_0 \in N$ such that $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$; that is, $\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$ for all $m, n \geq n_0$.

Definition 7 (see [6]). A sequence $\{x_n\}$ in d-metric spaces (X, d) is said to be d-converged to $x \in X$ provided that

$$\lim d(x_n, x) = \lim d(x, x_n) = 0. \tag{3}$$

In this case, x is called a d-limit of $\{x_n\}$ and we write $x_n \to x$.

Definition 8 (see [6]). A d-metric space (X, d) is called d-complete if every d-Cauchy sequence in X converges with respect to x in X.

Lemma 9. *Every converging sequence in a d-metric space is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a sequence which converges to some x, and let $\epsilon > 0$ be arbitrarily given. Then there exists $n_0 \in N$ with $d(x_n, x) < \epsilon/2$ for all $n \geq n_0$. For $m, n \geq n_0$, then we obtain that $d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\{x_n\}$ is a Cauchy sequence. $\square$

Lemma 10. *Limits in dislocated metric spaces are unique.*

Proof. Let x and y be limits of the sequence $\{x_n\}$. Then $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ as $n \to \infty$. By the triangle inequality of Definition 5, we conclude that $d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0$ as $n \to \infty$. Hence $d(x, y) = 0$ and using the properties (2) of Definition 5, we conclude that $x = y$. $\square$

Lemma 11. *Limits in dislocated quasimetric spaces are unique.*

Proof. Let x and y be limits of the sequence $\{x_n\}$. Then $x_n \to x$ and $x_n \to y$ as $n \to \infty$. By the triangle inequality, it follows that $d(x, y) \leq d(x, x_n) + d(x_n, y)$. As $n \to \infty$, we have $d(x, y) \leq d(x, x) + d(y, y)$. Similarly, $d(y, x) \leq d(x, x) + d(y, y)$. Hence $|d(x, y) - d(y, x)| \leq 0$; that is, $d(x, y) = d(y, x)$. Also $d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0$ as $n \to \infty$. That is, $d(x, y) = 0$, and $d(x, y) = d(y, x) = 0$. By the property (2) of Definition 4, we conclude that $x = y$. $\square$

Example 12. Let $X = R^+$. Define $d : X \times X \to R^+$ by $d(x, y) = |x - y|$. Then the pair (X, d) is a dislocated metric space. We define an arbitrary sequence $\{x_n\}$ in X; if $k > N$, there exists an positive integer N that satisfies $|x_k - a| < \epsilon/2$. Then, for any $m, n > N$, we have $d(x_n, x_m) = |x_n - x_m| \leq |x_n - a| + |x_m - a| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus, $\{x_n\}$ is a Cauchy sequence in X. Also as $n \to \infty$, then $\{x_n\} \to \infty \in X$. Hence, every Cauchy sequence in X is convergent with respect to d. Thus, (X, d) is a complete dislocated metric space.

3. Main Results

In this section, now we establish that common fixed points for mapping satisfying contractive condition are proved in the frame of dislocated metric spaces.

Definition 13 (see [9]). There exist $\phi(t)$ that satisfy the condition ϕ', if one lets $\phi : [0, +\infty] \to [0, \infty)$ be nondecreasing and non-negative, then $\lim \phi_n(t) = 0$, for a given $t > 0$.

Lemma 14 (see [9]). *If ϕ satisfy the condition ϕ', then $\phi(t) < t$, for a given $t > 0$.*

Lemma 15 (see [11]). *Let $F : R_+^3 \to R_+$, and satisfy the condition ϕ'; for all $u, v \geq 0$, if $u \leq F(v, v, u)$ or $u \leq F(v, u, v)$ or $u \leq F(u, v, v)$, then $u \leq \phi(v)$.*

Theorem 16. *Let (X, d) be a complete dislocated metric space, and let $f, g : X \to X$ be self-mapping, if*

(1) *either f or g is continuous;*

(2) *there exists F satisfying the condition ϕ', for all $x, y \in X$, such that*

$$d\left(f\left(x\right), g\left(y\right)\right) \leq F\left(d\left(x, y\right), d\left(x, f\left(x\right)\right), d\left(y, g\left(y\right)\right)\right), \quad (4)$$

then f, g have unique common fixed points.

Proof. Let g be continuous, x_0 arbitrary in X, x_n, y_n the sequence of X, and

$$x_n = \left(fg\right)^n\left(x_0\right) = fg\left(x_n - 1\right), \qquad y_n = g\left(fg\right)^{n-1}\left(x_0\right),$$
$$n = 1, 2, \ldots \quad (5)$$

Obviously,

$$y_n = g\left(x_n - 1\right), \qquad f\left(y_n\right) = x_n,$$
$$gf\left(y_n\right) = g\left(x_n\right) = y_{n+1}, \quad (6)$$
$$n = 1, 2, \ldots. $$

By the given condition, we have

$$d\left(x_{n+1}, y_{n+1}\right) = d\left(fg\left(x_n\right), g\left(x_n\right)\right)$$
$$\leq F\left(d\left(g\left(x_n\right), x_n\right), d\left(g\left(x_n\right), fg\left(x_n\right)\right), \quad (7)$$
$$d\left(x_n, g\left(x_n\right)\right)\right).$$

