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Research Article

g-Bases in Hilbert Spaces

Xunxiang Guo

Department of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Correspondence should be addressed to Xunxiang Guo, guoxunxiang@yahoo.com

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The concept of g-basis in Hilbert spaces is introduced, which generalizes Schauder basis in Hilbert spaces. Some results about g-bases are proved. In particular, we characterize the g-bases and g-orthonormal bases. And the dual g-bases are also discussed. We also consider the equivalent relations of g-bases and g-orthonormal bases. And the property of g-minimal of g-bases is studied as well. Our results show that, in some cases, g-bases share many useful properties of Schauder bases in Hilbert spaces.

1. Introduction

In 1946, Gabor [1] introduced a fundamental approach to signal decomposition in terms of elementary signals. In 1952, Duffin and Schaeffer [2] abstracted Gabor's method to define frames in Hilbert spaces. Frame was reintroduced by Daubechies et al. [3] in 1986. Today, frame theory is a central tool in many areas such as characterizing function spaces and signal analysis. We refer to [4–10] for an introduction to frame theory and its applications. The following are the standard definitions on frames in Hilbert spaces. A sequence $\{f_i\}_{i\in N}$ of elements of a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \le \sum_{i\in N} |\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{1.1}$$

The numbers A, B are called the *lower* (resp., *upper*) frame bounds. The frame is a *tight frame* if $A = B$ and a *normalized tight frame* if $A = B = 1$.

In [11], Sun raised the concept of g-frame as follows, which generalized the concept of frame extensively. A sequence $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is called a g-frame for H with respect to $\{H_i : i \in N\}$, which is a sequence of closed subspaces of a Hilbert space V, if there exist two positive constants A and B such that for any $f \in H$

$$A\|f\|^2 \leq \sum_{i\in N} \|\Lambda_i f\|^2 \leq B\|f\|^2. \tag{1.2}$$

We simply call $\{\Lambda_i : i \in N\}$ a g-frame for H whenever the space sequence $\{H_i : i \in N\}$ is clear. The tight g-frame, normalized tight g-frame, g-Riesz basis are defined similarly. We call $\{\Lambda_i : i \in N\}$ a *g-frame sequence*, if it is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i\in N}$. We call $\{\Lambda_i : i \in N\}$ a g-Bessel *sequence*, if only the right inequality is satisfied. Recently, g-frames in Hilbert spaces have been studied intensively; for more details see [12–17] and the references therein.

It is well known that frames are generalizations of bases in Hilbert spaces. So it is natural to view g-frames as generalizations of the so-called g-bases in Hilbert spaces, which will be defined in the following section. And that is the main object which will be studied in this paper. In Section 2, we will give the definitions and lemmas. In Section 3, we characterize the g-bases. In Section 4, we discuss the equivalent relations of g-bases and g-orthonormal bases. In Section 5, we study the property of g-minimal of g-bases. Throughout this paper, we use N to denote the set of all natural numbers, Z to denote the set of all integer numbers, and C to denote the field of complex numbers. The sequence of $\{H_j : j \in N\}$ always means a sequence of closed subspace of some Hilbert space V.

2. Definitions and Lemmas

In this section, we introduce the definitions and lemmas which will be needed in this paper.

Definition 2.1. For each Hilbert space sequence $\{H_i\}_{i\in N}$, we define the space $l^2(\oplus H_i)$ by

$$l^2(\oplus H_i) = \left\{ \{f_i\}_{i\in N} : f_i \in H_i,\ i \in N,\ \sum_{i=1}^{+\infty} \|f_i\|^2 < +\infty \right\}. \tag{2.1}$$

With the inner product defined by $\langle \{f_i\}, \{g_i\} \rangle = \sum_{i=1}^{+\infty} \langle f_i, g_i \rangle$, it is easy to see that $l^2(\oplus H_i)$ is a Hilbert space.

Definition 2.2. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ is called *g-complete* with respect to $\{H_j\}$ if $\{f : \Lambda_j f = 0, \text{ for all } j\} = \{0\}$.

Definition 2.3. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ is called *g-linearly independent* with respect to $\{H_j\}$ if $\sum_{j=1}^{\infty} \Lambda_j^* g_j = 0$, then $g_j = 0$, where $g_j \in H_j$ $(j = 1, 2, \ldots)$.

Definition 2.4. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ is called *g-minimal* with respect to $\{H_j\}$ if for any sequence $\{g_j : j \in N\}$ with $g_j \in H_j$ and any $m \in N$ with $g_m \neq 0$, one has $\Lambda_m^* g_m \notin \overline{\text{span}}_{i\neq m}\{\Lambda_i^* g_i\}$.

Definition 2.5. $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ and $\{\Gamma_j \in B(H, H_j)\}_{j=1}^{\infty}$ are called *g-biorthonormal* with respect to $\{H_j\}$, if

$$\left\langle \Lambda_j^* g_j, \Gamma_i^* g_i \right\rangle = \delta_{j,i} \langle g_j, g_i \rangle, \quad \forall j, i \in N,\ g_j \in H_j,\ g_i \in H_i. \tag{2.2}$$

Definition 2.6. We say $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ is *g-orthonormal basis* for H with respect to $\{H_j\}$, if it is g-biorthonormal with itself and for any $f \in H$ one has

$$\sum_{j \in N} \|\Lambda_j f\|^2 = \|f\|^2. \tag{2.3}$$

Definition 2.7. We call $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ a *g-basis* for H with respect to $\{H_j\}$ if for any $x \in H$ there is a unique sequence $\{g_j\}$ with $g_j \in H_j$ such that $x = \sum_{j=1}^{\infty} \Lambda_j^* g_j$.

The following result is about pesudoinverse, which plays an important role in some proofs.

Lemma 2.8 (see [5]). *Suppose that $T : K \to H$ is a bounded surjective operator. Then there exists a bounded operator (called the pseudoinverse of T) $T^{\dagger} : H \to K$ for which*

$$TT^{\dagger} f = f, \quad \forall f \in H. \tag{2.4}$$

The following lemmas characterize g-frame sequence and g-Bessel sequence in terms of synthesis operators.

Lemma 2.9 (see [14]). *A sequence $\{\Lambda_j : j \in N\}$ is a g-frame sequence for H with respect to $\{H_j : j \in N\}$ if and only if*

$$Q : \{g_j : j \in N\} \longrightarrow \sum_{j \in N} \Lambda_j^* g_j \tag{2.5}$$

is a well-defined bounded linear operator from $l^2(\oplus H_j)$ into H with closed range.

