Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 142759, 7 pages
http://dx.doi.org/10.1155/2013/142759

Research Article

Strong Convergence Theorems for a Common Fixed Point of a Family of Asymptotically k-Strict Pseudocontractive Mappings

H. Zegeye[1] and N. Shahzad[2]

[1] Department of Mathematics, University of Botswana, Private Bag 00704, Gaborone, Botswana
[2] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to N. Shahzad; nshahzad@kau.edu.sa

Received 21 September 2012; Accepted 11 December 2012

Academic Editor: Cristina Marcelli

We provide an iterative process which converges strongly to a common fixed point of finite family of asymptotically k-strict pseudocontractive mappings in Banach spaces. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let E be a real normed linear space with dual E^*. A *gauge function* $\varphi : [0,\infty] := R^+ \to R^+$ is a continuous and strictly increasing function satisfying $\varphi(0) = 0$ and $\varphi(t) \to \infty$, as $t \to \infty$. The generalized duality mapping from E to 2^{E^*} associated with the gauge function φ (see, e.g., [1]) is defined by

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|), \quad \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in E, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. In the case that $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the *normalized duality mapping*.

Following Browder [2], we say that a Banach space E has *a weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping J_φ is single valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weak* to $J_\varphi(x)$). It is known that l_p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-2}$, for all $1 < p < \infty$.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is called *asymptotically k-strict pseudocontractive*, with sequence $\{l_n\} \subseteq [1,\infty)$, $\lim_{n\to\infty} l_n = 1$ (see, e.g., [3–6]) if for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in [0,1)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le l_n\|x-y\|^2 - k\|(I-T^n)x - (I-T^n)y\|^2, \tag{2}$$

for all $n \ge 1$.

If I denotes the identity operator, then (2) can be equivalently written as

$$\langle (I-T^n)x - (I-T^n)y, j(x-y)\rangle \ge k\|(I-T^n)x - (I-T^n)y\|^2 - (l_n - 1)\|x-y\|^2, \tag{3}$$

for all $n \ge 1$.

If $E = H$, a real Hilbert space, it is shown by Osilike et al. [4] that (2) (and hence (3)) is equivalent to the inequality

$$\|T^n x - T^n y\|^2 \le (1 + 2(l_n - 1))\|x-y\|^2 + \lambda\|(I-T^n)x - (I-T^n)y\|^2, \tag{4}$$

where $\lambda = (1-2k)$. T is called *uniformly Lipschitz* if there exists $L \ge 0$ such that $\|T^n x - T^n y\| \le L\|x-y\|$ for all $x, y \in D(T)$. It is shown in [4] that an asymptotically k-strict pseudocontractive mapping is uniformly Lipschitz.

The class of asymptotically k-strict pseudocontractive mappings was first introduced in Hilbert spaces by Liu [5]. He proved the following theorem.

Theorem Q (see [5]). *Let K be a closed convex and bounded subset of a Hilbert space H. Let $T : K \to K$ be completely continuous asymptotically k-strict pseudocontractive mapping for some $0 \leq k < 1$ with sequence $\{l_n\} \subset [0, \infty)$ such that $\sum(l_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the modified Mann's iteration method:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \quad n \geq 1, \tag{5}$$

where $\{\alpha_n\}$ is a real sequence satisfying $\epsilon \leq \alpha_n \leq 1 - k - \epsilon$ for all $n \geq 1$ and some $\epsilon > 0$. Then, $\{x_n\}$ converges strongly to a fixed point of T.

The iteration scheme (5) is called *modified Mann's iterative processes* which was introduced by Schu [7, 8] and has been used by several authors (see, e.g., [3–5, 9–17]). We observe that Liu [5] proved *strong convergence* of scheme (5) to a fixed point of asymptotically k-strict pseudocontractive mapping T with additional assumption that T is *completely continuous*, where $T : C \to C$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence, say $\{x_{n_j}\}$ of $\{x_n\}$ such that the sequence $\{Tx_{n_j}\}$ converges strongly to some element of the range of T.

In [12], Kim and Xu studied weak convergence theorem for the class of asymptotically k-strict pseudocontractive mappings in the frame work of Hilbert spaces. In fact, they proved the following.

