Research Article

Exact Travelling Wave Solutions for Isothermal Magnetostatic Atmospheres by Fan Subequation Method

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The equations of magnetohydrostatic equilibria for a plasma in a gravitational field are investigated analytically. An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. These equations transform to a single nonlinear elliptic equation for the magnetic vector potential u. This equation depends on an arbitrary function of u that must be specified. With choices of the different arbitrary functions, we obtain analytical solutions of elliptic equation using the Fan subequation method.

1. Introduction

The equations of magnetostatic equilibria have been used extensively to model the solar magnetic structure [1–4]. An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. The force balance consists of the $J \wedge B$ force (B is the magnetic field induction and J is the electric current density), the gravitational force, and gas pressure gradient force. However, in many models, the temperature distribution is specified a priori and direct reference to the energy equations is eliminated. In solar physics, the equations of magnetostatic have been used to model diverse phenomena, such as the slow evolution stage

of solar flares, or the magnetostatic support of prominences [5, 6]. The nonlinear equilibrium problem has been solved in several cases [7–9].

Recently, Fan and Hon [10] developed an algebraic method, belonging to the sub-equation method to seek more new solutions of nonlinear partial differential equations (NLPDEs) that can be expressed as polynomial in an elementary function which satisfies a more general sub-equation, called Fan sub-equation, than other sub-equations like Riccati equation, auxiliary ordinary equation, elliptic equation, and generalized Riccati equation. As we know, the more general analytical exact solutions of the sub-equation are proposed, the more general corresponding exact solutions of NLPDEs will be obtained. Thus, it is very important how to obtain more new solutions to the sub-equation. Fortunately, the Fan sub-equation method can construct more general exact solutions to the sub-equation that can capture all the solutions of the Riccati equation, auxiliary ordinary equation, elliptic equation, and generalized Riccati equation. Some works using the Fan's technique are presented in [1, 11–16].

In this paper, we obtain the exact travelling wave solutions for the Liouville and sinh-Poisson equations using the Fan sub-equation method. These two models are special cases of magnetostatic atmospheres model. Also in these cases there is force balance between differents forces.

2. The Basic Idea of Fan Subequation Method

In this section, we outline the main steps of Fan sub-equation method [11].

Step 1. For a given nonlinear partial differential equation

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \tag{2.1}$$

we consider its travelling wave solutions $u(x, t) = u(\xi)$, $\xi = x - ct$, then (2.1) is reduced to a nonlinear ordinary differential equation

$$N\left(u(\xi), -cu'(\xi), u'(\xi), c^2 u''(\xi), u''(\xi), \ldots\right) = 0, \tag{2.2}$$

where a prime denotes the derivative with respect to the variable ξ.

Step 2. Expand the solution of (2.2) in the form

$$u(\xi) = \sum_{i=0}^{n} A_i \phi^i, \quad A_n \neq 0, \tag{2.3}$$

where A_i $(i = 0, 1, \ldots, n)$ are constants to be determined later and the new variable ϕ satisfies the Fan sub-equation

$$\phi'(\xi) = \epsilon \sqrt{\sum_{j=0}^{4} w_j \phi^j}, \tag{2.4}$$

where $\epsilon = \pm 1$ and w_j $(j = 0, \ldots, 4)$ are constants.

Thus, the derivatives with respect to the variable ξ become the derivatives with respect to the variable ϕ as follows:

$$\frac{du}{d\xi} = \epsilon \sqrt{\sum_{j=0}^{4} w_j \phi^j} \frac{du}{d\phi}, \qquad \frac{d^2u}{d\xi^2} = \frac{1}{2} \sqrt{\sum_{j=1}^{4} j w_j \phi^{j-1} \frac{du}{d\phi} + \sum_{j=0}^{4} w_j \phi^j \frac{d^2u}{d\phi^2}}. \tag{2.5}$$

Step 3. Determine n by substituting (2.3) with (2.4) into (2.2) and balancing the linear term of the highest order with the nonlinear term in (2.2).

Step 4. Substituting (2.3) and (2.4) into (2.2) again and collecting all coefficients of ϕ^i $(i = 0, 1, 2, \ldots, n)$, then setting these coefficients to zero will give a set of algebraic equations with respect to A_i $(i = 0, 1, \ldots, n)$.