By Lemma 15, we have

$$d\left(x_{n+1}, y_{n+1}\right) \leq \phi\left(d\left(x_n, g\left(x_n\right)\right)\right). \quad (8)$$

Also

$$d\left(x_n, g\left(x_n\right)\right) = d\left(f\left(y_n\right), g\left(x_n\right)\right) = d\left(y_{n+1}, x_n\right)$$
$$\leq F\left(d\left(y_n, x_n\right), d\left(y_n, f\left(y_n\right)\right), d\left(x_n, g\left(x_n\right)\right)\right)$$
$$= F\left(d\left(y_n, f\left(y_n\right)\right), d\left(y_n, f\left(y_n\right)\right),$$
$$d\left(x_n, g\left(x_n\right)\right)\right). \quad (9)$$

Therefore

$$d\left(x_n, g\left(x_n\right)\right) = d\left(y_{n+1}, x_n\right) \leq \phi\left(d\left(y_n, f\left(y_n\right)\right)\right)$$
$$= \phi\left(d\left(y_n, x_n\right)\right). \quad (10)$$

By Lemma 14, we have

$$d\left(x_{n+1}, y_{n+1}\right) \leq \phi^2\left(d\left(x^n, y_n\right)\right). \quad (11)$$

Hence, by induction, for all $n \in N$, we obtain

$$d\left(x_{n+1}, y_{n+1}\right) \leq \phi^{2n}\left(d\left(x_1, y_1\right)\right) = \phi^{2n}\left(d\left(x_1, g\left(x_0\right)\right)\right). \quad (12)$$

Similarly

$$d\left(y_{n+1}, x_n\right) \leq \phi^{2n-1}\left(d\left(x_1, y_1\right)\right) = \phi^{2n-1}\left(d\left(x_1, g\left(x_0\right)\right)\right). \quad (13)$$

If $n \geq 2$, we have

$$d\left(x_{n+1}, x_n\right) \leq d\left(x_{n+1}, y_{n+1}\right) + d\left(y_{n+1}, x_n\right)$$
$$\leq \phi^{2n}\left(d\left(x_1, g\left(x_0\right)\right)\right) + \phi^{2n-1}\left(d\left(x_1, g\left(x_0\right)\right)\right)$$
$$\leq 2\phi^{2n-1}\left(d\left(x_1, g\left(x_0\right)\right)\right). \quad (14)$$

Note that by the condition ϕ' we know, for $n, m \in N$ such that $m > n$, we have

$$d\left(x_n, x_{n+m}\right)$$
$$\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + \cdots + d\left(x_{n+m-1}, x_{n+m}\right)$$
$$\leq \phi^{2n-1}\left(d\left(x_1, g\left(x_0\right)\right)\right) + 2\phi^{2(n+1)-1}\left(d\left(x_1, g\left(x_0\right)\right)\right)$$
$$+ \cdots + 2\phi^{2(n+m-1)-1}\left(d\left(x_1, g\left(x_0\right)\right)\right)$$
$$\leq \sum_{i=2n-1}^{2(n+m-1)-1} \phi^i\left(d\left(x_1, g\left(x_0\right)\right)\right)$$
$$\leq \sum_{i=2n-1}^{\infty} \phi^i\left(d\left(x_1, g\left(x_0\right)\right)\right) \longrightarrow 0. \quad (15)$$

Hence, $d(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$. This forces that $\{x_n\}$ is a Cauchy sequence in X. But X is a completely dislocated metric space; hence, $\{x_n\}$ is d-converges. Call the d-limit $x_* \in X$. Then, $x_n \rightarrow x_*$ as $n \rightarrow \infty$. By the continuity of g, as $n \rightarrow \infty$, $y_n = y_* = g(x_*)$. So as $n \rightarrow \infty$, $d(x_{n+1}, y_{n+1}) \leq d(x_*, y_*) \leq 0$, x_* is the fixed point of g.

By the given condition, we have

$$d\left(f\left(x_*\right), g\left(x_*\right)\right) = d\left(f\left(x_*\right), x_*\right)$$
$$\leq F\left(d\left(x_*, x_*\right), d\left(x_*, f\left(x_*\right)\right),$$
$$d\left(x_*, g\left(x_*\right)\right)\right) \quad (16)$$
$$= F\left(0, d\left(x_*, f\left(x_*\right)\right), 0\right).$$

Hence, $d(f(x_*), x_*) \leq \phi(0) = 0 \Rightarrow f(x_*) = x_*$, so x_* is a common fixed point of f, g.