Theorem KX (see [12]). *Let K be a closed and convex subset of a Hilbert space H. Let $T : K \to K$ be an asymptotically k-strict pseudocontractive mapping for some $0 \leq k < 1$ with sequence $\{l_n\} \subset [0, \infty)$ such that $\sum(l_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the modified Mann's iteration method:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \quad n \geq 1, \tag{6}$$

where $\{\alpha_n\}$ is a real sequence satisfying $k + \lambda \leq \alpha_n \leq 1 - \lambda$, for all $n \geq 1$, and $\lambda \in (0, 1)$. Then, $\{x_n\}$ converges weakly to a fixed point of T.

In 2007, Osilike et al. [13] extended Theorem KX by proving *weak convergence* of scheme (6) to a fixed point of T in the frame work of q uniformly smooth Banach spaces which are also uniformly convex under suitable control conditions.

In 2011, Zhang and Xie [17] extended Theorem of Osilike et al. [13] to a more general real uniformly convex Banach space E with Fréchet differentiable norm. In addition, they proved *strong convergence* of scheme (5) to a fixed point of asymptotically k-strict pseudocontractive mapping provided that $\liminf_{n \to \infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{p \in F(T)} \|x_n - p\|$.

However, we observe that the convergence obtained above is either *weak* or requiring *additional assumption* like $\liminf_{n \to \infty} d(x_n, F(T)) = 0$ or T is completely continuous. But the requirement that $\liminf_{n \to \infty} d(x_n, F(T)) = 0$ is not easy to verify, as $F(T)$ is in general unknown, and there is also an example of asymptotically k-strict pseudocontractive mapping which is not completely continuous as shown below.

An example of asymptotically k-strict pseudocontractive mapping which is not completely continuous.

Example 1. Let $E = l_2 = \{\overline{x} = \{x_i\}_{i=1}^{\infty}, x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\overline{B} = \{\overline{x} \in l_2 : \|\overline{x}\| \leq 1\}$. Define $T : \overline{B} \to \overline{B}$ by $T\overline{x} = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots)$, where $\{a_k\}_{k=1}^{\infty}$ is a real sequence satisfying $0 < a_k < 1$, $k \geq 2$, and $\prod_{k=2}^{\infty} a_k = 1/2$. Then it is shown in [13] that T is asymptotically k-strict pseudocontractive mapping.

Now, we show that T is not completely continuous. Let $\{x_n\}$ be a sequence in $\overline{B}$ defined by $x_1 = (1, 0, 0, \ldots)$, $x_2 = (0, 1, 0, 0, \ldots)$, $x_3 = (0, 0, 1, 0, 0, \ldots), \ldots$. Then $\{x_n\} \subset \overline{B}$ and $\{Tx_n\} = \{y_n\}$ is given by $y_1 = (0, 1, 0, 0, \ldots)$, $y_2 = (0, 0, a_2, 0, 0, \ldots)$, $y_3 = (0, 0, 0, a_3, 0, 0, \ldots), \ldots$. Hence, since $a_k \to 1$, as $k \to \infty$, there is no subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{Tx_{n_i}\}$ converges strongly to a point in $\overline{B}$, as $\|Tx_{n_i} - Tx_{n_j}\| = \|y_{n_i} - y_{n_j}\| = \sqrt{|a_{n_i}|^2 + |a_{n_j}|^2} \nrightarrow 0$, as $i, j \to \infty$. Therefore, T is not completely continuous.

Thus, one question is raised naturally: can we obtain a scheme that converges strongly to a fixed point of asymptotically k-strict pseudocontractive mappings without those additional assumptions?

It is our purpose in this paper to provide an iterative scheme $\{x_n\}$ which converges strongly to a common fixed point of finite family of asymptotically k-strict pseudocontractive mappings in Banach spaces. The assumption that $\liminf_{n \to \infty} d(x_n, F(T)) = 0$ or T is completely continuous is not required.

2. Preliminaries

We need the following definitions from [18]. The Banach space E is said to be *uniformly convex* if, given $\varepsilon > 0$, there exists $\delta > 0$, such that, for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, $\|(1/2)(x + y)\| \leq 1 - \delta$. It is well known that L_p, ℓ_p, and Sobolev spaces W_m^p, $(1 < p < \infty)$, are uniformly convex.

A Banach space E is said to have a *Fréchet differentiable norm* if for all $x \in B = \{x \in E : \|x\| = 1\}$

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{7}$$

exists and is attained uniformly in $y \in B$. It is well known that uniformly smooth Banach spaces has a Fréchet differentiable norm.

In order to prove our results, we need the following lemmas.

