Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 456530, 4 pages
http://dx.doi.org/10.1155/2013/456530

Research Article

Dynamics of a Family of Nonlinear Delay Difference Equations

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Received 16 March 2013; Accepted 18 April 2013

Academic Editor: Zhenkun Huang

We study the global asymptotic stability of the following difference equation: $x_{n+1} = f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_s}; x_{n-m_1}, x_{n-m_2}, \ldots, x_{n-m_t})$, $n = 0, 1, \ldots$, where $0 \le k_1 < k_2 < \cdots < k_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{k_1, k_2, \ldots, k_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, the initial values are positive, and $f \in C(E^{s+t}, (0, +\infty))$ with $E \in \{(0, +\infty), [0, +\infty)\}$. We give sufficient conditions under which the unique positive equilibrium $\overline{x}$ of that equation is globally asymptotically stable.

1. Introduction

In this note, we consider a nonlinear difference equation and deal with the question of whether the unique positive equilibrium $\overline{x}$ of that equation is globally asymptotically stable. Recently, there has been much interest in studying the global attractivity, the boundedness character, and the periodic nature of nonlinear difference equations; for example, see [1–22].

Amleh et al. [1] studied the characteristics of the difference equation:

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}. \tag{E1}$$

They confirmed a conjecture in [13] and showed that the unique positive equilibrium $\overline{x} = p + 1$ of (E1) is globally asymptotically stable provided $p > 1$.

Fan et al. [8] investigated the following difference equation:

$$x_{n+1} = f(x_n, x_{n-k}). \tag{E2}$$

They showed that the length of finite semicycle of (E2) is less than or equal to k and gave sufficient conditions under which every positive solution of (E2) converges to the unique positive equilibrium.

Kulenović et al. [11] investigated the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots, \tag{E3}$$

where the initial values $x_{-1}, x_0 \in R_+ \equiv (0, +\infty)$ and p_n is the period-two sequence

$$p_n = \begin{cases} \alpha, & \text{if } n \text{ is even}, \\ \beta, & \text{if } n \text{ is odd}, \end{cases} \quad \text{with } \alpha, \beta \in R_+. \tag{1}$$

Sun and Xi [20] studied the more general equation

$$x_{n+1} = f(x_{n-s}, x_{n-t}), \quad n = 0, 1, 2, \ldots, \tag{E4}$$

where $s, t \in \{0, 1, 2, \ldots\}$ with $s < t$, the initial values $x_{-t}, x_{-t+1}, \ldots, x_0 \in R_+$ and gave sufficient conditions under which every positive solution of (E4) converges to the unique positive equilibrium.

In this paper, we study the global asymptotic stability of the following difference equation:

$$x_{n+1} = f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_s}; x_{n-m_1}, x_{n-m_2}, \ldots, x_{n-m_t}), \quad n = 0, 1, \ldots, \tag{2}$$

where $0 \le k_1 < k_2 < \cdots < k_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{k_1, k_2, \ldots, k_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, the initial values are positive and $f \in C(E^{s+t}, (0, +\infty))$ with $E \in \{(0, +\infty), [0, +\infty)\}$ and $a = \inf_{(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t) \in E^{s+t}} \times f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) \in E$ satisfies the following conditions:

(H_1) $f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t)$ is decreasing in u_i for any $i \in \{1, 2, \ldots, s\}$ and increasing in v_j for any $j \in \{1, 2, \ldots, t\}$.

(H_2) Equation (2) has the unique positive equilibrium, denoted by $\overline{x}$.

(H_3) The function $f(a, a, \ldots, a; x, x, \ldots, x)$ has only fixed point in the interval $(a, +\infty)$, denoted by A.

(H_4) For any $y \in E$, $f(y, \ldots, y; x, \ldots, x)/x$ is nonincreasing in $x \in (0, +\infty)$.

(H_5) If $(x, y) \in E \times E$ is a solution of the system

$$\begin{aligned} y &= f(x, \ldots, x; y, \ldots, y), \\ x &= f(y, \ldots, y; x, \ldots, x), \end{aligned} \tag{3}$$

then $x = y$.

