Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 897198, 15 pages
doi:10.1155/2012/897198

Research Article

On Coupled Fixed Point Theorems for Nonlinear Contractions in Partially Ordered G-Metric Spaces

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Received 31 July 2012; Revised 2 October 2012; Accepted 2 October 2012

Academic Editor: Simeon Reich

Two concepts—one of the coupled fixed point and the other of the generalized metric space—play a very active role in recent research on the fixed point theory. The definition of coupled fixed point was introduced by Bhaskar and Lakshmikantham (2006) while the generalized metric space was introduced by Mustafa and Sims (2006). In this work, we determine some coupled fixed point theorems for mixed monotone mapping satisfying nonlinear contraction in the framework of generalized metric space endowed with partial order. We also prove the uniqueness of the coupled fixed point for such mappings in this setup.

1. Introduction

Fixed point theory is a very useful tool in solving variety of problems in the control theory, economic theory, nonlinear analysis and, global analysis. The Banach contraction principle [1] is the most famous, most simplest, and one of the most versatile elementary results in the fixed point theory. A huge amount of literature is witnessed on applications, generalizations, and extensions of this principle carried out by several authors in different directions, for example, by weakening the hypothesis, using different setups, and considering different mappings.

Recently, the idea of generalized metric spaces was introduced and studied by Mustafa and Sims [2] originated from the concept of metric spaces. Some fixed point theorem in this setup was first determined by Mustafa et al. [3]; particularly, the Banach contraction principle was established in this work. Since then several fixed point, coupled fixed point, and triple fixed point theorems in the framework of generalized metric spaces have been investigated in [4–11].

In the recent past, many authors obtained important fixed point theorems in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering (see [12–24]).

The aim of this paper is to determine some coupled fixed point theorems for nonlinear contractions in the framework of partially ordered generalized metric spaces.

2. Definitions and Preliminary Results

We will assume throughout this paper that the symbol $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real and natural numbers, respectively. In this section, we recall some definitions and preliminary results which we will use throughout the paper. Mustafa and Sims [2] have recently introduced the concept of generalized metric space as follows.

Let X be a nonempty set and a mapping $G : X \times X \times X \to \mathbb{R}$. Then G is called a *generalized metric* (for short, *G-metric*) on X and (X, G) a *generalized metric space* or simply *G-metric space* if the following conditions are satisfied:

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $G(x, x, y) > 0$, for all $x, y \in X$ and $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ and $y \neq z$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

We remark that every G-metric on X defines a metric d_G on X by $d_G(x, y) = G(x, y, y) + G(y, x, x)$, for all $x, y \in X$.

Example 2.1 (see [2]). Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, \infty)$ defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{2.1}$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \tag{2.2}$$

for all $x, y, z \in X$, is a G-metric on X.

The concepts of convergence and Cauchy sequences and continuous functions in G-metric space are studied in [2].

Let (X, G) be a G-metric space. Then, a sequence (x_n) is said to be *convergent* in (X, G) or simply *G-convergent* to $x \in X$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x) < \epsilon$, for all $n, m \geq N$.

Let (X, G) be a G-metric space. Then, (x_n) is said to be *Cauchy* in (X, G) or simply *G-Cauchy* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_k) < \epsilon$, for all $n, m, k \geq N$. A G-metric space (X, G) is said to be *complete* if every G-Cauchy sequence is G-convergent.

Let (X, G) be a G-metric space and $f : X \to X$ a mapping. Then, f is said to *G-continuous* at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, we have $(f(x_n))$ G-convergent to $f(x)$.

Proposition 2.2 (see [2]). *Let (X, G) be a G-metric space and (x_n) a sequence in X. Then, for all $x \in X$, the following statements are equivalent:*

(i) *(x_n) is G-convergent to x,*

(ii) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*

(iii) *$G(x_n, x, x) \to 0$ as $n \to \infty$,*

(iv) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Proposition 2.3 (see [2]). *Let (X, G) be a G-metric space and (x_n) a sequence in X. Then, the following statements are equivalent:*

(i) *(x_n) is G-Cauchy,*

(ii) *For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Lemma 2.4 (see [2]). *If (X, G) is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.*

Let (X, G) be a G-metric space and $F : X \times X \to X$ a mapping. Then, a map F is said to be *continuous* [10] in (X, G) if for every G-convergent sequences $x_n \to x$ and $y_n \to y$, $(F(x_n, y_n))$ is G-convergent to $F(x, y)$.