Lemma 2 (see [19]). *Let C be a nonempty close convex subset of a real Banach space E which has the Fréchet differentiable norm. For $x \in E$, let ρ be defined for $0 < t < \infty$ by*

$$\rho(t) = \sup_{y \in B} \left| \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x) \rangle \right|. \tag{8}$$

Then, $\lim_{t \to 0} \rho(t) = 0$ *and*

$$\|x + h\|^2 \leq \|x\|^2 + 2\langle h, j(x) \rangle + \|h\| \rho(\|h\|), \quad \forall h \in E \setminus \{0\}. \tag{9}$$

It is shown in [19] that if $E = H$, a real Hilbert space, then $\rho(t) = t$, for $t > 0$. In our general setting, throughout this paper we assume that $\rho(t) \le 2t$.

Lemma 3. *Let E be a real Banach space. Then the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + \langle y, j(x + y)\rangle, \quad \forall x, y \in H,\ j(x + y) \in J(x + y). \tag{10}$$

Lemma 4 (see [20]). *Let E be a uniformly convex Banach space and $B_R(0)$ a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k\|^2 \le \sum_{i=0}^{k} \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g(\|x_i - x_j\|), \tag{11}$$

for each $\alpha_i \in (0, 1)$ and for $x_i \in B_R(0) := \{x \in E : \|x\| \le R\}$, $i = 0, 1, 2, \ldots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$.

Lemma 5 (see [21]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \ge n_0, \tag{12}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \le 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 6 (see [17]). *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E which has the Fréchet differentiable norm. Let $T : C \to C$ be an asymptotically k-strict pseudocontractive mapping with fixed point of T, $F(T) := \{x \in C : Tx = x\} \ne \emptyset$. Then $(I - T)$ is demiclosed at zero, that is, if $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, then $x = T(x)$.*

Lemma 7 (see [22]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in N$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{13}$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

3. Main Results

We now prove our main theorem.

Theorem 8. *Let C be a nonempty, closed, and convex subset of a real uniformly convex Banach space E which has Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping from E into E^*. Let $T_i : C \to C$ be asymptotically k_i-strict pseudocontractive mappings for $0 \le k_i < 1$ with sequences $\{l_{n,i}\} \subset [1, \infty)$, for $i = 1, 2, \ldots, N$. Assume that $F := \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)((1 - \beta_n) x_n + \beta_n S_n x_n), \quad n \ge 1, \tag{14}$$

where $S_n := \theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \cdots + \theta_{n,N} T_N^n$, such that $\theta_{n,1} + \theta_{n,2} + \cdots + \theta_{n,N} = 1$, for each $n \ge 1$, $\{\alpha_n\}, \{\theta_{n,i}\} \subset (0, c) \subset (0, 1)$, satisfying $\liminf_n \theta_{n,i} > 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n\to\infty}((l_{n,i} - 1)/\alpha_n) = 0$, for $i = 1, 2, \ldots, N$ and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (a, b, and c constants), for $k = \min_{1\le i\le N}\{k_i\}$, Then the sequence $\{x_n\}$ generated by (14) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$.

Proof. Fix $x^* \in F$. Let $y_n = (1 - \beta_n)x_n + \beta_n S_n x_n$ and $l_n := \max\{l_{n,i} : i = 1, 2, \ldots, N\}$. Then, using Lemma 2 and (3) we have that

$$\begin{aligned}
\|y_n - x^*\|^2
&= \|(x_n - x^*) - \beta_n(x_n - S_n x_n)\|^2 \\
&= \|(x_n - x^*) - \beta_n(x_n - (\theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \cdots + \theta_{n,N} T_N^n) x_n)\|^2 \\
&\le \|x_n - x^*\|^2 - 2\beta_n \langle \theta_{n,1}(x_n - T_1^n x_n) + \theta_{n,2}(x_n - T_2^n x_n) + \cdots + \theta_{n,N}(x_n - T_N^n x_n), j(x_n - x^*)\rangle \\
&\quad + \beta_n \|x_n - S_n x_n\| \rho(\beta_n \|x_n - S_n x_n\|) \\
&\le \|x_n - x^*\|^2 - 2\beta_n \theta_{n,1} \langle x_n - T_1^n x_n, j(x_n - x^*)\rangle - 2\beta_n \theta_{n,2} \langle x_n - T_2^n x_n, j(x_n - x^*)\rangle \\
&\quad - \cdots - 2\beta_n \theta_{n,N} \langle x_n - T_N^n x_n, j(x_n - x^*)\rangle + 2\beta_n^2 \|x_n - S_n x_n\|^2 \\
&\le \|x_n - x^*\|^2 - 2\beta_n \theta_{n,1} [k\|x_n - T_1^n x_n\|^2 - (l_n - 1)\|x_n - x^*\|^2] \\
&\quad - 2\beta_n \theta_{n,2} [k\|x_n - T_2^n x_n\|^2 - (l_n - 1)\|x_n - x^*\|^2] \\
&\quad - \cdots - 2\beta_n \theta_{n,N} [k\|x_n - T_N^n x_n\|^2 - (l_n - 1)\|x_n - x^*\|^2] + 2\beta_n^2 \|x_n - S_n x_n\|^2 \\
&\le [1 + 2\beta_n(l_n - 1)]\|x_n - x^*\|^2 - 2\beta_n \theta_{n,1} k\|x_n - T_1^n x_n\|^2 - 2\beta_n \theta_{n,2} k\|x_n - T_2^n x_n\|^2 \\
&\quad - \cdots - 2\beta_n \theta_{n,N} k\|x_n - T_N^n x_n\|^2 + 2\beta_n^2 \|x_n - S_n x_n\|^2.
\end{aligned} \tag{15}$$