Lemma 2.10 (see [12]). *A sequence $\{\Lambda_j : j \in N\}$ is a g-Bessel sequence for H with respect to $\{H_j : j \in N\}$ if and only if*

$$Q : \{g_j : j \in N\} \longrightarrow \sum_{j \in N} \Lambda_j^* g_j \tag{2.6}$$

is a well-defined bounded linear operator from $l^2(\oplus H_j)$ into H.

The following is a simple property about g-basis, which gives a necessary condition for g-basis in terms of g-complete and g-linearly independent.

Lemma 2.11. *If $\{\Lambda_j : j \in N\}$ is a g-basis for H with respect to $\{H_j : j \in N\}$, then $\{\Lambda_j : j \in N\}$ is g-complete and g-linearly independent with respect to $\{H_j : j \in N\}$.*

Proof. Suppose $\Lambda_i f = 0$ for each i. Then for each $g_i \in H_i$, we have $\langle \Lambda_i f, g_i \rangle = \langle f, \Lambda_i^* g_i \rangle = 0$. Hence $f \perp \text{span}\{\Lambda_i(Hi) : i \in N\}$. Therefore $f \perp \overline{\text{span}}\{\Lambda_i(H_i) : i \in N\} = H$. So $f = 0$. So $\{\Lambda_i : i \in N\}$ is g-complete. Now suppose $\sum_{i=1}^{+\infty} \Lambda_i^* g_i = 0$. Since $\sum_{i=1}^{+\infty} \Lambda_i^* 0 = 0$ and $\{\Lambda_i : i \in N\}$ is a *g-basis*, so $g_i = 0$ for each i. Hence $\{\Lambda_i : i \in N\}$ is g-linearly independent. □

The following remark tells us that g-basis is indeed a generalization of Schauder basis of Hilbert space.

Remark 2.12. If $\{x_i\}_{i\in N}$ is a Schauder basis of Hilbert space H, then it induces a g-basis $\{\Lambda_{x_i} : i \in N\}$ of H with respect to the complex number field C, where Λ_{x_i} is defined by $\Lambda_{x_i} f = \langle f, x_i\rangle$. In fact, it is easy to see that $\Lambda^*_{x_i} c = c \cdot x_i$ for any $c \in C$, so for any $x \in H$, there exists a unique sequence of constants $\{a_n : n \in N\}$ such that $x = \sum_{i\in N} a_i x_i = \sum_{i\in N} \Lambda^*_i a_i$.

Definition 2.13. Suppose $\{\Lambda_j : j \in N\}$ is a g-Riesz basis of H with respect to $\{H_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ is a g-Riesz basis of Y with respect to $\{H_j : j \in N\}$. If there is a homomorphism $S : H \to Y$ such that $\Gamma_j = \Lambda_j S^*$ for each $j \in N$, then we say that $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are *equivalent*.

Definition 2.14. If $\{\Lambda_j\}$ is a g-basis of H with respect to $\{H_j\}$, then for any $x \in H$, there exists a unique sequence $\{g_j : j \in N\}$ such that $g_j \in H_j$ and $x = \sum_{j=1}^{\infty} \Lambda^*_j g_j$. We define a map $\Gamma_j : H \to H_j$, by $\Lambda_j x = g_j$ for each j. Then $\{\Gamma_j\}$ is well defined. We call it the *dual sequence* of $\{\Lambda_j\}$, in case that $\{\Gamma_j\}$ is also a g-basis, we call it the *dual g-basis* of $\{\Lambda_j\}$.

The following results link g-Riesz basis with g-basis.

Lemma 2.15 (see [11]). *A g-Riesz basis $\{\Lambda_j : j \in N\}$ is an exact g-frame. Moreover, it is g-biorthonormal with respect to its dual $\{\widetilde{\Lambda}_j : j \in N\}$.*

Lemma 2.16. *Let $\Lambda_j \in B(H, Hj)$, $j \in N$. Then the following statements are equivalent.*

(1) *The sequence $\{\Lambda_j\}_{j\in N}$ is a g-Riesz basis for H with respect to $\{H_j\}_{j\in N}$.*

(2) *The sequence $\{\Lambda_j\}_{j\in N}$ is a g-frame for H with respect to $\{H_j\}_{j\in N}$ and $\{\Lambda_j\}_{j\in N}$ is g-linearly independent.*

(3) *The sequence $\{\Lambda_j\}_{j\in N}$ is a g-basis and a g-frame with respect to $\{H_j\}_{j\in N}$.*

Proof. The equivalent between statements (1) and (2) is shown in Theorem 2.8 of [12]. By Lemma 2.11, we know that if $\{\Lambda_j\}_{j\in N}$ is a g-basis, then it is g-linearly independent, so (3) implies (2). If $\{\Lambda_j\}_{j\in N}$ is a g-frame for H, then for every $x \in H$, $x = \sum_{j\in N} \Lambda^*_j \widetilde{\Lambda}_j x$, where $\{\widetilde{\Lambda}_j\}_{j\in N}$ is the canonical dual *g-frame* of $\{\Lambda_j\}_{j\in N}$. Hence for every $x \in H$, there exists a sequence $\{g_j : j \in N\}$, $g_j \in H_j$, such that $x = \sum_{j\in N} \Lambda^*_j g_j$. Since $\{\Lambda_j\}_{j\in N}$ is g-linearly independent, the sequence is unique. Hence $\{\Lambda_j\}_{j\in N}$ is a g-basis for H. So (2) implies (3). □

From Lemma 2.16, it is easy to get the following well-known result, which is proved more directly.

Corollary 2.17. *Suppose $\{\Lambda_j\}_{j\in N}$ is a g-Riesz basis for H, then $\{\Lambda_j\}_{j\in N}$ has a unique dual g-frame.*

Proof. It has been shown that every g-frame has a dual g-frame in [11], so it suffices to show the uniqueness of dual g-frame for g-Riesz bases. Suppose $\{\Gamma_j : j \in N\}$ and $\{\eta_j : j \in N\}$ are dual g-frames of $\{\Lambda_j\}_{j\in N}$. Then for every $x \in H$, we have $x = \sum_{j\in N} \Lambda^*_j \Gamma_j x = \sum_{j\in N} \Lambda^*_j \eta_j x$. Hence $\sum_{j\in N} \Lambda^*_j (\Gamma_j - \eta_j) x = 0$. But $\{\Lambda_j\}_{j\in N}$ is g-linearly independent by Lemma 2.16, so $(\Gamma_j - \eta_j) x = 0$, that is, $\Gamma_j x = \eta_j x$ for each $j \in N$. Thus, $\Gamma_j = \eta_j$ for each $j \in N$, which implies that the dual g-frame of $\{\Lambda_j\}_{j\in N}$ is unique. □

The following lemma generalizes the similar result in frames to g-frames.

