

*Research Article*

## **Smooth Solutions of a Class of Iterative Functional Differential Equations**

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By Faà di Bruno's formula, using the fixed-point theorems of Schauder and Banach, we study the existence and uniqueness of smooth solutions of an iterative functional differential equation  $x'(t) = 1/(c_0x^{[0]}(t) + c_1x^{[1]}(t) + \dots + c_mx^{[m]}(t))$ .

### **1. Introduction**

There has been a lot of monographs and research articles to discuss the kinds of solutions of functional differential equations since the publication of Jack Hale's paper [1]. Several papers discussed the iterative functional differential equations of the form

$$x'(t) = H(x^{[0]}(t), x^{[1]}(t), \dots, x^{[m]}(t)), \quad (1.1)$$

where  $x^{[0]}(t) = t$ ,  $x^{[1]}(t) = x(t)$ ,  $x^{[k]}(t) = x(x^{[k-1]}(t))$ ,  $k = 2, \dots, m$ . More specifically, Eder [2] considered the functional differential equation

$$x'(t) = x^{[2]}(t) \quad (1.2)$$

and proved that every solution either vanishes identically or is strictly monotonic. Furthermore, Fečkan [3] and Wang [4] studied the equation

$$x'(t) = f(x^{[2]}(t)) \quad (1.3)$$

with different conditions. Staněk [5] considered the equation

$$x'(t) = x(t) + x^{[2]}(t) \quad (1.4)$$

and obtained every solution either vanishes identically or is strictly monotonic. Si and his coauthors [6, 7] studied the following equations:

$$\begin{aligned} x'(t) &= x^{[m]}(t), \\ x'(t) &= \frac{1}{x^{[m]}(t)}, \end{aligned} \quad (1.5)$$

$$x'(t) = \frac{1}{c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \cdots + c_m x^{[m]}(t)} \quad (1.6)$$

and established sufficient conditions for the existence of analytic solutions. Especially in [8, 9], the smooth solutions of the following equations:

$$\begin{aligned} x'(t) &= \sum_{j=1}^m a_j x^{[j]}(t) + F(t), \\ x'(t) &= \sum_{j=1}^m a_j(t) x^{[j]}(t) + F(t), \end{aligned} \quad (1.7)$$

have been studied by the fixed-point theorems of Schauder and Banach.

A smooth function is taken to mean one that has a number of continuous derivatives and for which the highest continuous derivative is also Lipschitz. Let  $x \in C^n$  if  $x', \dots, x^{(n)}$  are continuous,  $C^n(I, I)$  is the set in which  $x \in C^n$  and maps a closed interval  $I$  into  $I$ . As in [9], we, using the same symbols, denote the norm

$$\|x\|_n = \sum_{k=0}^n \|x^{(k)}\|, \quad \|x\| = \max_{t \in I} \{|x(t)|\}, \quad (1.8)$$

then  $C^n(I, R)$  with  $\|\cdot\|_n$  is a Banach space, and  $C^n(I, I)$  is a subset of  $C^n(I, R)$ . For given  $M_i > 0$  ( $i = 1, 2, \dots, n+1$ ), let

$$\begin{aligned} \Omega(M_1, \dots, M_{n+1}; I) = \Big\{ x \in C^n(I, I) : &\left| x^{(i)}(t) \right| \leq M_i, i = 1, 2, \dots, n; \\ &\left| x^{(n)}(t_1) - x^{(n)}(t_2) \right| \leq M_{n+1} |t_1 - t_2|, \quad t, t_1, t_2 \in I \Big\}. \end{aligned} \quad (1.9)$$