On the other hand using Lemma 4 we get that

$$\begin{aligned}\|x_n - S_n x_n\|^2 &= \|x_n - (\theta_{n,1}T_1^n + \theta_{n,2}T_2^n + \cdots + \theta_{n,N}T_N^n)\, x_n\|^2 \\ &= \|\theta_{n,1}(x_n - T_1^n x_n) + \theta_{n,2}(x_n - T_2^n x_n) \\ &\quad + \cdots + \theta_{n,N}(x_n - T_N^n x_n)\|^2 \\ &\le \theta_{n,1}\|x_n - T_1^n x_n\|^2 + \theta_{n,2}\|x_n - T_2^n x_n\|^2 \\ &\quad + \cdots + \theta_{n,N}\|x_n - T_N^n x_n\|^2.\end{aligned} \tag{16}$$

Now substituting (16) into (15) we obtain that

$$\begin{aligned}\|y_n - x^*\|^2 \le{} & [1 + 2\beta_n(l_n - 1)]\,\|x_n - x^*\|^2 \\ & - 2\beta_n\theta_{n,1}(k - \beta_n)\,\|x_n - T_1^n x_n\|^2 \\ & - 2\beta_n\theta_{n,2}(k - \beta_n)\,\|x_n - T_2^n x_n\|^2 \\ & - \cdots - 2\beta_n\theta_{n,N}(k - \beta_n)\,\|x_n - T_N^n x_n\|^2\end{aligned} \tag{17}$$

$$\le [1 + 2\beta_n(l_n - 1)]\,\|x_n - x^*\|^2, \tag{18}$$

since $(k - \beta_n) \ge 0$ for each $n \ge 1$. Then now, from (14) and (18) we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + (1 - \alpha_n)(y_n - x^*)\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\,\|y_n - x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 \\ &\quad + (1 - \alpha_n)\left[1 + 2\beta_n(l_n - 1)\,\|x_n - x^*\|^2\right] \\ &\le \alpha_n\|u - x^*\|^2 \\ &\quad + (1 - \alpha_n + \epsilon\alpha_n)\,\|x_n - x^*\|^2, \quad \forall n \ge N_0, \\ &\le \alpha_n\|u - x^*\|^2 \\ &\quad + (1 - \alpha_n(1 - \epsilon))\,\|x_n - x^*\|^2, \quad \forall n \ge N_0,\end{aligned} \tag{19}$$

where N_0 is a positive integer such that $2(1 - \alpha_n)\beta_n(l_n - 1)/\alpha_n < \epsilon$, for all $n \ge N_0$, for some $\epsilon > 0$. Therefore, by induction,

$$\|x_{n+1} - x^*\|^2 \le \max\left\{\|x_{N_0} - x^*\|^2, (1 - \epsilon)^{-1}\|u - x^*\|^2\right\}, \quad \forall n \ge N_0, \tag{20}$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ are bounded.