Lemma 2.18. *Suppose $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are both g-Bessel sequences for H with respect to $\{H_j : j \in N\}$. Then the following statements are equivalent.*

(1) *For any $x \in H$, $x = \sum_{j\in N} \Lambda_j^*\Gamma_j x$.*

(2) *For any $x \in H$, $x = \sum_{j\in N} \Gamma_j^*\Lambda_j x$.*

(3) *For any $x, y \in H$, $\langle x, y\rangle = \sum_{j\in N}\langle \Lambda_j x, \Gamma_j x\rangle$.*

Moreover, any of the above statements implies that $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are dual g-frames for each other.

Proof. (1) ⇒ (2): Since $\{\Lambda_j : j \in N\}$ is a g-Bessel sequence, $\{\Lambda_j x\}_{j\in N} \in l^2(\oplus H_j)$ for any $x \in H$. Since $\{\Gamma_j : j \in N\}$ is a g-Bessel sequence, the series $\sum_{j\in N} \Gamma_j^*\Lambda_j x$ is convergent by Lemma 2.10. Let $\tilde{x} = \sum_{j\in N} \Gamma_j^*\Lambda_j x$. Then for any $y \in H$, we have

$$\begin{aligned}\langle x, y\rangle &= \left\langle x, \sum_{j\in N}\Lambda_j^*\Gamma_j y\right\rangle \\ &= \sum_{j\in N}\left\langle x, \Lambda_j^*\Gamma_j y\right\rangle = \sum_{j\in N}\left\langle \Gamma_j^*\Lambda_j x, y\right\rangle \\ &= \left\langle \sum_{j\in N}\Gamma_j^*\Lambda_j x, y\right\rangle = \langle \tilde{x}, y\rangle.\end{aligned} \tag{2.7}$$

So $x = \tilde{x}$, that is, (2) is established.

(2) ⇒ (3): Since for any $x \in H$, we have $x = \sum_{j\in N} \Gamma_j^*\Lambda_j x$; hence for any $x, y \in H$,

$$\langle x, y\rangle = \left\langle \sum_{j\in N}\Gamma_j^*\Lambda_j x, y\right\rangle = \sum_{j\in N}\left\langle \Gamma_j^*\Lambda_j x, y\right\rangle = \sum_{j\in N}\langle \Lambda_j x, \Gamma_j x\rangle. \tag{2.8}$$

(3) ⇒ (1): From the proof of (1) ⇒ (2), we know that for any $x \in H$, $\sum_{j\in N} \Gamma_j^*\Lambda_j x$ is convergent. Let $\tilde{x} = \sum_{j\in N} \Gamma_j^*\Lambda_j x$, then for any $y \in H$, we have

$$\langle y, x\rangle = \sum_{j\in N}\langle \Lambda_j y, \Gamma_j x\rangle = \langle y, \sum_{j\in N} \Gamma_j^*\Lambda_j x\rangle = \langle y, \tilde{x}\rangle. \tag{2.9}$$

So $x = \tilde{x}$, that is, (1) is true.

If any one of the three statements is true, then for any x H, we have

$$\begin{aligned}\|x\|^2 &= \left\langle \sum_{j\in N} \Lambda_j^* \Gamma_j x, x \right\rangle = \sum_{j\in N} \langle \Lambda_j x, \Gamma_j x\rangle \\ &\leq \sum_{j\in N} \|\Lambda_j x\| \|\Gamma_j x\| \leq \left(\sum_{j\in N} \|\Lambda_j x\|^2 \right)^{1/2} \left(\sum_{j\in N} \|\Gamma_j x\|^2 \right)^{1/2} \\ &\leq \left(B_1 \|x\|^2\right)^{1/2} \left(\sum_{j\in N} \|\Gamma_j x\|^2 \right)^{1/2},\end{aligned} \tag{2.10}$$

where B_1 is the bound for the g-Bessel sequence $\{\Lambda_j : j \in N\}$. So

$$\sum_{j\in N} \|\Gamma_j x\|^2 \geq \frac{1}{B_1} \|x\|^2, \tag{2.11}$$

which implies that the g-Bessel sequence $\{\Gamma_j : j \in N\}$ is a g-frame. Similarly, $\{\Lambda_j : j \in N\}$ is also a g-frame. And that they are dual g-frames for each other is obvious by the equality that for any $x \in H$, $x = \sum_{j\in N} \Lambda_j^* \Gamma_j x$. □

3. Characterizations of g-Bases

In this section, we characterized g-bases.

Theorem 3.1. *Suppose that $\{\Lambda_j \in B(H, H_j)\}_{j=1}^{\infty}$ is a g-frame sequence with respect to $\{H_j\}$ and it is g-linearly independent with respect to $\{H_j\}$. Let $Y = \{\{g_j\}_{j=1}^{\infty} \mid g_j \in H_j, \sum_{j=1}^{\infty} \Lambda_j^* g_j$ is convergent$\}$. If for any $\{g_j\} \in Y$, set $\|\{g_j\}\|_Y = \mathrm{Sup}_N \| \sum_{j=1}^{N} g_j \|$, then*

(1) *Y is a Banach space,*

(2) *when $\{\Lambda_j\}$ is a g-basis with respect to $\{H_j\}$ as well, $S : Y \to H, S(\{g_j\}) = \sum_{j=1}^{\infty} \Lambda_j^* g_j$ is a linear bounded and invertible operator, that is, S is a homeomorphism between H and Y.*

Proof. (1) Let $\{g_j\} \in Y$, then $\sum_{j=1}^{N} \Lambda_j^* g_j$ is convergent as $N \to \infty$. Hence $\{\sum_{j=1}^{N} \Lambda_j^* g_j\}_{N=1}^{\infty}$ is a convergent sequence, so it is bounded. So $\|\{g_j\}\|_Y < \infty$. It is obvious that for $a \in C$, $\{g_j\}$, $\{h_j\} \in Y$, we have $\|\{g_j\} + \{h_j\}\|_Y \leq \|\{g_j\}\|_Y + \|\{h_j\}\|_Y$ and $\|a \cdot \{g_j\}\|_Y = |a| \cdot \|\{g_j\}\|_Y$. If $\|\{g_j\}\|_Y = 0$, then for any N, $\| \sum_{j=1}^{N} \Lambda_j^* g_j \| = 0$, which implies that $\sum_{j=1}^{N} \Lambda_j^* g_j = 0$. Since $\{\Lambda_j\}$ is g-linearly independent with respect to $\{H_j\}$, we get that $g_j = 0$, for $j = 1, 2, \ldots, N$. Since