For convenience, we will make use of the notation

$$x_{ij}(t) = x^{(i)}(x^{[j]}(t)), \quad x_{*jk}(t) = (x^{[j]}(t))^{(k)}, \quad (1.10)$$

where  $i, j$ , and  $k$  are nonnegative integers. Let  $I$  be a closed interval in  $R$ . By induction, we may prove that

$$x_{*jk}(t) = P_{jk}(x_{10}(t), \dots, x_{1,j-1}(t); \dots; x_{k0}(t), \dots, x_{k,j-1}(t)), \quad (1.11)$$

$$\beta_{jk} = P_{jk} \left( \overbrace{x'(\xi), \dots, x'(\xi)}^{j \text{ terms}}; \dots; \overbrace{x^{(k)}(\xi), \dots, x^{(k)}(\xi)}^{j \text{ terms}} \right), \quad (1.12)$$

$$H_{jk} = P_{jk} \left( \overbrace{1, \dots, 1}^{j \text{ terms}}; \overbrace{M_2, \dots, M_2}^{j \text{ terms}}; \dots; \overbrace{M_k, \dots, M_k}^{j \text{ terms}} \right), \quad (1.13)$$

where  $P_{jk}$  is a uniquely defined multivariate polynomial with nonnegative coefficients. The proof can be found in [8].

In order to seek a solution  $x(t)$  of (1.6), in  $C^n(I, I)$  such that  $\xi$  is a fixed point of the function  $x(t)$ , that is,  $x(\xi) = \xi$ , it is natural to seek an interval  $I$  of the form  $[\xi - \delta, \xi + \delta]$  with  $\delta > 0$ .

Let us define

$$\begin{aligned} X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I) \\ = \left\{ x \in \Omega(1, M_2, \dots, M_{n+1}; I) : x(\xi) = \xi_0 = \xi, x^{(i)}(\xi) = \xi_i, i = 1, 2, \dots, n \right\}. \end{aligned} \quad (1.14)$$

## 2. Smooth Solutions of (1.6)

In this section, we will prove the existence theorem of smooth solutions for (1.6). First of all, we have the inequalities in the following for all  $x(t), y(t) \in X$ :

$$|x^{[j]}(t_1) - x^{[j]}(t_2)| \leq |t_1 - t_2|, \quad t_1, t_2 \in I, \quad j = 0, 1, \dots, m, \quad (2.1)$$

$$\|x^{[j]} - x^{[j]}\| \leq j \|x - y\|, \quad j = 1, \dots, m, \quad (2.2)$$

$$\|x - y\| \leq \delta^n \|x^{(n)} - y^{(n)}\|, \quad (2.3)$$

and the proof can be found in [9].

**Theorem 2.1.** *Let  $I = [\xi - \delta, \xi + \delta]$ , where  $\xi$  and  $\delta$  satisfy*

$$\xi \geq \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \quad 0 < \delta \leq \xi - \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \quad (2.4)$$

where  $|c_0| > \sum_{i=0}^m |c_i|$ , then (1.6) has a solution in

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \quad (2.5)$$

provided the following conditions hold:

(i)

$$\xi_1 = \xi^{-1} \left( \sum_{i=0}^m c_i \right)^{-1}, \quad (2.6)$$

$$\begin{aligned} \xi_k &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \left( \xi \sum_{i=0}^m c_i \right)^{-s-1} \left( \frac{\sum_{i=0}^m c_i \beta_{i1}}{1!} \right)^{s_1} \\ &\quad \times \left( \frac{\sum_{i=0}^m c_i \beta_{i2}}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m c_i \beta_{ik-1}}{(k-1)!} \right)^{s_{k-1}}, \end{aligned} \quad (2.7)$$

where  $k = 2, \dots, n$ , and the sum is over all nonnegative integer solutions of the Diophantine equation  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ ,

(ii)

$$\begin{aligned} \sum \frac{(k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} (\xi - \delta)^{-s-1} &\left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \frac{\sum_{i=0}^m |c_i| H_{i1}}{1!} \right)^{s_1} \\ &\times \left( \frac{\sum_{i=0}^m |c_i| H_{i2}}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m |c_i| H_{ik-1}}{(k-1)!} \right)^{s_{k-1}} \leq M_k, \quad k = 2, \dots, n, \end{aligned} \quad (2.8)$$

where  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ ,

(iii)