Furthermore, from (14), Lemma 3, and (17) we get that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad = \|\alpha_n(u - x^*) + (1 - \alpha_n)(y_n - x^*)\|^2 \\ &\quad \le (1 - \alpha_n)\,\|y_n - x^*\|^2 \\ &\qquad + 2\alpha_n\langle u - x^*, J(x_{n+1} - x^*)\rangle \\ &\quad \le (1 - \alpha_n)[1 + 2\beta_n(l_n - 1)]\,\|x_n - x^*\|^2 \\ &\qquad + 2\alpha_n\langle u - x^*, J(x_{n+1} - x^*)\rangle \\ &\qquad - 2\beta_n\theta_{n,1}(k - \beta_n)(1 - \alpha_n)\,\|x_n - T_1^n x_n\|^2 \\ &\qquad - 2\beta_n\theta_{n,2}(k - \beta_n)(1 - \alpha_n)\,\|x_n - T_2^n x_n\|^2 \\ &\qquad - \cdots - 2\beta_n\theta_{n,N}(k - \beta_n)(1 - \alpha_n)\,\|x_n - T_N^n x_n\|^2 \\ &\quad \le (1 - \alpha_n)\,\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, J(x_{n+1} - x^*)\rangle \\ &\qquad + 2\beta_n M(l_n - 1) \\ &\qquad - 2\beta_n\theta_{n,1}(1 - \alpha_n)(k - \beta_n)\,\|x_n - T_1^n x_n\|^2 \\ &\qquad - 2\beta_n\theta_{n,2}(k - \beta_n)(1 - \alpha_n)\,\|x_n - T_2^n x_n\|^2 \\ &\qquad - \cdots - 2\beta_n\theta_{n,N}(k - \beta_n)(1 - \alpha_n)\,\|x_n - T_N^n x_n\|^2\end{aligned} \tag{21}$$

$$\begin{aligned}&\le (1 - \alpha_n)\,\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, J(x_{n+1} - x^*)\rangle \\ &\quad + 2\beta_n M(l_n - 1),\end{aligned} \tag{22}$$

for some $M > 0$.

Now, the rest of the proof is divided into two parts.

Case 1. Suppose that there exists $N_1 \ge 0$ such that $\{||x_n - x^*||\}$ is decreasing for all $n \ge N_1$. Then we have that $\{||x_n - x^*||)\}$ is convergent. Then from (21) and the assumptions on $\{\beta_n\}$, $\{\alpha_n\}$, and $\{l_n\}$ we have that $\beta_n\theta_{n,i}(1 - c)(k - \beta_n)||x_n - T_i^n x_n||^2 \to 0$, as $n \to \infty$, which implies that

$$x_n - T_i^n x_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{23}$$

for $i = 1, 2, \ldots, N$. Then from (14) we obtain that

$$x_{n+1} - y_n = \alpha_n(u - y_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{24}$$

Again, from (23) we get that

$$\begin{aligned}\|y_n - x_n\| &= \|\beta_n(S_n x_n - x_n)\| \le \|S_n x_n - x_n\| \\ &\le \theta_{n,1}\,\|T_1^n x_n - x_n\| + \theta_{n,2}\,\|T_2^n x_n - x_n\| \\ &\quad + \cdots + \theta_{n,N}\,\|T_N^n x_n - x_n\| \longrightarrow 0,\end{aligned} \tag{25}$$

as $n \to \infty$. Thus, (24) and (25) imply that

$$x_{n+1} - x_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{26}$$

Therefore, since each T_i, for $i = 1, 2, \ldots, N$, is uniformly L-Lipschitzian and

$$\begin{aligned}\|x_n - T_i x_n\| &\le \|x_n - x_{n+1}\| \\ &\quad + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + \left\|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\right\| \\ &\quad + \left\|T_i^{n+1} x_n - T_i x_n\right\| \\ &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + L\|x_{n+1} - x_n\| \\ &\quad + \|T_i(T_i^n x_n) - T_i x_n\| \\ &= (1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\ &\quad + \|T_i(T_i^n x_n) - T_i x_n\|,\end{aligned} \tag{27}$$

we have from (23), (26), and uniform continuity of T_i that

$$\|x_n - T_i x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{28}$$

for each $i = 1, 2, \ldots, N$. Furthermore, the fact that $\{x_n\}$ is bounded and E is reflexive implies that we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightharpoonup z$ and

$$\begin{aligned}&\limsup_{n\to\infty} \langle u - x^*, J(x_{n+1} - x^*)\rangle \\ &\qquad = \lim_{i\to\infty} \left\langle u - x^*, J\left(x_{n_i+1} - x^*\right)\right\rangle.\end{aligned} \tag{29}$$