N is arbitrary, so $\{g_j\} = \{0\}$. Thus $\|\cdot\|_Y$ is a norm on Y. Suppose $\{G^k\}_{k=1}^{\infty} \in Y$ is a Cauchy sequence, where $G^k = \{g_j^k\}_{j=1}^{\infty}$. Then

$$\lim_{k,l\to\infty}\left\|G^k - G^l\right\|_Y = \lim_{k,l\to\infty}\left\|\left\{g_j^k - g_j^l\right\}\right\|_Y, \tag{3.1}$$

$$\lim_{k,l\to\infty}\operatorname*{Sup}_N\left\|\sum_{j=1}^{N}\Lambda_j^*\left(g_j^k - g_j^l\right)\right\| = 0. \tag{3.2}$$

For any fixed j, we have

$$\begin{aligned}\left\|\Lambda_j^*\left(g_j^k - g_j^l\right)\right\| &= \left\|\sum_{t=1}^{j}\Lambda_t^*\left(g_t^k - g_t^l\right) - \sum_{t=1}^{j-1}\Lambda_t^*\left(g_t^k - g_t^l\right)\right\| \\ &\le \left\|\sum_{t=1}^{j}\Lambda_t^*\left(g_t^k - g_t^l\right)\right\| + \left\|\sum_{t=1}^{j-1}\Lambda_t^*\left(g_t^k - g_t^l\right)\right\| \le 2\left\|G^k - G^l\right\|_Y.\end{aligned} \tag{3.3}$$

Now let $T : l^2(\oplus H_j) \to H, T(\{g_j\}) = \sum_{j=1}^{\infty}\Lambda_j^* g_j$. Since $\{\Lambda_j\}$ is a g-frame sequence with respect to $\{H_j\}$, T is a well-defined linear bounded operator with closed range by Lemma 2.9. Since $\{\Lambda_j\}$ is g-linearly independent with respect to $\{H_j\}$, T is injective. Hence $T^* : H \to l^2(\oplus H_j)$ is surjective. So by Lemma 2.8, there is a bounded operator L, the pseudoinverse of T^*, such that $T^*L = I_{l^2(\oplus H_j)}$, which implies that $L^*T = I_{l^2(\oplus H_j)}$. Let $\{\delta_j\}$ denote the canonical basis of $l^2(N)$, then for any $g_j \in H_j$, $\{g_j\delta_j\} \in l^2(\oplus H_j)$. So $\{g_j\delta_j\} = L^*T(\{g_j\delta_j\}) = L^*(\Lambda_j^* g_j)$; hence

$$\|g_j\| = \|\{g_j\delta_j\}\| = \left\|L^*\left(\Lambda_j^* g_j\right)\right\| \le \|L\|\left\|\Lambda_j^* g_j\right\|. \tag{3.4}$$

So by inequalities (3.3), we get

$$\left\|\left(g_j^k - g_j^l\right)\right\| \le \|L\|\left\|\Lambda_j^*\left(g_j^k - g_j^l\right)\right\| \le 2\|L\|\left\|G^k - G^l\right\|_Y. \tag{3.5}$$

So for any fixed j, $\{g_j^k\}_{j=1}^{\infty}$ is a Cauchy sequence. Suppose $\lim_{k\to\infty} g_j^k = g_j$. From (3.2), we know that, for any $\varepsilon > 0$, there exists $L_0 > 0$, such that whenever $k, l \ge L_0$, we have

$$\operatorname*{Sup}_Q\left\|\sum_{j=1}^{Q}\Lambda_j^*\left(g_j^k - g_j^l\right)\right\| < \varepsilon. \tag{3.6}$$

Fix $l \ge L_0$, since $\lim_{k\to\infty} g_j^k = g_j$, so whenever $l \ge L_0$,

$$\operatorname*{Sup}_Q\left\|\sum_{j=1}^{Q}\Lambda_j^*\left(g_j - g_j^l\right)\right\| \le \varepsilon. \tag{3.7}$$

Since $G^{L_0} = \{g_j^{L_0}\}_{j=1}^{\infty} \in Y$, $\sum_{j=1}^{\infty} \Lambda_j^* g_j^{L_0}$ is convergent. So there exists $K_0 > 0$, such that whenever $M > P \geq K_0$, we have $\| \sum_{j=P+1}^{M} \Lambda_j^* g_j^{L_0} \| < \varepsilon$. So when $M > P > \max\{L_0, K_0\}$, we have

$$\begin{aligned}\left\| \sum_{j=P+1}^{M} \Lambda_j^* g_j \right\| &= \left\| \sum_{j=1}^{M} \Lambda_j^* \left(g_j - g_j^{L_0} \right) - \sum_{j=1}^{P} \Lambda_j^* \left(g_j - g_j^{L_0} \right) + \sum_{j=P+1}^{M} \Lambda_j^* g_j^{L_0} \right\| \\ &\leq \left\| \sum_{j=1}^{M} \Lambda_j^* \left(g_j - g_j^{L_0} \right) \right\| + \left\| \sum_{j=1}^{P} \Lambda_j^* \left(g_j - g_j^{L_0} \right) \right\| + \left\| \sum_{j=P+1}^{M} \Lambda_j^* g_j^{L_0} \right\| \leq 3\varepsilon. \end{aligned} \tag{3.8}$$

So $\sum_{j=1}^{\infty} \Lambda_j^* g_j$ is convergent, thus $G = \{g_j\} \in Y$. Let $l \to \infty$ in (3.7), we get that $\|G_l - G\|_Y \to 0$. Hence Y is a complete normed space, that is, Y is a Banach space.