Step 5. Solve these algebraic equations to obtain A_i $(i = 0, 1, 2, \ldots, n)$. Substituting these results into (2.3) yields the general form of travelling wave solutions.

Step 6. For each solution to (2.4) which depends on the special conditions chosen for the w_0, w_1, w_2, w_3, and w_4, it follows from (2.3) obtained from the above steps that the corresponding exact solution of (2.2) can be constructed.

3. Basic Equations

The relevant magnetohydrostatic equations consist of the equilibrium equation

$$J \wedge B - \rho \nabla \Phi - \nabla P = 0, \tag{3.1}$$

which is coupled with Maxwells equations

$$J = \frac{\nabla \wedge B}{\mu}, \qquad \nabla \cdot B = 0, \tag{3.2}$$

where P, ρ, μ, and Φ are the gas pressure, the mass density, the magnetic permeability, and the gravitational potential, respectively. It is assumed that the temperature is uniform in space

and that the plasma is an ideal gas with equation of state $p = \rho R_0 T_0$, where R_0 is the gas constant and T_0 is the temperature. Then the magnetic field B can be written as

$$B = \nabla u \wedge e_x + B_x e_x = \left(B_x, \frac{\partial u}{\partial z}, \frac{-\partial u}{\partial y} \right). \tag{3.3}$$

The form of (3.3) for B ensures that $\nabla \cdot B = 0$ and there is no mono pole or defect structure.

Equation (3.1) requires the pressure and density to be of the form [4]

$$P(y, z) = P(u)e^{-z/h}, \qquad \rho(y, z) = \frac{1}{(gh)} P(u)e^{-z/h}, \tag{3.4}$$

where $h = R_0 T_0 / g$ is the scale height. Substituting (3.2)–(3.4) into (3.1), we obtain

$$\nabla^2 u + f(u)e^{-z/h} = 0, \tag{3.5}$$

where

$$f(u) = \mu \frac{dP}{du}. \tag{3.6}$$

Equation (3.6) gives

$$P(u) = P_0 + \frac{1}{\mu} \int f(u) du, \tag{3.7}$$

where P_0 is constant. Substituting (3.7) into (3.4), we obtain

$$P(y, z) = \left(P_0 + \frac{1}{\mu} \int f(u) du \right) e^{-z/h},$$

$$\rho(y, z) = \frac{1}{gh} \left(P_0 + \frac{1}{\mu} \int f(u) du \right) e^{-z/h}. \tag{3.8}$$

Using transformation $x_1 + ix_2 = e^{-z/l} e^{iy/l}$, (3.5) reduces to

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + l^2 f(u) e^{(2/l - 1/h)z} = 0. \tag{3.9}$$

These equations have been given in [2].

4. Applications of the Fan Subequation Method

In this section, we will employ the Fan sub-equation method for solving (3.9) for specific forms of the function $f(u)$.

4.1. Liouville Equation

We first consider Liouville equation, which is a special case of (3.9), namely,

$$u_{xx} + u_{tt} - \alpha^2 l^2 e^{-2u} = 0. \tag{4.1}$$

In order to apply the Fan sub-equation method, we use the wave transformation $u(x,t) = u(\xi)$, $\xi = x - ct$ and transform (4.1) into the form

$$\left(1 + c^2\right)u'' = \alpha^2 l^2 e^{-2u}. \tag{4.2}$$

We next use the transformation $v = e^{-2u}$ and obtain the nonlinear ordinary differential equation

$$\left(1 + c^2\right)vv'' - \left(1 + c^2\right)v'^2 + 2\alpha^2 l^2 v^3 = 0. \tag{4.3}$$