Now, from (26) we get that $x_{n_i} \rightharpoonup z$ and from Lemma 6 we have that $z \in F(T_i)$, for each $i = 1, 2, \ldots, N$. Hence, $z \in \cap_{i=1}^N F(T_i)$. Therefore, putting $x^* = z$ in (29) and using the fact that J is weakly sequentially continuous we immediately obtain that $\limsup_{n\to\infty}\langle u - z, J(x_{n+1} - z)\rangle = \lim_{i\to\infty}\langle u - z, J(x_{n_i+1} - z)\rangle = \langle u - z, J(z - z)\rangle = 0$. Again, putting $x^* = z$ in inequality (22), we get that

$$\begin{aligned}\|x_{n+1} - z\|^2 \le\ &(1-\alpha_n)\|x_n - z\|^2 \\ &+ 2\alpha_n \langle u - z, J(x_{n+1} - z)\rangle \\ &+ 2\beta_n M(l_n - 1),\end{aligned} \tag{30}$$

and, hence, it follows from (30) and Lemma 5 that $\|x_n - z\| \to 0$, as $n \to \infty$. Consequently, $x_n \to z$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\left\|x_{n_i} - x^*\right\| < \left\|x_{n_i+1} - x^*\right\|, \tag{31}$$

for all $i \in N$. Then, by Lemma 7, there exists a nondecreasing sequence $\{m_j\} \subset N$ such that $m_j \to \infty$, $\|x_{m_j} - x^*\| \le \|x_{m_j+1} - x^*\|$ and $\|x_j - x^*\| \le \|x_{m_j+1} - x^*\|$ for all $j \in N$. Then from (21) and following the method of Case 1, we get that

$$\left\|x_{m_j} - T_i^{m_j} x_{m_j}\right\| \longrightarrow 0, \quad \text{as } j \longrightarrow \infty, \tag{32}$$

for each $i = 1, 2, \ldots, N$. Thus, again following the method of Case 1, we obtain that $x_{m_j+1} - x_{m_j} \to 0$ and $x_{m_j} - T_i x_{m_j} \to 0$, as $j \to \infty$, for each $i = 1, 2, \ldots, N$ and there exists $z^* \in \cap_{i=1}^N F(T_i)$ such that

$$\limsup_{j\to\infty} \left\langle u - z^*, J\left(x_{m_j+1} - z^*\right)\right\rangle = 0. \tag{33}$$

Then now, putting $x^* = z^*$ in (22) we have that

$$\begin{aligned}\left\|x_{m_j+1} - z^*\right\|^2 \le\ &\left(1 - \alpha_{m_j}\right)\left\|x_{m_j} - z^*\right\|^2 \\ &+ 2\alpha_{m_j}\left\langle u - z^*, J\left(x_{m_j+1} - z^*\right)\right\rangle \\ &+ 2\beta_{m_j} M\left(l_{m_j} - 1\right).\end{aligned} \tag{34}$$

Since $\|x_{m_j} - z^*\|^2 \le \|x_{m_j+1} - z^*\|^2$, (34) implies that

$$\begin{aligned}\alpha_{m_j}\left\|x_{m_j} - z^*\right\|^2 \le\ &\left\|x_{m_j} - z^*\right\|^2 - \left\|x_{m_j+1} - z^*\right\|^2 \\ &+ 2\alpha_{m_j}\left\langle u - z^*, J\left(x_{m_j+1} - z^*\right)\right\rangle \\ &+ 2\beta_{m_j} M\left(l_{m_j} - 1\right).\end{aligned} \tag{35}$$

Moreover, since $\alpha_{m_j} > 0$, inequality (35) gives that

$$\begin{aligned}\left\|x_{m_j} - z^*\right\|^2 \le\ &2\left\langle u - z^*, J\left(x_{m_j+1} - z^*\right)\right\rangle \\ &+ \frac{2\beta_{m_j} M\left(l_{m_j} - 1\right)}{\alpha_{m_j}}.\end{aligned} \tag{36}$$

Then, from (33) and the fact that $2\beta_{m_j} M(l_{m_j} - 1)/\alpha_{m_j} \to 0$, we obtain that $\|x_{m_j} - z^*\| \to 0$, as $j \to \infty$. This together with (34) gives that $\|x_{m_j+1} - z^*\| \to 0$, as $j \to \infty$. But $\|x_j - z^*\| \le \|x_{m_j+1} - z^*\|$, for all $j \in N$; thus we obtain that $x_j \to z^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to an element of F and the proof is complete. □

If, in Theorem 8, we assume a single asymptotically k-strict pseudocontractive mapping we get the following corollary.