Quite recently, Bhaskar and Lakshmikantham [14] defined and studied the concepts of mixed monotone property and coupled fixed point in partially ordered metric space.

Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ a mapping. Then, a map F is said to have *mixed monotone property* if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} &x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies } F(x_1, y) \leq F(x_2, y), \\ &y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies } F(x, y_1) \geq F(x, y_2). \end{aligned} \tag{2.3}$$

An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{2.4}$$

The following class of functions are considered in [25]. Denote with Φ the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy that

(i) φ is continuous and nondecreasing,

(ii) $\varphi(t) = 0$ if and only if $t = 0$,

(iii) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$, for all $t, s \in [0, \infty)$.

By Ψ we denote the set of all functions $\psi : [0, \infty) \to (0, \infty)$ which satisfy $\lim_{t \to r} \psi(t) > 0$, for all $r > 0$ and $\lim_{t \to 0^+} \psi(t) = 0$.

For example, functions $\varphi_1, \varphi_2, \varphi_3, \varphi_3 \in \Phi$, where $\varphi_1(t) = kt$ $(k > 0)$, $\varphi_2(t) = t/t+1$, $\varphi_3(t) = \ln(t+1)$, and $\varphi_4(t) = \min\{t, 1\}$, and the functions $\psi_1, \psi_2, \psi_3 \in \Psi$, where $\psi_1(t) = kt$, $\psi_2(t) = \ln(2t+1)/2$, and

$$\psi_3(t) = \begin{cases} 1, & \text{if } t = 0,\ t = 1, \\ \dfrac{t}{t+1}, & \text{if } 0 < t < 1, \\ \dfrac{t}{2}, & \text{if } t > 1. \end{cases} \tag{2.5}$$

3. Main Results

In this section, we establish some coupled fixed point results by considering a map on generalized metric spaces endowed with a partial order.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set, and let G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$, and a mapping $F : X \times X \to X$ such that*

$$\varphi(G(F(x,y), F(u,v), F(s,t))) \leq \frac{1}{2}\varphi(G(x,u,s) + G(y,v,t)) - \psi\left(\frac{G(x,u,s) + G(y,v,t)}{2}\right), \tag{3.1}$$

for all $x, y, u, v, s, t \in X$ with $x \succeq u \succeq s$ and $y \preceq v \preceq t$ where either $u \neq s$ or $v \neq t$. Suppose F has a mixed monotone property and also suppose that either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence (x_n) is G-convergent to x, then $x_n \preceq x$, for all n,*

(ii) *if a nonincreasing sequence (y_n) is G-convergent to y, then $y_n \succeq y$, for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled point; that is, there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We can choose $x_1, y_1 \in X$ such that $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Write

$$x_{n+1} = F(x_n, y_n), \qquad y_{n+1} = F(y_n, x_n), \tag{3.2}$$

for all $n \geq 1$. Due to the mixed monotone property of F, we can find $x_2 \succeq x_1 \succeq x_0$ and $y_2 \preceq y_1 \preceq y_0$. By straightforward calculation, we obtain

$$\begin{gathered} x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n+1} \preceq \cdots, \\ y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_{n+1} \succeq \cdots. \end{gathered} \tag{3.3}$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned}
&\varphi(G(x_{n+1},x_{n+1},x_n))\\
&\quad=\varphi(G(F(x_n,y_n),F(x_n,y_n),F(x_{n-1},y_{n-1})))\\
&\quad\le\frac{1}{2}\varphi(G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1}))-\psi\left(\frac{G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1})}{2}\right),
\end{aligned}\tag{3.4}$$

and similarly

$$\begin{aligned}
&\varphi(G(y_{n+1},y_{n+1},y_n))\\
&\quad=\varphi(G(F(y_n,x_n),F(y_n,x_n),F(y_{n-1},x_{n-1})))\\
&\quad\le\frac{1}{2}\varphi(G(y_n,y_n,y_{n-1})+G(x_n,x_n,x_{n-1}))-\psi\left(\frac{G(y_n,y_n,y_{n-1})+G(x_n,x_n,x_{n-1})}{2}\right).
\end{aligned}\tag{3.5}$$