2. Main Result

Theorem 1. *Assume that (H_1)–(H_5) hold. Then the unique positive equilibrium $\overline{x}$ of (2) is globally asymptotically stable.*

Proof. Let $l = \max\{m_t, k_s\}$. Since

$$\begin{aligned} a &= \inf_{(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t) \in E^{s+t}} f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) \\ &\in E, \end{aligned} \tag{4}$$

we have

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}) > f(\overline{x} + 1, \overline{x}, \ldots, \overline{x}) \ge a. \tag{5}$$

Claim 1. $f(A, \ldots, A; a, \ldots, a) < \overline{x} < A$.

Proof of Claim 1. Assume on the contrary that $\overline{x} \ge A$. Then it follows from (H_1), (H_3), and (H_4) that

$$\begin{aligned} A &= f(a, \ldots, a; A, \ldots, A) > f(\overline{x}, \ldots, \overline{x}; A, \ldots, A) \\ &= \frac{f(\overline{x}, \ldots, \overline{x}; A, \ldots, A)}{A} A \ge \frac{f(\overline{x}, \ldots, \overline{x})}{\overline{x}} A \\ &= A. \end{aligned} \tag{6}$$

This is a contradiction. Therefore $\overline{x} < A$. Obviously

$$f(A, \ldots, A; a, \ldots, a) < f(\overline{x}, \ldots, \overline{x}; \overline{x}, \ldots, \overline{x}) = \overline{x}. \tag{7}$$

Claim 1 is proven.

Claim 2. For any $M \ge A$, $J = [a, M]$ is an invariable interval of (2).

Proof of Claim 2. For any $x_0, x_{-1}, \ldots, x_{-l} \in J$, we have from (H_4) that

$$\begin{aligned} a &\le x_1 \\ &= f(x_{-k_1}, x_{-k_2}, \ldots, x_{-k_s}; x_{-m_1}, x_{-m_2}, \ldots, x_{-m_t}) \\ &\le \frac{f(a, \ldots, a; M, \ldots, M)}{M} M \le \frac{f(a, \ldots, a; A, \ldots, A)}{A} M \\ &= M. \end{aligned} \tag{8}$$

By induction, we may show that $x_n \in J$ for any $n \ge 1$. Claim 2 is proven.

Let $m_0 = a$, $M_0 = M \ge A$ and for any $i \ge 0$,

$$\begin{aligned} m_{i+1} &= f(M_i, \ldots, M_i; m_i, \ldots, m_i), \\ M_{i+1} &= f(m_i, \ldots, m_i; M_i, \ldots, M_i). \end{aligned} \tag{9}$$

Claim 3. For any $n \ge 0$, we have

$$\begin{gathered} m_n \le m_{n+1} < \overline{x} < M_{n+1} \le M_n, \\ \lim_{n \to \infty} M_n = \lim_{n \to \infty} m_n = \overline{x}. \end{gathered} \tag{10}$$

Proof of Claim 3. From Claim 2, we obtain

$$\begin{aligned} m_0 \le m_1 &= f(M_0, \ldots, M_0; m_0, \ldots, m_0) \\ &< f(\overline{x}, \ldots, \overline{x}) = \overline{x} \\ &< f(m_0, \ldots, m_0; M_0, \ldots, M_0) \\ &= M_1 \le M_0, \\ m_1 &= f(M_0, \ldots, M_0; m_0, \ldots, m_0) \\ &\le f(M_1, \ldots, M_1; m_1, \ldots, m_1) = m_2 \\ &< f(\overline{x}, \ldots, \overline{x}) = \overline{x} \\ &< f(m_1, \ldots, m_1; M_1, \ldots, M_1) = M_2 \\ &\le f(m_0, \ldots, m_0; M_0, \ldots, M_0) \\ &= M_1. \end{aligned} \tag{11}$$

By induction, we have that for $n \ge 0$,

$$m_n \le m_{n+1} < \overline{x} < M_{n+1} \le M_n. \tag{12}$$

Set

$$\beta = \lim_{n \to \infty} m_n \quad \text{and} \quad \alpha = \lim_{n \to \infty} M_n. \tag{13}$$

Then

$$\begin{aligned}\beta &= f(\alpha,\ldots,\alpha;\beta,\ldots,\beta),\\ \alpha &= f(\beta,\ldots,\beta;\alpha,\ldots,\alpha).\end{aligned} \tag{14}$$

This with (H_2) and (H_5) implies $\alpha = \beta = \overline{x}$. Claim 3 is proven.