$$\begin{aligned} \sum \frac{(n-1)! s!}{s_1! s_2! \cdots s_{n-1}! 1!^{s_1} 2!^{s_2} \cdots (n-1)!^{s_{n-1}}} &\times \left[ (s+1)(\xi - \delta)^{-s-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1+1} \left( \sum_{i=0}^m |c_i| H_{i2} \right)^{s_2} \right. \\ &\times \cdots \times \left( \sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} + s_1(\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\ &\times \left( \sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \left( \sum_{i=0}^m |c_i| H_{i3} \right)^{s_3} \cdots \left( \sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} \\ &+ \cdots + s_{n-1}(\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left( \sum_{i=0}^m |c_i| H_{in-2} \right)^{s_{n-2}} \\ &\times \left. \left( \sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}-1} \left( \sum_{i=0}^m |c_i| H_{in} \right) \right] \\ &\leq M_{n+1}, \end{aligned} \quad (2.9)$$

where  $s_1 + 2s_2 + \cdots + (n-1)s_{n-1} = n-1$  and  $s = s_1 + s_2 + \cdots + s_{n-1}$ ,

*Proof.* Define an operator  $T$  from  $X$  into  $C^n(I, I)$  by

$$(Tx)(t) = \xi + \int_{\xi}^t \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)} ds. \quad (2.10)$$

We will prove that for any  $x \in X$ ,  $Tx \in X$ ,

$$|(Tx)(t) - \xi| = \left| \int_{\xi}^t \frac{1}{\sum_{i=0}^m c_i x^{[i]}(s)} ds \right| \leq \left( (\xi - \delta) \left( |c_0| - \sum_{i=1}^m |c_i| \right) \right)^{-1} |t - \xi| \leq \delta, \quad (2.11)$$

where the second inequality is from (2.4) and  $x(I) \subseteq I$ . Thus,  $(Tx)(I) \subseteq I$ .

It is easy to see that

$$(Tx)'(t) = \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)}, \quad (2.12)$$

and by Faà di Bruno's formula, for  $k = 2, \dots, n$ , we have

$$\begin{aligned} (Tx)^{(k)}(t) &= \left( \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} \right)^{(k-1)} = \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{(\sum_{i=0}^m c_i x^{[i]}(t))^{s+1}} \left( \frac{(\sum_{i=0}^m c_i x^{[i]}(t))'}{1!} \right)^{s_1} \\ &\quad \times \left( \frac{(\sum_{i=0}^m c_i x^{[i]}(t))''}{2!} \right)^{s_2} \cdots \left( \frac{(\sum_{i=0}^m c_i x^{[i]}(t))^{(k-1)}}{(k-1)!} \right)^{s_{k-1}} \\ &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{(\sum_{i=0}^m c_i x^{[i]}(t))^{s+1}} \left( \frac{\sum_{i=0}^m c_i x_{*i1}(t)}{1!} \right)^{s_1} \\ &\quad \times \left( \frac{\sum_{i=0}^m c_i x_{*i2}(t)}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m c_i x_{*ik-1}(t)}{(k-1)!} \right)^{s_{k-1}}, \end{aligned} \quad (2.13)$$

where the sum is over all nonnegative integer solutions of the Diophantine equation  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ .

Furthermore, note  $(Tx)(\xi) = \xi$ , by (2.6) and (2.7),

$$\begin{aligned}
 (Tx)'(\xi) &= \frac{1}{\sum_{i=0}^m c_i x^{[i]}(\xi)} = \frac{1}{\xi \sum_{i=0}^m c_i} = \xi_1, \\
 (Tx)^{(k)}(\xi) &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{(\xi \sum_{i=0}^m c_i)^{s+1}} \left( \frac{\sum_{i=0}^m c_i x_{*i1}(\xi)}{1!} \right)^{s_1} \\
 &\quad \times \left( \frac{\sum_{i=0}^m c_i x_{*i2}(\xi)}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m c_i x_{*ik-1}(\xi)}{(k-1)!} \right)^{s_{k-1}} \\
 &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{(\xi \sum_{i=0}^m c_i)^{s+1}} \left( \frac{\sum_{i=0}^m c_i \beta_{i1}}{1!} \right)^{s_1} \\
 &\quad \times \left( \frac{\sum_{i=0}^m c_i \beta_{i2}}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m c_i \beta_{ik-1}}{(k-1)!} \right)^{s_{k-1}} \\
 &= \xi_k, \quad k = 2, \dots, n,
 \end{aligned} \tag{2.14}$$

where  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ . Thus,  $(Tx)^{(k)}(\xi) = \xi_k$  for  $k = 0, 1, \dots, n$ ,

$$|(Tx)'(t)| = \left| \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} \right| \leq \left( (\xi - \delta) \left( |c_0| - \sum_{i=1}^m |c_i| \right) \right)^{-1} \leq 1 = M_1, \tag{2.15}$$