(2) If $\{\Lambda_j\}$ is a g-basis, then it is g-complete and g-linearly independent with respect to $\{H_j\}$ by the Lemma 2.11, then the operator $S : Y \to H$, $S(\{g_j\}) = \sum_{j=1}^{\infty} \Lambda_j^* g_j$ not only is well defined but also is one to one and onto. And for any $\{g_j\} \in Y$, we have

$$\begin{aligned}\|S(\{g_j\})\| &= \left\| \sum_{j=1}^{\infty} \Lambda_j^* g_j \right\| = \lim_{N \to \infty} \left\| \sum_{j=1}^{N} \Lambda_j^* g_j \right\| \\ &\leq \operatorname{Sup}_{Q} \left\| \sum_{j=1}^{Q} \Lambda_j^* g_j \right\| = \|\{g_j\}\|_Y. \end{aligned} \tag{3.9}$$

So S is bounded operator. Since Y is a Banach space, by the Open Mapping Theorem, we get that S is a homeomorphism. □

Theorem 3.2. *Suppose $\{\Lambda_j : j \in N\}$ is a g-basis of H with respect to $\{H_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ is its dual sequence. If $\{\Lambda_j : j \in N\}$ is also a g-frame sequence of H with respect to $\{H_j : j \in N\}$, then*

(1) $\forall x \in H$, *let* $S_N x = \sum_{j=1}^{N} \Lambda_j^* \Gamma_j x$, *then* $\operatorname{Sup}_N \|S_N x\| < \infty$,

(2) $C = \operatorname{Sup}_N \|S_N\| < \infty$,

(3) $|||x||| = \operatorname{Sup}_N \|S_N x\|$ *is a norm on H and* $\| \cdot \| \leq ||| \cdot ||| \leq C \| \cdot \|$.

Proof. (1) Let Y and S be as defined in Theorem 3.1. Then for any $x \in H$, $S^{-1}x = \{\Gamma_j x : j \in N\}$. So

$$\begin{aligned}\operatorname{Sup}_{N} \|S_N x\| &= \operatorname{Sup}_{N} \left\| \sum_{j=1}^{N} \Lambda_j^* \Gamma_j x \right\| = \|\{\Gamma_j x\}\|_Y \\ &= \left\| S^{-1} x \right\| \leq \left\| S^{-1} \right\| \|x\| < \infty. \end{aligned} \tag{3.10}$$

(2) Since $\|S_N x\| \leq \operatorname{Sup}_Q \|S_Q x\| \leq \|S^{-1}\| \|x\|$, $\|S_N\| \leq \|S^{-1}\|$. Thus $C = \operatorname{Sup}_N \|S_N\| < \infty$.

(3) It is obvious that $|||\cdot|||$ is a seminorm. It is sufficient to show that $\|\cdot\| \le |||\cdot||| \le C\|\cdot\|$. For any $x \in H$, we have

$$|||x||| = \operatorname*{Sup}_{N}\|S_N x\| \le \operatorname*{Sup}_{N}\|S_N\|\|x\| = C\|x\|. \tag{3.11}$$

On the other hand,

$$\begin{aligned}\|x\| &= \left\|\sum_{j=1}^{\infty}\Lambda_j^*\Gamma_j x\right\| = \lim_{N\to\infty}\left\|\sum_{j=1}^{N}\Lambda_j^*\Gamma_j x\right\| \\ &\le \operatorname*{Sup}_{Q}\left\|\sum_{j=1}^{Q}\Lambda_j^*\Gamma_j x\right\| = \operatorname*{Sup}_{Q}\|S_Q x\| = |||x|||.\end{aligned} \tag{3.12}$$

□

Theorem 3.3. *Suppose $\{\Lambda_j \in B(H,H_j) : j \in N\}$ is a g-frame with respect to $\{H_j : j \in N\}$. Then $\{\Lambda_j : j \in N\}$ is a g basis with respect to $\{H_j : j \in N\}$ if and only if there exists a constant C such that for any $g_j \in H_j$, any $m,n \in N$ and $m \le n$, one has*

$$\left\|\sum_{j=1}^{m}\Lambda_j^* g_j\right\| \le C\cdot\left\|\sum_{j=1}^{n}\Lambda_j^* g_j\right\|. \tag{3.13}$$

Proof. ⇒: Suppose $\{\Lambda_j : j \in N\}$ is a g-basis with respect to $\{H_j : j \in N\}$. Then for any $x \in H$, there exists a unique sequence $\{g_j : j \in N\}$ with $g_j \in H_j$ for each $j \in N$ such that $x = \sum_{j=1}^{\infty}\Lambda_j^* g_j$. Let $|||x||| = \operatorname{Sup}_n\|\sum_{j=1}^{n}\Lambda_j^* g_j\|$. Then by Theorem 3.2, $|||\cdot|||$ is a norm on H and it is equivalent to $\|\cdot\|$. So there exists a constant C such that, for any $x \in H$, $|||x||| \le C\cdot\|x\|$. Hence for any $n \in N$, any $g_j \in H_j$, $j = 1,2,\dots,n$, we choose $x = \sum_{j=1}^{n}\Lambda_j^* g_j$, then for any $m \le n$, we have

$$\left\|\sum_{j=1}^{m}\Lambda_j^* g_j\right\| \le C\cdot\left\|\sum_{j=1}^{n}\Lambda_j^* g_j\right\|. \tag{3.14}$$

⇐: Let $\mathbb{A} = \{\sum_{k\in N}\Lambda_k^* g_k, g_k \in H_k,$ and $\sum_{k\in N}\Lambda_k^* g_k$ is *covergent*$\}$. First, we show that $\mathbb{A} = H$. Since $\{\Lambda_j \in B(H,H_j) : j \in N\} \subset B(H,H_j)$ is a g-frame, $\mathbb{A}$ is dense in H. It is sufficient to show that $\mathbb{A}$ is closed. Suppose $\{y_k\} \subset \mathbb{A}$ and $\lim_{k\to\infty}y_k = y$. Denote $y_k = \sum_{j\in N}\Lambda_j^* g_j^{(k)}$. Then for any $j \in N$ and any $n \le m \le j$, we have, for any $k,l \in N$,

$$\begin{aligned}\left\|\Lambda_j^* g_j^{(k)} - \Lambda_j^* g_j^{(l)}\right\| &\le 2C \cdot \left\|\sum_{s=1}^{m} \Lambda_s^*\left(g_s^{(k)} - g_s^{(l)}\right)\right\| \\ &\le 2C^2 \cdot \left\|\sum_{s=1}^{n} \Lambda_s^*\left(g_s^{(k)} - g_s^{(l)}\right)\right\| \\ &\le 2C^2 \cdot \left(\left\|\sum_{s=1}^{n} \Lambda_j^* g_j^{(k)} - y_k\right\| + \|y_k - y\|\right) + 2C^2 \cdot \left(\|y - y_l\| + \left\|y_l - \sum_{j=1}^{n} \Lambda_s^* g_s^{(l)}\right\|\right).\end{aligned} \tag{3.15}$$