Using Step 3 given above, we get $n = 2$, therefore the solution of (4.3) can be expressed as

$$v(\xi) = A_0 + A_1\phi + A_2\phi^2. \tag{4.4}$$

Following Step 4, we obtain a system of nonlinear algebraic equations for A_0, A_1, and A_2:

$$2\alpha^2 l^2 A_0{}^3 - \epsilon^2 A_1{}^2 w_0 - c^2 \epsilon^2 A_1{}^2 w_0 + 2\epsilon^2 A_0 w_0 + 2c^2 \epsilon^2 A_0 A_2 w_0$$

$$+ \frac{1}{2}\epsilon^2 A_0 A_1 w_1 + \frac{1}{2}c^2 \epsilon^2 A_0 A_1 w_1 = 0,$$

$$6\alpha l^2 A_0{}^2 A_1 - 2\epsilon^2 A_1 A_2 w_0 - 2c^2 \epsilon^2 A_1 A_2 w_0 - \frac{1}{2}\epsilon^2 A_1{}^2 w_1 + 3\epsilon^2 A_0 A_2 w_1$$

$$+ 3c^2 \epsilon^2 A_0 A_1{}^2 + \epsilon^2 A_0 A_1 w_2 + c^2 \epsilon^2 A_0 A_1 w_2 = 0,$$

$$6\alpha^2 l^2 A_0 A_1{}^2 + 6\alpha^2 l^2 A_0{}^2 A_2 - 2\epsilon^2 A_2{}^2 w_0 - \frac{1}{2}\epsilon^2 A_1 A_2 w_1 - \frac{1}{2}c^2 \epsilon^2 A_1 A_2 w_1$$

$$+ 4\epsilon^2 A_0 A_2 w_2 + 4c^2 \epsilon^2 A_0 A_2 w_2 + \frac{3}{2}\epsilon^2 A_0 A_1 w_3 + \frac{3}{2}c^2 \epsilon^2 A_0 A_1 w_3 = 0,$$

$$2\alpha^2 l^2 A_1{}^3 + 12\alpha^2 l^2 A_0 A_1 A_2 - \epsilon^2 A_2{}^2 w_1 - c^2 \epsilon^2 A_2{}^2 w_1 + \epsilon^2 A_1 A_2 w_2 + c^2 \epsilon^2 A_1 A_2 w_2$$

$$+ \frac{1}{2}\epsilon^2 A_1{}^2 w_3 + \frac{1}{2}c^2 \epsilon^2 A_1{}^2 w_3 + 5\epsilon^2 A_0 A_2 w_3 + 5c^2 \epsilon^2 A_0 A_2 w_3$$

$$+ 2\epsilon^2 A_0 A_1 w_4 + 2c^2 \epsilon^2 A_0 A_1 w_4 = 0,$$

$$6\alpha^2 l^2 A_1{}^2 A_2 + 6\alpha^2 l^2 A_0 A_2{}^2 + \frac{5}{2}\epsilon^2 A_1 A_2 w_3 + \frac{5}{2}\epsilon^2 c^2 A_1 A_2 w_3 + \epsilon^2 A_1{}^2 w_4$$

$$+ c^2 \epsilon^2 A_1{}^2 w_4 + 6\epsilon^2 A_0 A_2 w_4 + 6c^2 \epsilon^2 A_0 A_2 w_4 = 0,$$

$$6\alpha^2 l^2 A_1 A_2{}^2 + \epsilon^2 A_2{}^2 w_3 + c^2 \epsilon^2 A_2{}^2 w_1 + \epsilon^2 A_1 A_2 w_2 + c^2 \epsilon^2 A_2{}^2 w_3$$

$$+ 4\epsilon^2 A_1 A_2 w_4 + 4c^2 \epsilon^2 A_1 A_2 w_4 = 0,$$

$$2\alpha^2 l^2 A_2{}^3 + 2\epsilon^2 A_2{}^2 w_4 + 2c^2 \epsilon^2 A_2{}^2 w_4 = 0.$$

$$\text{(4.5)}$$

Case 1. When $w_0 = w_1 = w_3 = 0$, $w_2 > 0$, $w_4 < 0$, (2.4) admits a hyperbolic function solution

$$\phi = \sqrt{-\frac{w_2}{w_4}}\ \text{sech}(\sqrt{w_2}\ \xi). \tag{4.6}$$

Thus (4.4) yields the following new solitary wave solution of (2.1) of bell-type

$$v_1(\xi) = \frac{(1+c^2)w_2}{\alpha^2 l^2}\text{sech}^2(\sqrt{w_2}\ \xi), \tag{4.7}$$

where $w_2 > 0$, $w_4 < 0$, $\alpha \neq 0$, $l \neq 0$, and c are arbitrary constants. Reverting back to the original variables x and t, we obtain the solution of (4.1) in the form

$$u_1(x,t) = -\frac{1}{2}\ln\left[\frac{(1+c^2)w_2}{\alpha^2 l^2}\text{sech}^2\{\sqrt{w_2}\ (x-ct)\}\right]. \tag{4.8}$$