By (2.8), we have

$$\begin{aligned}
 |(Tx)^{(k)}(t)| &\leq \sum \frac{(k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{|\sum_{i=0}^m c_i x^{[i]}(t)|^{s+1}} \left( \frac{\sum_{i=0}^m |c_i| |x_{*i1}(t)|}{1!} \right)^{s_1} \\
 &\quad \times \left( \frac{\sum_{i=0}^m |c_i| |x_{*i2}(t)|}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m |c_i| |x_{*ik-1}(t)|}{(k-1)!} \right)^{s_{k-1}} \\
 &\leq \sum \frac{(k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} (\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \frac{\sum_{i=0}^m |c_i| H_{i1}}{1!} \right)^{s_1} \\
 &\quad \times \left( \frac{\sum_{i=0}^m |c_i| H_{i2}}{2!} \right)^{s_2} \cdots \left( \frac{\sum_{i=0}^m |c_i| H_{ik-1}}{(k-1)!} \right)^{s_{k-1}} \\
 &\leq M_k, \quad k = 2, \dots, n,
 \end{aligned} \tag{2.16}$$

where  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ .

Finally,

$$\begin{aligned}
& \left| (Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2) \right| \\
& \leq \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1^{s_1}2^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
& \quad \times \left| \left( \sum_{i=0}^m c_i x^{[i]}(t_1) \right)^{-s-1} \left( \sum_{i=0}^m c_i x_{*i1}(t_1) \right)^{s_1} \left( \sum_{i=0}^m c_i x_{*i2}(t_1) \right)^{s_2} \cdots \left( \sum_{i=0}^m c_i x_{*in-1}(t_1) \right)^{s_{n-1}} \right. \\
& \quad \left. - \left( \sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{-s-1} \left( \sum_{i=0}^m c_i x_{*i1}(t_2) \right)^{s_1} \left( \sum_{i=0}^m c_i x_{*i2}(t_2) \right)^{s_2} \cdots \left( \sum_{i=0}^m c_i x_{*in-1}(t_2) \right)^{s_{n-1}} \right| \\
& \leq \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1^{s_1}2^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
& \quad \times \left[ \left| \frac{1}{(\sum_{i=0}^m c_i x^{[i]}(t_1))^{s+1}} - \frac{1}{(\sum_{i=0}^m c_i x^{[i]}(t_2))^{s+1}} \right| \left( \sum_{i=0}^m |c_i| |x_{*i1}(t_1)| \right)^{s_1} \right. \\
& \quad \times \left. \cdots \left( \sum_{i=0}^m |c_i| |x_{*in-1}(t_1)| \right)^{s_{n-1}} + \left| \left( \sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{-s-1} \right| \right. \\
& \quad \times \left| \left( \sum_{i=0}^m c_i x_{*i1}(t_1) \right)^{s_1} - \left( \sum_{i=0}^m c_i x_{*i1}(t_2) \right)^{s_1} \right| \left( \sum_{i=0}^m |c_i| |x_{*i2}(t_1)| \right)^{s_2} \cdots \\
& \quad \times \left( \sum_{i=0}^m |c_i| |x_{*in-1}(t_1)| \right)^{s_{n-1}} + \cdots + \left| \left( \sum_{i=0}^m c_i x^{[i]}(t_2) \right)^{-s-1} \right| \left( \sum_{i=0}^m |c_i| |x_{*i1}(t_2)| \right)^{s_1} \\
& \quad \times \left( \sum_{i=0}^m |c_i| |x_{*i2}(t_2)| \right)^{s_2} \cdots \left( \sum_{i=0}^m |c_i| |x_{*in-2}(t_2)| \right)^{s_{n-2}} \\
& \quad \times \left. \left| \left( \sum_{i=0}^m c_i x_{*in-1}(t_1) \right)^{s_{n-1}} - \left( \sum_{i=0}^m c_i x_{*in-1}(t_2) \right)^{s_{n-1}} \right| \right] \\
& \leq \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1^{s_1}2^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
& \quad \times \left[ (s+1)(\xi-\delta)^{-s-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1+1} \left( \sum_{i=0}^m |c_i| H_{i2} \right)^{s_2} \right. \\
& \quad \times \cdots \times \left( \sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} + s_1(\xi-\delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \left( \sum_{i=0}^m |c_i| H_{i3} \right)^{s_3} \cdots \left( \sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}} \\
& \quad + \cdots + s_{n-1}(\xi-\delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left( \sum_{i=0}^m |c_i| H_{in-2} \right)^{s_{n-2}} \\
& \quad \times \left. \left( \sum_{i=0}^m |c_i| H_{in-1} \right)^{s_{n-1}-1} \left( \sum_{i=0}^m |c_i| H_{in} \right) \right] |t_1 - t_2|,
\end{aligned} \tag{2.17}$$