Since $\lim_{k\to\infty} y_k = y$, so for any $\varepsilon > 0$, there exists $M > 0$, such that whenever $k \ge M$, we have $\|y - y_k\| \le \varepsilon/2C^2$. In the above inequality, let $n \to \infty$, we get

$$\begin{gathered}\left\|\Lambda_j^* g_j^{(k)} - \Lambda_j^* g_j^{(l)}\right\| \le \varepsilon \quad \text{for any } j \in N \text{ and } k, l \ge M, \\ \left\|\sum_{s=1}^{m} \Lambda_s^*\left(g_s^{(k)} - g_s^{(l)}\right)\right\| \le \frac{\varepsilon}{2C} \quad \text{for any } m \in N \text{ and any } k, l \ge M.\end{gathered} \tag{3.16}$$

Since $\{\Lambda_j : j \in N\}$ is a g-frame sequence, by inequality (3.4), we have that $\|g_j^{(k)} - g_j^{(l)}\| \le \|L\|\|\Lambda_j^*(g_j^{(k)} - g_j^{(l)})\| \le \|L\|\varepsilon$ for any $j \in N$ and any $k, l \ge M$. So $\{g_j^{(k)}\}_{k\in N}$ is convergent for each $j \in N$. Suppose $\lim_{k\to\infty} g_j^{(k)} = g_j$. Then

$$\begin{gathered}\left\|\Lambda_j^*\left(g_j^{(k)} - g_j\right)\right\| \le \varepsilon \quad \text{for any } j \in N \text{ and } k \ge M, \\ \left\|\sum_{s=1}^{m} \Lambda_s^*\left(g_s^{(k)} - g_s\right)\right\| \le \frac{\varepsilon}{2C} \quad \text{for any } m \in N \text{ and any } k \ge M.\end{gathered} \tag{3.17}$$

Since

$$\left\|y - \sum_{j=1}^{m} \Lambda_j^* g_j\right\| \le \|y - y_k\| + \left\|y_k - \sum_{j=1}^{m} \Lambda_j^* g_j^{(k)}\right\| + \left\|\sum_{j=1}^{m} \Lambda_j^* g_j^{(k)} - \sum_{j=1}^{m} \Lambda_j^* g_j\right\|, \tag{3.18}$$

so $\sum_{j=1}^{m} \Lambda_j^* g_j$ converges to y, which implies that $y \in \mathbb{A}$. Thus $\mathbb{A}$ is a closed set. Now we will show that $\{\Lambda_j : j \in N\}$ is g-linearly independent. Suppose that $\sum_{j\in N} \Lambda_j^* g_j = 0$, where $g_j \in H_j$ for each $j \in N$. Since for any $n \in N$ and any $j \le n$, we have $\|\Lambda_j^* g_j\| \le C \cdot \|\sum_{s=1}^{n} \Lambda_s^* g_s\|$, hence $\|\Lambda_j^* g_j\| = 0$ for any $j \le n$. But from inequality (3.4), we have $\|g_j\| \le \|L\|\|\Lambda_j^* g_j\|$. So for each $j \le n$, $g_j = 0$. Since n is arbitrary, $g_j = 0$ for any $j \in N$. Thus $\{\Lambda_j : j \in N\}$ is g-linearly independent. So $\{\Lambda_j : j \in N\}$ is a g-basis. □

4. Equivalent Relations of g-Bases

In this section, the equivalent relations of g-bases were discussed.

Theorem 4.1. *Suppose that $\{\Lambda_j : j \in N\}$ is a g-basis of H with respect to $\{H_j : j \in N\}$, and $S : H \to Y$ is a homeomorphism. Then $\{\Lambda_j S^* : j \in N\}$ is a g-basis of Y with respect to $\{H_j\}$.*

Proof. For any $y \in Y$, $S^{-1}y \in H$. Since $\{\Lambda_j : j \in N\}$ is a g-basis of H, there exists a unique sequence $\{g_j : j \in N\}$ and $g_j \in H_j$ for each $j \in N$ such that $S^{-1}y = \sum_{j\in N} \Lambda_j^* g_j$. So $y = \sum_{j\in N} S\Lambda_j^* g_j = \sum_{j\in N} (\Lambda_j S^*)^* g_j$. Suppose there is another sequence $\{h_j : j \in N\}$ and $h_j \in H_j$ for each $j \in N$ such that $y = \sum_{j\in N} (\Lambda_j S^*)^* h_j$, then $y = \sum_{j\in N} S\Lambda_j^* h_j$. So $S^{-1}y = \sum_{j\in N} \Lambda_j^* h_j$. But the expansion for $S^{-1}y$ is unique, so $h_j = g_j$ for each $j \in N$. Hence $\{\Lambda_j S^*\}$ is a g-basis of Y with respect to $\{H_j\}$. □

Theorem 4.2. *Suppose $\{\Lambda_j : j \in N\}$ is a g-basis of Hilbert space H with respect to $\{H_j : j \in N\}$, $\{\Gamma_j : j \in N\}$ is a g-basis of Hilbert space Y with respect to $\{H_j : j \in N\}$, and $\{G_j : j \in N\}$, and $\{L_j : j \in N\}$ are dual sequences of $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$, respectively. If $\{G_j : j \in N\}$ or $\{L_j : j \in N\}$ is a g-basis, then the following statements are equivalent.*

(1) *$\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are equivalent.*

(2) *$\sum_{j\in N} \Lambda_j^* g_j$ is convergent if and only if $\sum_{j\in N} \Gamma_j^* g_j$ is convergent, where $g_j \in H_j$ for each $j \in N$.*

Moreover, any one of the above statements implies that both $\{G_j : j \in N\}$ and $\{L_j : j \in N\}$ are g-bases and they are also equivalent.

Proof. (1) ⇒ (2): Suppose there is an invertible bounded operator $S : H \to Y$ such that $\Lambda_j = \Gamma_j S^*$ for each $j \in N$ and $\sum_{j\in N} \Lambda_j^* g_j$ is convergent. Then $\sum_{j\in N} S\Gamma_j^* g_j$ is convergent. So $S^{-1}(\sum_{j\in N} S\Gamma_j^* g_j) = \sum_{j\in N} \Gamma_j^* g_j$ is convergent. Conversely, if $\sum_{j\in N} \Gamma_j^* g_j$ is convergent, then $\sum_{j\in N} S^{-1}\Lambda_j^* g_j$ is convergent. So $S(\sum_{j\in N} S^{-1}\Lambda_j^* g_j) = \sum_{j\in N} \Lambda_j^* g_j$ is convergent.