Uniqueness. Let y_* be another common fixed point of f, g. Then by the given condition, we have

$$d\left(x_*, y_*\right) = d\left(f\left(x_*\right), g\left(y_*\right)\right)$$
$$\leq F\left(d\left(x_*, y_*\right), d\left(y_*, f\left(y_*\right)\right), d\left(x_*, g\left(x_*\right)\right)\right)$$
$$= F\left(d\left(x_*, y_*\right), 0, 0\right). \quad (17)$$

Since $d(x_*, y_*) \leq \phi(0) = 0 \Rightarrow x_* = y_*$, x_* is the unique fixed point of f; similarly, we prove that x_* is also the unique fixed point of g. Thus the fixed point of f, g is unique, and we prove the theorem. □

Theorem 17. *Let (X, d) be a complete dislocated metric space; and let $f, g : X \to X$ be continuous mapping, if*

(1) *there exists F satisfying the condition ϕ', for all $x, y \in X$, if $x \neq y$, such that*

$$d(f(x), g(y)) \leq F(d(x, y), d(x, f(x)), d(y, g(y))), \tag{18}$$

(2) *there exists $x_0 \in X$ such that $\{(fg)^n(x_0)\}$ have a condensation point, then f, g have a unique common fixed point.*

Proof. Let x_n, y_n be the sequence of X, and for all n, $x_n \neq y_n$,

$$x_n = (fg)^n(x_0) = fg(x_n - 1), \qquad y_n = g(fg)^{n-1}(x_0),$$
$$n = 1, 2, \ldots. \tag{19}$$

Obviously,

$$y_n = g(x_n - 1), \qquad f(y_n) = x_n,$$
$$gf(y_n) = g(x_n) = y_{n+1}, \tag{20}$$
$$n = 1, 2, \ldots.$$

Suppose that x_* is the condensation point of $\{x_n\}$; there exists the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} = x_*$. Since g is continuous, $\lim g(x_{n_i}) = g(x_*) = y_*$.

Consider

$$d(f(y_*), g(x_*))$$
$$= d(f(y_*), y_*)$$
$$\leq F(d(y_*, x_*), d(y_*, f(y_*)), d(x_*, g(x_*)))$$
$$= F(d(y_*, x_*), d(y_*, f(y_*)), d(x_*, y_*)), \tag{21}$$
$$d(f(y_*), y_*) \leq \phi(d(x_*, y_*))$$
$$\Longleftrightarrow d(f(y_*), y_*) < d(x_*, y_*).$$

Also consider

$$d(x_{n+1}, y_{n+1})$$
$$= d(fg(x_n), g(x_n))$$
$$\leq F(d(g(x_n), x_n), d(g(x_n), fg(x_n)), d(x_n, g(x_n)))$$
$$\leq \phi(d(x_n, g(x_n))),$$
$$d(x_n, g(x_n))$$
$$= d(f(y_n), g(x_n)) = d(y_{n+1}, x_n)$$
$$\leq F(d(y_n, x_n), d(y_n, f(y_n)), d(x_n, g(x_n)))$$
$$= F(d(y_n, f(y_n)), d(y_n, f(y_n)), d(x_n, g(x_n)))$$
$$\leq \phi(d(y_n, f(y_n)))$$
$$= \phi(d(y_n, x_n)). \tag{22}$$

Thus

$$d(x_{n+1}, y_{n+1}) \leq \phi^2(d(x_n, y_n)) < d(x_n, y_n). \tag{23}$$

Hence, we know that $\{d(x_{n+1}, y_{n+1})\}$ is decreasing. Let $\lim d(x_{n+1}, y_{n+1}) = \epsilon$ and $\lim d(y_{n_i+1}, x_{n_i}) = d(y_*, x_*) \leq \lim d(y_{n_i}, x_{n_i}) = \epsilon$. Since $\{x_{n_i}\}$ is the subsequence of $\{x_n\}$, we have

$$d(f(y_*), y_*) = \lim d(f(y_{n_i+1}), y_{n_i+1})$$
$$= \lim(d(x_{n_i}, y_{n_i})) = \epsilon. \tag{24}$$

Hence, we conclude that $d(x_*, y_*) \leq d(f(y_*), y_*)$, a contradiction. So $x_* = y_*$, y_* is the fixed point of g. Similarly, y_* is the fixed point of f.

Uniqueness. Let y' be another common fixed point of f, g. Then by the given condition, we have

$$d(y_*, y') = d(f(y_*), g(y'))$$
$$\leq F(d(y_*, y'), d(y', f(y')), d(y_*, g(y_*)))$$
$$= F(d(y_*, y'), 0, 0). \tag{25}$$

Since $d(y_*, y') \leq \phi(0) = 0 \Rightarrow y_* = y'$, y_* is the unique fixed point of f. Similarly, we prove that y_* is also the unique fixed point of g. Thus the fixed point of f, g is unique. $\qquad \square$
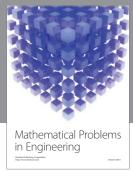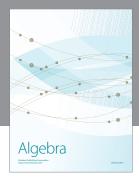
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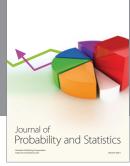
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[11] D. F. Xia, M. F. Fu, and B. Jiang, "Common fixed points for two self-mappings on symmetric sets and complete metric spaces," *Advances in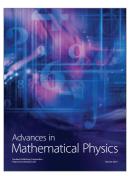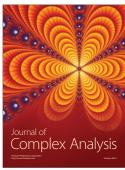 Mathematics*, vol. 36, no. 4, pp. 415–420, 2007.