where  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ . By (2.9), we see that

$$\left| (Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2) \right| \leq M_{n+1}|t_1 - t_2|. \quad (2.18)$$

Now, we can say that  $T$  is an operator from  $X$  into itself.

Next, we will show that  $T$  is continuous. Let  $x, y \in X$ , then

$$\begin{aligned} \|Tx - Ty\|_n &= \|Tx - Ty\| + \left\| (Tx)' - (Ty)' \right\| + \sum_{k=2}^n \left\| (Tx)^{(k)} - (Ty)^{(k)} \right\| \\ &= \max_{t \in I} \left| \int_{\xi}^t \left( \frac{1}{\sum_{i=0}^m c_i x^{[i]}(s)} - \frac{1}{\sum_{i=0}^m c_i y^{[i]}(s)} \right) ds \right| \\ &\quad + \max_{t \in I} \left| \frac{1}{\sum_{i=0}^m c_i x^{[i]}(t)} - \frac{1}{\sum_{i=0}^m c_i y^{[i]}(t)} \right| \\ &\quad + \sum_{k=2}^n \max_{t \in I} \left| \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}} \right. \\ &\quad \times \left[ \frac{1}{(\sum_{i=0}^m c_i x^{[i]}(t))^{s+1}} \left( \sum_{i=0}^m c_i x_{*i1}(t) \right)^{s_1} \left( \sum_{i=0}^m c_i x_{*i2}(t) \right)^{s_2} \cdots \right. \\ &\quad \times \left( \sum_{i=0}^m c_i x_{*ik-1}(t) \right)^{s_{k-1}} - \frac{1}{(\sum_{i=0}^m c_i y^{[i]}(t))^{s+1}} \left( \sum_{i=0}^m c_i y_{*i1}(t) \right)^{s_1} \\ &\quad \times \left. \left( \sum_{i=0}^m c_i y_{*i2}(t) \right)^{s_2} \cdots \left( \sum_{i=0}^m c_i y_{*ik-1}(t) \right)^{s_{k-1}} \right] \right| \\ &\leq \delta(\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\ &\quad + (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\ &\quad + \sum_{k=2}^n \max_{t \in I} \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}} \\ &\quad \times \left[ \left( \sum_{i=0}^m c_i x^{[i]}(t) \right)^{-s-1} - \left( \sum_{i=0}^m c_i y^{[i]}(t) \right)^{-s-1} \right] \left( \sum_{i=0}^m |c_i| |x_{*i1}(t)| \right)^{s_1} \cdots \\ &\quad \times \left( \sum_{i=0}^m |c_i| |x_{*ik-1}(t)| \right)^{s_{k-1}} + \left| \left( \sum_{i=0}^m c_i y^{[i]}(t) \right)^{-s-1} \right| \\ &\quad \times \left| \left( \sum_{i=0}^m c_i x_{*i1}(t) \right)^{s_1} - \left( \sum_{i=0}^m c_i y_{*i1}(t) \right)^{s_1} \right| \end{aligned}$$