Case 2. When $w_1 = w_3 = 0$, $w_0 = w_2{}^2/4w_4$, $w_2 < 0$, $w_4 > 0$, (2.4) admits two hyperbolic function solutions

$$\phi = \pm\sqrt{-\frac{w_2}{2w_4}}\ \tanh\left(\sqrt{\frac{-w_2}{2}}\ \xi\right), \tag{4.9}$$

and so (4.4) yields one family of solitary travelling wave solutions of (4.1) given by

$$u_2(x,t) = -\frac{1}{2}\ln\left[-\frac{(1+c^2)w_2}{2\alpha^2 l^2} + \frac{(1+c^2)w_2}{\alpha^2 l^2}\tanh^2\left(\sqrt{-\frac{w_2}{2}}\ (x-ct)\right)\right], \tag{4.10}$$

where $w_2 < 0$, $w_4 > 0$, $\alpha \neq 0$, $l \neq 0$, and c are arbitrary constants.

Case 3. When $w_0 = w_1 = 0$, $w_3 = \pm 2\sqrt{w_2 w_4}$, $w_2 > 0$, $w_4 > 0$, (2.4) has two kinds of exact solutions:

$$\phi = -\frac{\sqrt{w_2 w_4}}{2w_4}\ \text{sign}(w_3)\left[1 + \tanh\left(\frac{\sqrt{w_2}}{2}\ \xi\right)\right], \tag{4.11}$$

and (4.4) yields one family of solitary travelling wave solutions of (4.1) given by

$$
\begin{aligned}
u_3(x,t) = & -\frac{1}{2}\ln\left[\pm\frac{(1+c^2)w_2}{\alpha^2 l^2}\,\mathrm{sign}(w_3)\left[1+\tanh\left(\frac{\sqrt{w_2}}{2}(x-ct)\right)\right]\right.\\
& \left.-\frac{(1+c^2)w_2}{4\alpha^2 l^2}\left[1+\tanh\left(\frac{\sqrt{w_2}}{2}(x-ct)\right)\right]^2\right],
\end{aligned}
\tag{4.12}
$$

where $w_2 > 0$, $w_4 > 0$, $\alpha \neq 0$, $l \neq 0$, and c are arbitrary constants.

Case 4. When $w_1 = w_3 = 0$, (2.4) admits three Jacobian elliptic doubly periodic solutions

$$
\begin{aligned}
\phi &= \sqrt{\frac{-w_2 k^2}{w_4(2k^2-1)}}\ \mathrm{cn}\left(\sqrt{\frac{w_2}{2k^2-1}}\,\xi,k\right), \quad \text{for } w_0 = \frac{w_2^2 k^2 (k^2-1)}{w_4(2k^2-1)^2},\ w_2 > 0, w_4 < 0,\\
\phi &= \sqrt{\frac{-w_2}{w_4(2-k^2)}}\ \mathrm{dn}\left(\sqrt{\frac{w_2}{2-k^2}}\,\xi,k\right), \quad \text{for } w_0 = \frac{w_2^2(1-k^2)}{w_4(k^2-2)^2},\ w_2 > 0, w_4 < 0,\\
\phi &= \pm\sqrt{\frac{-w_2 k^2}{w_4(k^2+1)}}\ \mathrm{sn}\left(\sqrt{\frac{-w_2}{k^2+1}}\,\xi,k\right), \quad \text{for } w_0 = \frac{w_2^2 k^2}{w_4(k^2+1)^2},\ w_2 < 0, w_4 > 0,
\end{aligned}
\tag{4.13}
$$

and (4.4), respectively, yields two families of Jacobian elliptic doubly periodic wave solutions