(2) ⇒ (1): Without loss of generality, suppose $\{G_j : j \in N\}$ is a g-basis of H. Then for any $x \in H$, we have $x = \sum_{j\in N} \Lambda_j^* G_j x$, which is convergent in H, so $\sum_{j\in N} \Gamma_j^* G_j x$ is convergent in Y. Define operator $S : H \to Y$ by $Sx = \sum_{j\in N} \Gamma_j^* G_j x$. Then S is well defined and linear. If $Sx = 0$, that is, $\sum_{j\in N} \Gamma_j^* G_j x = 0$, then $G_j x = 0$ for each $j \in N$. So $x = \sum_{j\in N} \Lambda_j^* G_j x = 0$. Hence S is injective. For any $y \in Y$, $y = \sum_{j\in N} \Gamma_j^* L_j y$, which is convergent in Y. Then $\sum_{j\in N} \Lambda_j^* L_j y$ is convergent in H. Suppose $x = \sum_{j\in N} \Lambda_j^* L_j y$, but we know that $x = \sum_{j\in N} \Lambda_j^* G_j x$ and $\{\Lambda_j : j \in N\}$ is a g-basis, so $G_j x = L_j y$ for each $j \in N$. Hence $Sx = \sum_{j\in N} \Gamma_j^* G_j x = \sum_{j\in N} \Gamma_j^* L_j y = y$, which implies that S is surjective. Next, we want to verify that S is bounded.

For any $k \in N$, let $T_k : H \to Y$ be defined by $T_k x = \sum_{j=1}^{k} \Gamma_j^* G_j x$. Then it is obvious that T_k is well defined and linear. Since

$$\|T_k x\| = \left\| \sum_{j=1}^{k} \Gamma_j^* G_j x \right\| \leq \sum_{j=1}^{k} \left\| \Gamma_j^* G_j x \right\| \leq \sum_{j=1}^{k} \left\| \Gamma_j^* \right\| \|G_j\| \|x\|, \tag{4.1}$$

thus T_k is a bounded operator. It is easy to see that for any $x \in H$, $T_k x \to Sx (k \to \infty)$. So for any $\varepsilon > 0$, there exists k_0, such that whenever $k \geq k_0$, we have that $\|T_k x - Sx\| < \varepsilon$. Since for any $k \in N$, we have $\|T_k x\| \leq \|Sx\| + \|T_k x - Sx\|$, so for any $k \in N$, we have

$$\|T_k x\| \leq \operatorname{Sup}\left\{ \left\|T_j x\right\|, \|Sx\| + \varepsilon \mid j = 1, 2, \ldots, k_0 - 1 \right\} < \infty. \tag{4.2}$$

Hence, by the Banach-Steinhaus Theorem, $\mathrm{Sup}_k\|T_k\| < \infty$. Since for any $x \in H$, $T_k x \to Sx$ $(k \to \infty)$, we have

$$\|Sx\| \le \mathrm{Sup}_k\|T_k x\| \le \mathrm{Sup}_k\|T_k\|\|x\|. \tag{4.3}$$

So S is bounded. Hence S is a bounded invertible operator from H onto Y. Since for any $x \in H$, we have $\sum_{j\in N} S\Lambda_j^* G_j x = \sum_{j\in N}\Gamma_j^* G_j x$, that is, $\sum_{j\in N} S\Lambda_j^* G_j = \sum_{j\in N}\Gamma_j^* G_j$, so $\sum_{j\in N} G_j^*\Lambda_j S^* = \sum_{j\in N} G_j^*\Gamma_j$. Hence for any $y \in Y$, we have $\sum_{j\in N} G_j^*\Lambda_j S^* y = \sum_{j\in N} G_j^*\Gamma_j y$. Since $\{G_j : j \in N\}$ is a g-basis, so $\Lambda_j S^* = \Gamma_j$ for each $j \in N$, which implies that $\{\Lambda_j : j \in N\}$ and $\{G_j : j \in N\}$ are equivalent. In the case that any of the two statements is true, then there is an invertible operator $S : H \to Y$ such that $\Gamma_j = \Lambda_j S^*$ for each $j \in N$. Hence, for any $x \in H$,

$$\begin{aligned} x &= \sum_{j\in N}\Gamma_j^* L_j x = \sum_{j\in N} S\Lambda_j^* L_j x \\ &= S\left(\sum_{j\in N}\Lambda_j^* L_j x\right). \end{aligned} \tag{4.4}$$

So $S^{-1}x = \sum_{j\in N}\Lambda_j^* L_j x$. Thus $x = \sum_{j\in N}\Lambda_j^* L_j S x$, but $x = \sum_{j\in N}\Lambda_j^* G_j x$, and $\{\Lambda_j\}$ is a g-basis of H, it follows that $L_j S = G_j$ for each $j \in N$. By Theorem 4.1, we know that $\{L_j : j \in N\}$ is also a g-basis and $\{G_j : j \in N\}$ and $\{L_j : j \in N\}$ are equivalent. □

Theorem 4.3. *Suppose $\{\Lambda_j : j \in N\}$ is a g-orthonormal basis for H with respect to $\{H_j : j \in N\}$. Then $\{\Lambda_j : j \in N\}$ is a g-basis for H with respect to $\{H_j : j \in N\}$ and it is self-dual.*

Proof. By the definition of g-orthonormal bases, we know that $\{\Lambda_j : j \in N\}$ is a normalized tight g-frame. So for any $f \in H$, $f = \sum_{j\in N}\Lambda_j^*\Lambda_j f$. Since for any $i \ne j$ and for any $g_j \in H_j$, $h_i \in H_i$, we have

$$\left\langle \Lambda_i\Lambda_j^* g_j, h_i \right\rangle = \left\langle \Lambda_j^* g_j, \Lambda_i^* h_i \right\rangle = 0, \tag{4.5}$$

hence $\Lambda_i\Lambda_j^* g_j = 0$ for $i \ne j$. For $i = j$ and for any $g_i, h_i \in H_i$, we have

$$\langle \Lambda_i\Lambda_i^* g_i, h_i\rangle = \langle \Lambda_i^* g_i, \Lambda_i^* h_i\rangle = \langle g_i, h_i\rangle, \tag{4.6}$$

so for any $g_i \in H_i$, $\Lambda_i\Lambda_i^* g_i = g_i$. Thus if $f = \sum_{j\in N}\Lambda_j^* g_j$, then for any $i \in N$, we have $\Lambda_i f = \sum_{j\in N}\Lambda_i\Lambda_j^* g_j = \Lambda_i\Lambda_i^* g_i = g_i$. Thus for any $f \in H$, there is a unique sequence $\{g_j : j \in N\}$ such that $g_j \in H_j$ for any $j \in N$ and $f = \sum_{j\in N}\Lambda_j^* g_j$. So $\{\Lambda_j : j \in N\}$ is a g-basis. It is obvious that $\{\Lambda_j : j \in N\}$ is self-dual. □