$$\begin{aligned}
& \times \left( \sum_{i=0}^m |c_i| |x_{*i2}(t)| \right)^{s_2} \cdots \left( \sum_{i=0}^m |c_i| |x_{*ik-1}(t)| \right)^{s_{k-1}} \\
& + \cdots + \left| \left( \sum_{i=0}^m c_i y^{[i]}(t) \right)^{-s-1} \right| \left( \sum_{i=0}^m |c_i| |y_{*i1}(t)| \right)^{s_1} \cdots \left( \sum_{i=0}^m |c_i| |y_{*ik-2}(t)| \right)^{s_{k-2}} \\
& \times \left| \left( \sum_{i=0}^m c_i x_{*ik-1}(t) \right)^{s_{k-1}} - \left( \sum_{i=0}^m c_i y_{*ik-1}(t) \right)^{s_{k-1}} \right| \\
& \leq (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^m i |c_i| \right) \|x - y\| \\
& + \sum_{k=2}^n \sum \frac{(k-1)! s!}{s_1! s_2! \cdots s_{k-1}! 1!^{s_1} 2!^{s_2} \cdots (k-1)!^{s_{k-1}}} \\
& \times \left[ (s+1)(\xi - \delta)^{-s-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \right. \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left( \sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \\
& \quad + s_1(\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \cdots \left( \sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\
& \quad + \cdots + s_{k-1}(\xi - \delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{ik-2} \right)^{s_{k-2}} \left( \sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}-1} \left( \sum_{i=0}^m |c_i| H_{ik} \right) \\
& \quad \times \left. \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \right] \\
& + \delta^{n+1} (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^m i |c_i| \right) \|x^{(n)} - y^{(n)}\|,
\end{aligned} \tag{2.19}$$

where  $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$  and  $s = s_1 + s_2 + \cdots + s_{k-1}$ .

Moreover, we can find some constants  $P_k$  such that

$$\begin{aligned}
& \sum_{k=2}^n \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1^{s_1}2^{s_2}\cdots(k-1)^{s_{k-1}}} \\
& \times \left[ (s+1)(\xi-\delta)^{-s-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-2} \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \times \right. \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \left( \sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \\
& \quad + s_1(\xi-\delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1-1} \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{i2} \right)^{s_2+1} \cdots \left( \sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}} \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \\
& \quad + \cdots + s_{k-1}(\xi-\delta)^{-s-1} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-s-1} \left( \sum_{i=0}^m |c_i| H_{i1} \right)^{s_1} \cdots \\
& \quad \times \left( \sum_{i=0}^m |c_i| H_{ik-2} \right)^{s_{k-2}} \left( \sum_{i=0}^m |c_i| H_{ik-1} \right)^{s_{k-1}-1} \left( \sum_{i=0}^m |c_i| H_{ik} \right) \\
& \quad \times \left. \left( \sum_{i=0}^m |c_i| \|x^{[i]} - y^{[i]}\| \right) \right] \\
& \leq \sum_{k=1}^{n-1} P_k(\xi, \delta, c_i, H_{ij}) \|x^{(k)} - y^{(k)}\|,
\end{aligned} \tag{2.20}$$

where

$$P_k(\xi, \delta, c_i, H_{ij}) = P(\xi, \delta; c_1, \dots, c_m; H_{11}, \dots, H_{1k+1}; \dots; H_{m1}, \dots, H_{mk+1};) \tag{2.21}$$

are the positive constants depend on  $\xi, \delta, c_i$ , and  $H_{ij}$ ,  $i = 1, \dots, m$ ;  $j = 1, \dots, k+1$ . Then

$$\begin{aligned}
\|Tx - Ty\|_n & \leq (\xi-\delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^m i |c_i| \right) \|x - y\| \\
& + \sum_{k=1}^{n-1} P_k(\xi, \delta, c_i, H_{ij}) \|x^{(k)} - y^{(k)}\| \\
& + \delta^{n+1} (\xi-\delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^m i |c_i| \right) \|x^{(n)} - y^{(n)}\| \\
& \leq \Gamma \|x - y\|_n.
\end{aligned} \tag{2.22}$$

Here,

$$\begin{aligned} \Gamma = \max \left\{ (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^m i |c_i| \right); \max_{1 \leq k \leq n-1} \{ P_k(\xi, \delta, c_i, H_{ij}) \}; \right. \\ \left. \delta^{n+1} (\xi - \delta)^{-2} \left( |c_0| - \sum_{i=1}^m |c_i| \right)^{-2} \left( |c_0| + \sum_{i=1}^m i |c_i| \right) \right\}, \end{aligned} \quad (2.23)$$

$k = 1, \dots, n-1$ . So we can say that  $T$  is continuous.