Corollary 9. *Let C be a nonempty, closed, and convex subset of a real uniformly convex Banach space E which has Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping from E into E^*. Let $T : C \to C$ be an asymptotically k-strict pseudocontractive mapping for $0 \le k < 1$ with sequences $\{l_n\} \subset [1, \infty)$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\left((1 - \beta_n) x_n + \beta_n T^n x_n\right), \quad n \ge 1, \tag{37}$$

where $\{\alpha_n\} \subset (0, c) \subset (0, 1)$, satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n\to\infty}((l_n - 1)/\alpha_n) = 0$, and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (a, b, and c constants). Then the sequence $\{x_n\}$ generated by (37) *converges strongly to a fixed point of T.*

Proof. Putting $T = T_1 = T_2 = \cdots = T_N$ in (14), we get that $S_n = T^n$ and the scheme reduces to scheme (37) and following the method of proof of Theorem 8 we get that (see (21) and (22))

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq & (1-\alpha_n)\|x_n - x^*\|^2 \\ & + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*)\rangle \\ & - 2\beta_n(k-\beta_n)(1-\alpha_n)\|x_n - T^n x_n\|^2 \\ & + 2\beta_n M'(l_n - 1) \\ \leq & (1-\alpha_n)\|x_n - x^*\|^2 \\ & + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*)\rangle \\ & - 2\beta_n(k-\beta_n)(1-c)\|x_n - T^n x_n\|^2 \\ & + 2\beta_n M'(l_n - 1) \\ \leq & (1-\alpha_n)\|x_n - x^*\|^2 \\ & + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*)\rangle \\ & + 2\beta_n M'(l_n - 1),\end{aligned} \tag{38}$$

for some $M' > 0$. Now, considering cases, as in the proof of Theorem 8, we obtain the required result. □

Corollary 10. *Let K be a nonempty, closed, and convex subset of l_p, $1 < p < \infty$. Let $T_i : C \to C$ be asymptotically k_i-strict pseudocontractive mappings for $0 \leq k_i < 1$ with sequences $\{l_{n,i}\} \subset [1,\infty)$, for $i = 1, 2, \ldots, N$. Assume that $F := \cap_{i=1}^{n} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)((1-\beta_n)x_n + \beta_n S_n x_n), \quad n \geq 1, \tag{39}$$

where $S_n := \theta_{n,1}T_1^n + \theta_{n,2}T_2^n + \cdots + \theta_{n,N}T_N^n$, such that $\theta_{n,1} + \theta_{n,2} + \cdots + \theta_{n,N} = 1$, for each $n \geq 1$, $\{\alpha_n\}, \{\theta_{n,i}\} \subset (0,c) \subset (0,1)$, satisfying $\liminf_{n\to\infty}\theta_{n,i} > 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$, $\lim_{n\to\infty}((l_{n,i}-1)/\alpha_n) = 0$ and $\{\beta_n\} \subset [a,b] \subset (0,k)$ (for a, b, and c constants), for $k = \min_{1\leq i\leq N}\{k_i\}$. Then the sequence $\{x_n\}$ generated by (39) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$.

Proof. We note that l_p, $1 < p < \infty$, spaces are uniformly convex which have Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping from E into E^* (see, e.g., [18]). Thus, the result follows from Theorem 8. □

Corollary 11. *Let K be a nonempty, closed, and convex subset of l_p, $1 < p < \infty$. Let $T : C \to C$ be an asymptotically k-strict pseudocontractive mapping for some $0 \leq k < 1$ with sequences $\{l_n\} \subset [1,\infty)$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)((1-\beta_n)x_n + \beta_n T^n x_n), \quad n \geq 1, \tag{40}$$

where $\{\alpha_n\} \subset (0,c) \subset (0,1)$, and $\{\beta_n\} \subset [a,b] \subset (0,k)$ (for a, b, and c constants) satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$ and $\lim_{n\to\infty}((l_n - 1)/\alpha_n) = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

If in Theorem 8 we have that $E = H$, a real Hilbert space, then E is uniformly convex with Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping. Thus, we have the following corollary.

Corollary 12. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_i : C \to C$ be asymptotically k_i-strict pseudocontractive mappings for $0 \leq k_i < 1$ with sequences $\{l_{n,i}\} \subset [1,\infty)$, for $i = 1, 2, \ldots, N$. Assume that $F := \cap_{i=1}^{n} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)((1-\beta_n)x_n + \beta_n S_n x_n), \quad n \geq 1, \tag{41}$$

where $S_n := \theta_{n,1}T_1^n + \theta_{n,2}T_2^n + \cdots + \theta_{n,N}T_N^n$, such that $\theta_{n,1} + \theta_{n,2} + \cdots + \theta_{n,N} = 1$, for each $n \geq 1$, $\{\alpha_n\}, \{\theta_{n,i}\} \subset (0,c) \subset (0,1)$, satisfying $\liminf_n\theta_{n,i} > 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$, $\lim_{n\to\infty}((l_{n,i}-1)/\alpha_n) = 0$ and $\{\beta_n\} \subset [a,b] \subset (0,k)$ (for a, b, and c constants), for $k = \min_{1\leq i\leq N}\{k_i\}$. Then the sequence $\{x_n\}$ generated by (41) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$.