$$
u_4(x,t) = -\frac{1}{2}\ln\left[-\frac{(1+c^2)w_2}{2\alpha^2 l^2}+\frac{(1+c^2)w_2(2k^2-1)}{4\alpha^2 l^2(k^2-1)}\,\mathrm{cn}^2\left(\sqrt{\frac{w_2}{2k^2-1}}(x-ct),k\right)\right],
\tag{4.14}
$$

with $w_2 > 0$, $w_4 < 0$, $\alpha \neq 0$, $l \neq 0$, $k \in (\sqrt{2}/2, 1)$, and c being arbitrary constants. Similarly, from (4.4), respectively, we can obtain two families of Jacobian elliptic doubly periodic wave solutions

$$
u_5(x,t) = -\frac{1}{2}\ln\left[-\frac{(1+c^2)w_2}{2\alpha^2\beta^2}+\frac{(1+c^2)w_2(k^2-2)}{4\alpha^2 l^2(1-k^2)}\,\mathrm{dn}^2\left(\sqrt{\frac{w_2}{2-k^2}}(x-ct),k\right)\right],
\tag{4.15}
$$

with $w_2 > 0$, $w_4 < 0$, $\alpha \neq 0$, $l \neq 0$, $k \in (0,1)$, and c being arbitrary constants. Similarly, from (4.4), respectively, we can obtain two families of Jacobian elliptic doubly periodic wave solutions

$$
u_6(x,t) = -\frac{1}{2}\ln\left[-\frac{(1+c^2)w_2}{2\alpha^2 l^2}+\frac{(1+c^2)w_2(k^2+1)}{4\alpha^2 l^2}\,\mathrm{sn}^2\left(\sqrt{-\frac{w_2}{k^2+1}}(x-ct),k\right)\right]
\tag{4.16}
$$

with $w_2 < 0$, $w_4 > 0$, $\alpha \neq 0$, $l \neq 0$, $k \in (0,1)$, and c being arbitrary constants.

4.2. The sinh-*Poisson Equation*

Secondly, we consider sinh-Poisson equation which plays an important role in soliton model with BPS Bound. Also, this equation is a special case of (3.9) and is given by

$$u_{xx} + u_{tt} = \beta^2 \sinh(u). \tag{4.17}$$

In order to apply the Fan sub-equation method, we use the wave transformation $\xi = x - ct$ and convert (4.17) into the form

$$\left(1 + c^2\right)u'' = \beta^2 \sinh(u). \tag{4.18}$$

We next use the transformation $v = e^u$ and obtain the equation

$$2\left(1 + c^2\right)vv'' - 2\left(1 + c^2\right)v'^2 - \beta^2\left(v^3 - v\right) = 0. \tag{4.19}$$

Applying Step 3, we get $n = 2$, therefore the solution of (4.19) can be expressed as

$$v(\xi) = A_0 + A_1\phi + A_2\phi^2. \tag{4.20}$$

Then using Step 4, we obtain a system of nonlinear algebraic equations for A_0, A_1, and A_2:

$$
\begin{aligned}
& -l^2 A_0{}^3 - 2\epsilon^2 A_1{}^2 w_0 - 2c^2\epsilon^2 A_1{}^2 w_0 + 4\epsilon^2 A_0 A_2 w_0 + 4c^2\epsilon^2 A_0 A_2 w_0 \\
& \quad + \epsilon^2 A_0 A_1 w_1 + c^2\epsilon^2 A_0 A_1 w_1 = 0, \\
& -3l^2 A_0{}^2 A_1 - 4\epsilon^2 A_1 A_2 w_0 - 4c^2\epsilon^2 A_1 A_2 w_0 - \epsilon^2 A_1{}^2 w_1 - c^2\epsilon^2 A_1{}^2 w_1 + 6\epsilon^2 A_0 A_2 w_1 \\
& \quad + 6c^2\epsilon^2 A_0 A_2 w_1 + 2\epsilon^2 A_0 A_1 w_2 + 2c^2\epsilon^2 A_0 A_1 w_2 = 0, \\
& -3l^2 A_0 A_1{}^2 - 3l^2 A_0{}^2 A_2 - 4\epsilon^2 A_2{}^2 w_0 - 4c^2\epsilon^2 A_2{}^2 w_0 - \epsilon^2 A_1 A_2 w_1 - c^2\epsilon^2 A_1 A_2 w_1 \\
& \quad + 8\epsilon^2 A_0 A_2 w_2 + 8c^2\epsilon^2 A_0 A_2 w_2 + 3\epsilon^2 A_0 A_1 w_3 + 3c^2\epsilon^2 A_0 A_1 w_3 = 0, \\
& -l^2 A_1{}^3 - 6l^2 A_0 A_1 A_2 - 2\epsilon^2 A_2{}^2 w_1 - 2c^2\epsilon^2 A_2{}^2 w_1 + 2\epsilon^2 A_1 A_2 w_2 + 2c^2\epsilon^2 A_1 A_2 w_2 \\
& \quad + \epsilon^2 A_1{}^2 w_3 + c^2\epsilon^2 A_1{}^2 w_3 + 10\epsilon^2 A_0 A_2 w_3 + 10c^2\epsilon^2 A_0 A_2 w_3 \\
& \quad + 4\epsilon^2 A_0 A_1 w_4 + 4c^2\epsilon^2 A_0 A_1 w_4 = 0, \\
& -3l^2 A_1{}^2 A_2 - 3l^2 A_0 A_2{}^2 + 5\epsilon^2 A_1 A_2 w_3 + 5c^2\epsilon^2 A_1 A_2 w_3 + 2\epsilon^2 A_1{}^2 w_4 + 2c^2\epsilon^2 A_1{}^2 w_4 \\
& \quad + 12\epsilon^2 A_0 A_2 w_4 + 12c^2\epsilon^2 A_0 A_2 w_4 = 0, \\
& -3l^2 A_1 A_2{}^2 + 2\epsilon^2 A_2{}^2 w_3 + 2c^2\epsilon^2 A_2{}^2 w_3 + 8\epsilon^2 A_1 A_2 w_4 + 8c^2\epsilon^2 A_1 A_2 w_4 = 0, \\
& -l^2 A_2{}^3 + 4\epsilon^2 A_2{}^2 w_4 + 4c^2\epsilon^2 A_2{}^2 w_4 = 0.
\end{aligned}
\tag{4.21}
$$

Case 1. When $w_0 = w_1 = w_3 = 0$, $w_2 > 0$, $w_4 < 0$, (2.4) admits a hyperbolic function solution

$$\phi = \sqrt{-\frac{w_2}{w_4}} \operatorname{sech}(\sqrt{w_2}\ \xi) \tag{4.22}$$

and (4.20) yields the following new solitary wave solution of (4.17) of bell-type

$$u_1(x,t) = \ln\left[-\frac{4(1+c^2)w_2}{l^2}\operatorname{sech}^2(\sqrt{w_2}\ (x-ct))\right], \tag{4.23}$$

where $w_2 > 0$, $w_4 < 0$, $l \neq 0$, and c are arbitrary constants.

Case 2. When $w_1 = w_3 = 0$, $w_0 = w_2^2/4w_4$, $w_2 < 0$, $w_4 > 0$, (2.4) admits two hyperbolic function solutions

$$\phi = \pm\sqrt{-\frac{w_2}{2w_4}}\ \tanh\left(\sqrt{\frac{-w_2}{2}}\ \xi\right), \tag{4.24}$$

and (4.20) yields one family of solitary travelling wave solutions of (4.17) given by

$$u_2(x,t) = \ln\left[\frac{2(1+c^2)w_2}{l^2} - \frac{2(1+c^2)w_2}{l^2}\tanh^2\left(\sqrt{-\frac{w_2}{2}}(x-ct)\right)\right], \tag{4.25}$$

where $w_2 < 0$, $w_4 > 0$, $l \neq 0$, and c are arbitrary constants.

Case 3. When $w_0 = w_1 = 0$, $w_3 = \pm 2\sqrt{w_2 w_4}$, $w_2 > 0$, $w_4 > 0$, (2.4) has two kinds of exact solutions

$$\phi = -\frac{\sqrt{w_2 w_4}}{2w_4}\operatorname{sign}(w_3)\left[1 + \tanh\left(\frac{\sqrt{w_2}}{2}\xi\right)\right], \tag{4.26}$$

and (4.20) yields one family of solitary travelling wave solutions solitary travelling wave solutions of (4.17) given by

$$\begin{aligned}u_3(x,t) = \ \ln\Bigg[&\pm\frac{2(1+c^2)w_2}{l^2}\operatorname{sign}(w_3)\left[1 + \tanh\left(\frac{\sqrt{w_2}}{2}(x-ct)\right)\right]\\&-\frac{(1+c^2)w_2}{l^2}\left[1 + \tanh\left(\frac{\sqrt{w_2}}{2}(x-ct)\right)\right]^2\Bigg],\end{aligned} \tag{4.27}$$

where $w_2 > 0, w_4 > 0$, $l \neq 0$ and c are arbitrary constants.