It is easy to see that  $X$  is closed and convex. We now show that  $X$  is a relatively compact subset of  $C^n(I, I)$ . For any  $x = x(t) \in X$ ,

$$\|x\|_n \leq \|x\| + \sum_{k=1}^n \|x^{(k)}\| \leq |\xi| + \delta + 1 + \sum_{k=2}^n M_k. \quad (2.24)$$

Next, for any  $t_1, t_2$  in  $I$ , we have

$$|x(t_1) - x(t_2)| \leq |t_1 - t_2|. \quad (2.25)$$

Hence,  $X$  is bounded in  $C^n(I, I)$  and equicontinuous on  $I$ , and by the Arzela-Ascoli theorem, we know  $X$  is relatively compact in  $C^n(I, I)$ , since  $C^n(I, I)$  is the subset of  $C^n(I, R)$ , and we can say that  $X$  is relatively compact in  $C^n(I, R)$ .

From Schauder's fixed-point theorem, we conclude that

$$x(t) = \xi + \int_{\xi}^t \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \dots + c_m x^{[m]}(s)} ds, \quad (2.26)$$

for some  $x = x(t)$  in  $X$ . By differentiating both sides of the above equality, we see that  $x$  is the desired solution of (1.6). This completes the proof.  $\square$

**Theorem 2.2.** Let  $I = [\xi - \delta, \xi + \delta]$ , where  $\xi$  and  $\delta$  satisfy (2.4), then (1.6) has a unique solution in

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \quad (2.27)$$

provided the conditions (2.6)–(2.9) hold and  $\Gamma < 1$  in (2.23).

*Proof.* Since  $\Gamma < 1$ , we see that  $T$  defined by (2.10) is contraction mapping on the close subset  $X$  of  $C^n(I, I)$ . Thus, the fixed point  $x$  in the proof of Theorem 2.1 must be unique. This completes the proof.  $\square$

**Remark 2.3.** By Theorem 2.1 or Theorem 2.2, the existence and uniqueness of smooth solutions of an iterative functional differential equation of the form (1.6) can be obtained. If  $n \rightarrow +\infty$ , we can also find that the solution is  $C^\infty$ -smooth.

Now, we will show that the conditions in Theorem 2.1 do not self-contradict. Consider the following equation:

$$x'(t) = \frac{1}{t + (1/2)x(t) + (1/4)x(x(t))}, \quad (2.28)$$

where  $c_0 = 1$ ,  $c_1 = (1/2)$ ,  $c_2 = (1/4)$ , and  $\xi \geq 4$ . Moreover, we take  $0 < \delta \leq \xi - 4$ . Then, (2.4) is satisfied, and  $\xi, \delta$  define the interval  $I = [\xi - \delta, \xi + \delta]$ . Now, take  $\xi_0 = \xi$ ,

$$\begin{aligned} \xi_1 &= \frac{4}{7}\xi^{-1}, \\ \xi_2 &= -\frac{16}{49}\xi^{-2} \left(1 + \frac{1}{2}\xi_1 + \frac{1}{4}\xi_1^2\right), \\ \xi_3 &= \frac{4}{343}\xi^{-3} \left(4 + 2\xi_1 + \xi_1^2\right)^2 - \frac{4}{49}\xi^{-2}\xi_2 \left(2 + \xi_1 + \xi_1^2\right), \end{aligned} \quad (2.29)$$

then (2.6) and (2.7) are satisfied.

Finally, if we take

$$M_1 = 1, \quad M_2 = 28(\xi - \delta)^{-2}, \quad M_3 = 392(\xi - \delta)^{-3} + 16(\xi - \delta)^{-2}M_2 \quad (2.30)$$

as positive, and

$$M_4 = 8232(\xi - \delta)^{-4} + 576(\xi - \delta)^{-3}M_2 + 8(\xi - \delta)^{-2}(6M_2^2 + 5M_3), \quad (2.31)$$

then (2.8) and (2.9) are satisfied.