Claim 4. The equilibrium $\overline{x}$ of (2) is locally stable.

Proof of Claim 4. Let $M = A$ and m_n, M_n be the same as Claim 3. For any $\varepsilon > 0$ with $0 < \varepsilon < \min\{A - \overline{x}, \overline{x} - a\}$, there exists $n > 0$ such that

$$\overline{x} - \varepsilon < m_n < \overline{x} < M_n < \overline{x} + \varepsilon. \tag{15}$$

Set $0 < \delta = \min\{\overline{x} - m_n, M_n - \overline{x}\}$. Then for any $x_0, x_{-1}, \ldots, x_{-l} \in (\overline{x} - \delta, \overline{x} + \delta)$, we have

$$\begin{aligned}x_1 &= f\left(x_{-k_1},\ldots,x_{-k_s};x_{-m_1},\ldots,x_{-m_t}\right)\\ &\le f(m_n,\ldots,m_n;M_n,\ldots,M_n)\\ &= M_{n+1} \le M_n,\\ x_1 &= f\left(x_{-k_1},\ldots,x_{-k_s};x_{-m_1},\ldots,x_{-m_t}\right)\\ &\ge f(M_n,\ldots,M_n;m_n,\ldots,m_n)\\ &= m_{n+1} \ge m_n.\end{aligned} \tag{16}$$

In similar fashion,we can show that for any $k \ge 1$,

$$x_k \in [m_n, M_n] \subset (\overline{x} - \varepsilon, \overline{x} + \varepsilon). \tag{17}$$

Claim 4 is proven.

Claim 5. $\overline{x}$ is the global attractor of (2).

Proof of Claim 5. Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of (2), and let $M = \max\{x_1,\ldots,x_{l+1},A\}$ and m_n, M_n be the same as Claim 3. From Claim 2, we have $x_n \in [m_0, M_0] = [a, M]$ for any $n \ge 1$. Moreover, we have

$$\begin{aligned}&x_{l+2}\\ &\quad= f\left(x_{l+1-k_1},\ldots,x_{l+1-k_s};x_{l+1-m_1},\ldots,x_{l+1-m_t}\right)\\ &\quad\le f(m_0,\ldots,m_0;M_0,\ldots,M_0) = M_1,\\ &x_{l+2}\\ &\quad= f\left(x_{l+1-k_1},\ldots,x_{l+1-k_s};x_{l+1-m_1},\ldots,x_{l+1-m_t}\right)\\ &\quad\ge f(M_0,\ldots,M_0;m_0,\ldots,m_0) = m_1.\end{aligned} \tag{18}$$

In similar fashion, we may show $x_n \in [m_1, M_1]$ for any $n \ge l + 2$. By induction, we obtain

$$x_n \in [m_k, M_k] \quad \text{for } n \ge k(l+1)+1. \tag{19}$$

It follows from Claim 3 that $\lim_{n\to\infty} x_n = \overline{x}$. Claim 5 is proven.

From Claims 4 and 5, Theorem 1 follows. □

3. Applications

In this section, we will give two applications of Theorem 1.