Case 4. When $w_1 = w_3 = 0$, (2.4) admits three Jacobian elliptic doubly periodic solutions

$$\phi = \sqrt{\frac{-w_2 k^2}{w_4(2k^2 - 1)}} \ \text{cn}\left(\sqrt{\frac{w_2}{2k^2 - 1}}\ \xi, k\right), \quad \text{for } w_0 = \frac{w_2^2 k^2(k^2 - 1)}{w_4(2k^2 - 1)^2}, \ w_2 > 0, w_4 < 0,$$

$$\phi = \sqrt{\frac{-w_2}{w_4(2 - k^2)}} \ \text{dn}\left(\sqrt{\frac{w_2}{2 - k^2}}\ \xi, k\right), \quad \text{for } w_0 = \frac{w_2^2(1 - k^2)}{w_4(k^2 - 2)^2}, \ w_2 > 0, w_4 < 0, \qquad (4.28)$$

$$\phi = \pm\sqrt{\frac{-w_2 k^2}{w_4(k^2 + 1)}} \ \text{sn}\left(\sqrt{\frac{-w_2}{k^2 + 1}}\ \xi, k\right), \quad \text{for } w_0 = \frac{w_2^2 k^2}{w_4(k^2 + 1)^2}, \ w_2 < 0, w_4 > 0,$$

and (4.20), respectively, yields two families of Jacobian elliptic doubly periodic wave solutions

$$u_4(x, t) = \ln\left[\frac{2(1 + c^2)w_2}{l^2} + \frac{2(1 + c^2)(2k^2 - 1)w_2}{l^2(k^2 - 1)} \ \text{cn}^2\left(\sqrt{\frac{w_2}{2k^2 - 1}}(x - ct), k\right)\right], \qquad (4.29)$$

with $w_2 > 0$, $w_4 < 0$, $l \neq 0$, $k \in (\sqrt{2}/2, 1)$, and c being arbitrary constants. Similarly, from (4.20), respectively, we can obtain two families of Jacobian elliptic doubly periodic wave solutions

$$u_5(x, t) = \ln\left[\frac{2(1 + c^2)w_2}{l^2} - \frac{2(1 + c^2)w_2(2 - k^2)}{l^2(1 - k^2)} \ \text{dn}^2\left(\sqrt{\frac{w_2}{2 - k^2}}(x - ct), k\right)\right], \qquad (4.30)$$

with $w_2 > 0$, $w_4 < 0$, $\alpha \neq 0$, $l \neq 0$, $k \in (0, 1)$, and c being arbitrary constants. Likewise, from (4.20), respectively, we can get two families of Jacobian elliptic doubly periodic wave solutions

$$u_6(x, t) = \ln\left[\frac{2(1 + c^2)w_2}{l^2} - \frac{2(1 + c^2)w_2(k^2 + 1)}{l^2} \ \text{sn}^2\left(\sqrt{-\frac{w_2}{k^2 + 1}}(x - ct), k\right)\right], \qquad (4.31)$$

with $w_2 < 0$, $w_4 > 0$, $\alpha \neq 0$, $l \neq 0$, $k \in (0, 1)$, and c being arbitrary constants.

5. Concluding Remarks

In this paper, the Fan sub-equation method has been successfully used to obtain some exact travelling wave solutions for the Liouville and sinh-Poisson equations. These exact solutions include the hyperbolic function solutions, trigonometric function solutions. When the parameters are taken as special values, the solitary wave solutions are derived from the hyperbolic function solutions. Thus, this study shows that the Fan sub-equation method is quite efficient and practically well suited for use in finding exact solutions for nonlinear partial differential equations. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

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