Thus, we have shown that when  $\xi_0, \dots, \xi_3$  and  $M_1, \dots, M_4$  are defined as above, then there will be a solution for (2.28) in  $X(\xi; \xi_0, \dots, \xi_3; 1, \dots, M_4; I)$ .

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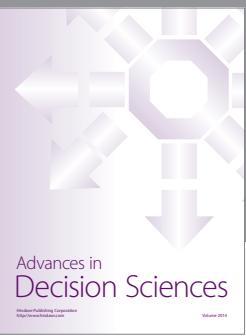
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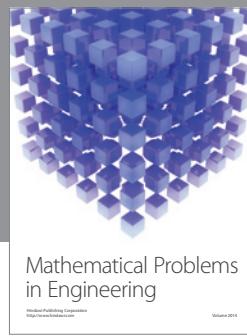
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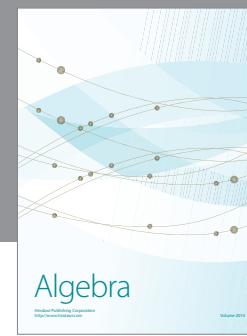
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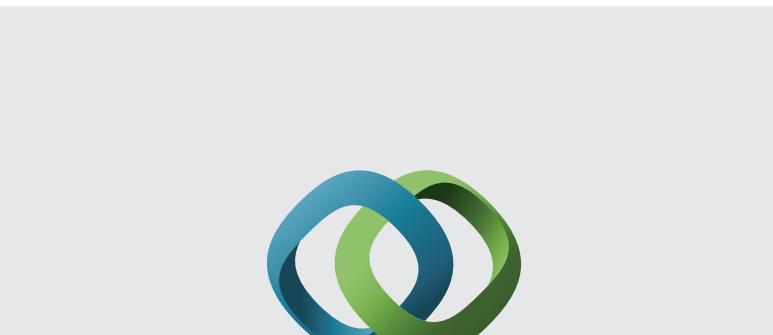
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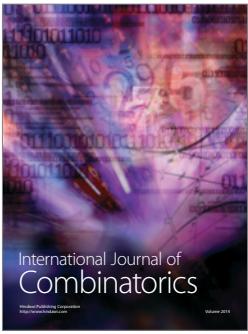


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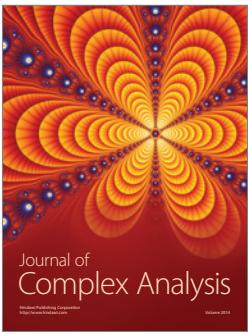
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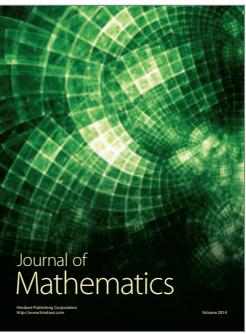
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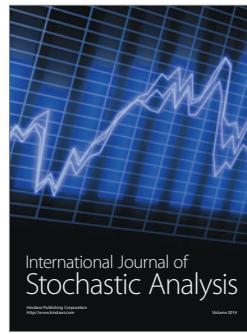
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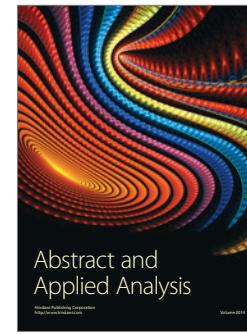
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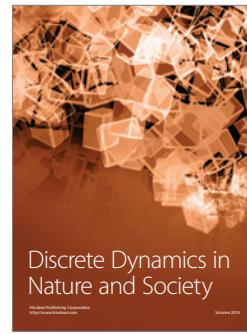
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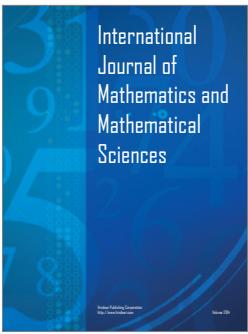
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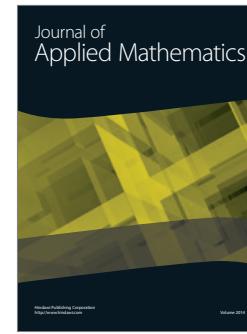
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