Theorem 4.4. *Suppose $\{\Lambda_j : j \in N\}$ is a g-orthonormal basis with respect to $\{H_j : j \in N\}$. Then $\sum_{j\in N}\Lambda_j^* g_j$ is convergent if and only if $\{g_j : j \in N\} \in l^2(\oplus H_j)$.*

Proof. Since $\{\Lambda_j : j \in N\}$ is a g-orthonormal basis, by Theorem 4.3, $\{\Lambda_j : j \in N\}$ is a g-basis and a g-frame. So $\{\Lambda_j : j \in N\}$ is a g-Riesz basis. Thus there exist constants $A, B > 0$, such that for any integer n, we have

$$A\sum_{j=1}^{n}\|g_j\|^2 \le \left\|\sum_{j=1}^{n}\Lambda_j^* g_j\right\|^2 \le B\sum_{j=1}^{n}\|g_j\|^2. \tag{4.7}$$

So $\sum_{j\in N}\Lambda_j^* g_j$ is convergent if and only if $\{g_j : j \in N\} \in l^2(\oplus H_j)$. □

From Theorems 4.2, 4.3, and 4.4, the following corollary is obvious.

Corollary 4.5. *Any two g-orthonormal bases are equivalent.*

5. The Property of g-Minimal of g-Bases

In this section, we studied the property of g-minimal of g-bases.

Theorem 5.1. *Suppose $\{\Lambda_j : j \in N\}$ is a g-frame sequence. Then*

(1) *if $\{\Lambda_j : j \in N\}$ is a g-basis, then $\{\Lambda_j : j \in N\}$ is g-minimal;*

(2) *if $\{\Lambda_j : j \in N\}$ is g-minimal, then $\{\Lambda_j : j \in N\}$ is g-linearly independent.*

Proof. (1). Since $\{\Lambda_j\}$ is a g-basis and it is also a g-frame sequence, it is easy to see that $\{\Lambda_j : j \in N\}$ is a g-frame. Hence $\{\Lambda_j : j \in N\}$ is a g-Riesz basis by (3) of Lemma 2.16. Suppose $\{\widetilde{\Lambda}_j : j \in N\}$ is the unique dual g-frame of $\{\Lambda_j : j \in N\}$. By Lemma 2.15, we know that $\{\Lambda_j\}$ and $\{\widetilde{\Lambda}_j : j \in N\}$ are g-biorthonormal, that is, $\langle \Lambda_j^* g_j, \widetilde{\Lambda}_j^* g_i\rangle = \delta_{ij}\langle g_j, g_i\rangle$, where $g_j \in H_j, g_i \in H_i$. For any $m \in N$ and any sequence $\{g_j : j \in N\}$ with $g_j \in H_j$ and $g_m \neq 0$, let $E_m = \overline{\text{span}}_{i\neq m}\{\widetilde{\Lambda}_i^* g_i\}$. Then for any $x \in E_m$, $\langle x, \widetilde{\Lambda}_m^* g_m\rangle = 0$, but $\langle \Lambda_m^* g_m, \widetilde{\Lambda}_m^* g_m\rangle = \langle g_m, g_m\rangle \neq 0$, so $\Lambda_m^* g_m \notin E_m$. Hence $\{\Lambda_j : j \in N\}$ is g-minimal.

(2). Suppose $\{\Lambda_j : j \in N\}$ is g-minimal. If $\sum_{j\in N}\Lambda_j^* g_j = 0$, where $g_j \in H_j$ for each $j \in N$, then $g_j = 0$ for any $j \in N$. In fact, if there exists $m \in N$ such that $g_m \neq 0$, then $\|\Lambda_m^* g_m\| \ge (1/L)\|g_m\| > 0$ by inequality (3.4), which implies that $\Lambda_m^* g_m \neq 0$. Since $\Lambda_m^* g_m = -\sum_{j\neq m}\Lambda_j^* g_j$, $\Lambda_m^* g_m \in \overline{\text{span}}_{j\neq m}\{\Lambda_j^* g_j\}$, which contradicts with the fact that $\{\Lambda_j : j \in N\}$ is g-minimal. □

Theorem 5.2. *Given sequence $\{\Lambda_j \in B(H, H_j) : j \in N\}$.*

(1) *If there exists a sequence $\{\Gamma_j \in B(H, H_j) : j \in N\}$, such that $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are biorthonormal, then $\{\Lambda_j : j \in N\}$ is g-minimal.*

(2) *If there exists a unique sequence $\{\Gamma_j \in B(H, H_j) : j \in N\}$ such that $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are biorthonormal, then $\{\Lambda_j : j \in N\}$ is g-minimal and g-complete.*

Proof. The proof of (1) is similar to the proof of (1) of Theorem 5.1, we omit the details. Now we prove (2): By (1) we know that $\{\Lambda_j : j \in N\}$ is g-minimal. So it only needs to show that $\{\Lambda_j : j \in N\}$ is g-complete. Suppose that $x \in H$ and $\langle x, \Lambda_j^* g_j\rangle = 0$ for any $j \in N$ and any $g_j \in H_j$. Since $\langle \Lambda_j^* g_j, \Gamma_j^* g_i\rangle = \delta_{ij}\langle g_j, g_i\rangle$, so $\langle \Lambda_j^* g_j, x + \Gamma_j g_i\rangle = \delta_{ij}\langle g_j, g_i\rangle$, which implies that $\{\delta_x + \Gamma_j : j \in N\}$ and $\{\Lambda_j : j \in N\}$ are biorthonormal, where $\delta_x^* \in B(V, H_j)$ for each $j \in N$

defined by $\delta_x^*(f) = x$ for any $f \in V$. But it is assumed that there exists a unique sequence $\{\Gamma_j \in B(H, H_j) : j \in N\}$ such that $\{\Lambda_j : j \in N\}$ and $\{\Gamma_j : j \in N\}$ are biorthonormal, so $\delta_x = 0$, hence $x = \delta_x^*(f) = 0$, which implies that $\overline{\text{span}}\{\Lambda_j^* g_j\} = H$. So $\{\Lambda_j : j \in N\}$ is g-complete. □

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