Example 2. Consider equation

$$\begin{aligned}x_{n+1} = p &+ \frac{\sum_{i=1}^{t} a_i x_{n-m_i}}{\sum_{k=1}^{s} b_k x_{n-n_k}}\\ &+ \sqrt{\frac{\sum_{i=1}^{t} a_i x_{n-m_i}}{\sum_{k=1}^{s} b_k x_{n-n_k}}}, \quad n = 0,1,\ldots,\end{aligned} \tag{20}$$

where $0 \le n_1 < n_2 < \cdots < n_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{n_1,n_2,\ldots,n_s\} \bigcap \{m_1,m_2,\ldots,m_t\} = \emptyset$, $p > 0$, $a_i > 0$ for any $i \in \{1,2,\ldots,t\}$ and $b_k > 0$ for any $k \in \{1,2,\ldots,s\}$, and the initial conditions $x_{-l},\ldots,x_0 \in (0,\infty)$ with $l = \max\{m_t, n_s\}$. Write $A = \sum_{i=1}^{t} a_i$ and $B = \sum_{k=1}^{s} b_k$. If $pB > A$, then the unique positive equilibrium $\overline{x}$ of (20) is globally asymptotically stable.

Proof. Let $E = (0,+\infty)$. It is easy to verify that (H_1), (H_2), and (H_4) hold for (20). Note that $a = \inf_{(u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_t)\in E^{s+t}} f(u_1,u_2,\ldots,u_s;v_1,v_2,\ldots,v_t) = p$. Then

$$x = f(a,a,\ldots,a;x,x,\ldots,x) = p + \frac{Ax}{Bp} + \sqrt{\frac{Ax}{Bp}} \tag{21}$$

has only solution

$$x = \sqrt{\left[\sqrt{pAB} + \sqrt{pAB + 4p^2B(Bp-A)}\right]/2(Bp-a)} \tag{22}$$

in the interval $(p,+\infty)$, which implies that (H_3) holds for (20). In addition, let

$$\begin{aligned}x &= p + \frac{xA}{yB} + \sqrt{\frac{xA}{yB}},\\ y &= p + \frac{yA}{xB} + \sqrt{\frac{yA}{xB}},\end{aligned} \tag{23}$$

then

$$\frac{x}{y} = \frac{p + xA/yB + \sqrt{xA/yB}}{p + yA/xB + \sqrt{yA/xB}}. \tag{24}$$

Therefore $x/y = 1$, which implies that (23) has unique solution

$$x = y = \overline{x} = p + \frac{A}{B} + \sqrt{A/B}. \tag{25}$$

Thus (H_5) holds for (20). It follows from Theorem 1 that the equilibrium $\overline{x} = p + A/B + \sqrt{A/B}$ of (20) is globally asymptotically stable. □

Example 3. Consider equation

$$x_{n+1} = \frac{q + \sum_{i=1}^{t} a_i x_{n-m_i}}{p + \sum_{k=1}^{s} b_k x_{n-n_k}}, \quad n = 0, 1, \ldots, \tag{26}$$

where $0 \le n_1 < n_2 < \cdots < n_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{n_1, n_2, \ldots, n_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, $p > 0$, $q > 0$, $a_i > 0$ for any $1 \le i \le t$ and $b_j > 0$ for any $1 \le j \le s$, and the initial conditions $x_{-l}, \ldots, x_0 \in (0, \infty)$ with $l = \max\{m_t, n_s\}$. Write $A = \sum_{i=1}^{t} a_i$ and $B = \sum_{k=1}^{s} b_k$. If $p > A$, then the unique positive equilibrium $\overline{x}$ of (26) is globally asymptotically stable.

Proof. Let $E = [0, +\infty)$. It is easy to verify that (H_1)–(H_4) hold for (26). In addition, the following equation

$$x = \frac{q + xA}{p + yB}, \qquad y = \frac{q + yA}{p + xB} \tag{27}$$

has unique solution

$$x = y = \overline{x} = \frac{A - p + \sqrt{(p - A)^2 + 4Bq}}{2B}, \tag{28}$$

which implies that (H_5) holds for (26). It follows from Theorem 1 that the equilibrium $\overline{x} = (A - p + \sqrt{(p - A)^2 + 4Bq})/2B$ of (26) is globally asymptotically stable. □

Acknowledgments

This project is supported by NNSF of China (11261005, 51267001) and NSF of Guangxi (2011GXNSFA018135, 2012GXNSFDA276040).

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