Corollary 13. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be an asymptotically k-strict pseudocontractive mapping for some $0 \leq k < 1$ with sequences $\{l_n\} \subset [1,\infty)$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)((1-\beta_n)x_n + \beta_n T^n x_n), \quad n \geq 1, \tag{42}$$

where $\{\alpha_n\} \subset (0,c) \subset (0,1)$, and $\{\beta_n\} \subset [a,b] \subset (0,k)$ (for a, b, and c constants) satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$ and $\lim_{n\to\infty}((l_n - 1)/\alpha_n) = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 14. We note that Corollary 9 generalizes several recent results of this nature. Particularly, it extends Theorem KX of [12], Theorem 2 of Liu [5], and corresponding theorem of Schu [7] in the sense that our convergence is *strong* in more general Banach spaces possessing weakly sequentially continuous duality mappings without the requirement that T be completely continuous.

Remark 15. Corollary 9 is an improvement of Theorem 3.2 of Osilike et al. [13] and Theorems 3.1 and 3.2 of Zhang and Xie [17] in the sense that our convergence is *strong* without the requirement that $\liminf_{n\to\infty}d(x_n, F(T)) = 0$, provided that E possesses weakly sequentially continuous duality mappings.

Acknowledgments

N. Shahzad gratefully acknowledges research support from the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

References

[1] J. M. A. Toledano, T. D. Benavides, and G. L. Acedo, *Measure of Noncoin Metric Fixed Point Theory*, Birkhäauser, Boston, Mass, USA, 1997.

[2] F. E. Browder, "Convergence theorems for sequences of nonlinear operators in Banach spaces," *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.

[3] M. O. Osilike, "Iterative approximation of fixed points of asymptotically demicontractive mappings," *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 12, pp. 1291–1300, 1998.

[4] M. O. Osilike, S. C. Aniagbosor, and B. G. Akuchu, "Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces," *Panamerican Mathematical Journal*, vol. 12, no. 2, pp. 77–88, 2002.

[5] Q. Liu, "Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 11, pp. 1835–1842, 1996.

[6] X. Qin, S. Y. Cho, and J. K. Kim, "Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense," *Fixed Point Theory and Applications*, vol. 2010, Article ID 186874, 14 pages, 2010.

[7] J. Schu, "Iterative construction of fixed points of asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 2, pp. 407–413, 1991.

[8] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[9] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[10] T. L. Hicks and J. D. Kubicek, "On the Mann iteration process in a Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 59, no. 3, pp. 498–504, 1977.

[11] Z. Huang, "Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 37, no. 3, pp. 1–7, 1999.

[12] T.-H. Kim and H.-K. Xu, "Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 9, pp. 2828–2836, 2008.

[13] M. O. Osilike, A. Udomene, D. I. Igbokwe, and B. G. Akuchu, "Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1334–1345, 2007.

[14] B. E. Rhoades, "Fixed point iterations using infinite matrices," *Transactions of the American Mathematical Society*, vol. 196, pp. 161–176, 1974.

[15] K.-K. Tan and H. K. Xu, "Fixed point iteration processes for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 122, no. 3, pp. 733–739, 1994.

[16] H. Zegeye, M. Robdera, and B. Choudhary, "Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 326–332, 2011.

[17] Y. Zhang and Z. Xie, "Convergence of modified Mann's iterative method for asymptotically k-strictly pseudocontractive mappings," *Fixed Point Theory and Applications*, vol. 2011, p. 100, 2011.

[18] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.

[19] P. Cholamjiak and S. Suantai, "Weak convergence theorems for a countable family of strict pseudocontractions in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 632137, 2010.

[20] H. Zegeye and N. Shahzad, "Convergence of Mann's type iteration method for Generalized asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 62, no. 11, pp. 4007–4014, 2011.

[21] H.-K. Xu, "Another control condition in an iterative method for nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 65, no. 1, pp. 109–113, 2002.

[22] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.