Adding (3.4) and (3.5), we get

$$\begin{aligned}
\varphi(G(x_{n+1},x_{n+1},x_n))&+\varphi(G(y_{n+1},y_{n+1},y_n))\\
&\le\varphi(G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1}))\\
&\quad-2\psi\left(\frac{G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1})}{2}\right).
\end{aligned}\tag{3.6}$$

Using the property $\varphi(t+s)\le\varphi(t)+\varphi(s)$ for all $t,s\in[0,\infty)$, we get

$$\begin{aligned}
&\varphi(G(x_{n+1},x_{n+1},x_n)+G(y_{n+1},y_{n+1},y_n))\\
&\quad\le\varphi(G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1}))-2\psi\left(\frac{G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1})}{2}\right),
\end{aligned}\tag{3.7}$$

which implies that

$$\varphi(G(x_{n+1},x_{n+1},x_n)+G(y_{n+1},y_{n+1},y_n))\le\varphi(G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1})).\tag{3.8}$$

Since φ is nondecreasing, we have

$$G(x_{n+1},x_{n+1},x_n)+G(y_{n+1},y_{n+1},y_n)\le G(x_n,x_n,x_{n-1})+G(y_n,y_n,y_{n-1}).\tag{3.9}$$

For all $n \in \mathbb{N}$, set

$$s_n = G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n), \tag{3.10}$$

then a sequence (s_n) is decreasing. Therefore, there exists some $s \geq 0$ such that

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left[G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n)\right] = s. \tag{3.11}$$

Now we have to show that $s = 0$. On the contrary, suppose that $s > 0$. Letting $n \to \infty$ in (3.7) (equivalently, s_n is G-convergent to s) and using the property of φ and ψ, we get

$$\varphi(s) = \lim_{n \to \infty} \varphi(s_n) \leq \lim_{n \to \infty} \left\{\varphi(s_{n-1}) - 2\psi\left(\frac{s_{n-1}}{2}\right)\right\} = \varphi(s) - 2 \lim_{s_{n-1} \to s} \psi\left(\frac{s_{n-1}}{2}\right) < \varphi(s), \tag{3.12}$$

which is a contradiction. Thus $s = 0$; from (3.11), we have

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left[G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n)\right] = 0. \tag{3.13}$$

Again, we have to show that (x_n) and (y_n) are Cauchy sequences in the G-metric space (X, G). On the contrary, suppose that at least one of (x_n) or (y_n) is not a Cauchy sequence in (X, G). Then there exists $\epsilon > 0$, for which we can find subsequences $(x_{k(j)})$, $(x_{l(j)})$ and $(y_{k(j)})$, $(y_{l(j)})$ of the sequences (x_n) and (y_n), respectively, with $k(j) > l(j) \geq j$, for all $j \in \mathbb{N}$ such that

$$\alpha_j = G(x_{k(j)}, x_{k(j)}, x_{l(j)}) + G(y_{k(j)}, y_{k(j)}, y_{l(j)}) \geq \epsilon. \tag{3.14}$$

We may also assume that

$$G(x_{k(j)-1}, x_{k(j)-1}, x_{l(j)}) + G(y_{k(j)-1}, y_{k(j)-1}, y_{l(j)}) < \epsilon, \tag{3.15}$$

by choosing $k(j)$ to be the smallest number exceeding $l(j)$, for which (3.14) holds. From (3.14) and (3.15) and using the rectangle inequality, we obtain

$$\begin{aligned}
\epsilon \leq \alpha_j &= G(x_{k(j)}, x_{k(j)}, x_{l(j)}) + G(y_{k(j)}, y_{k(j)}, y_{l(j)}) \\
&\leq G(x_{k(j)}, x_{k(j)}, x_{k(j)-1}) + G(x_{k(j)-1}, x_{k(j)-1}, x_{l(j)}) \\
&\quad + G(y_{k(j)}, y_{k(j)}, y_{k(j)-1}) + G(y_{k(j)-1}, y_{k(j)-1}, y_{l(j)}) \\
&< G(x_{k(j)}, x_{k(j)}, x_{k(j)-1}) + G(y_{k(j)}, y_{k(j)}, y_{k(j)-1}) + \epsilon.
\end{aligned} \tag{3.16}$$

Letting $j \to \infty$ in the above inequality and using (3.13), we get

$$\lim_{j \to \infty} \alpha_j = \lim_{j \to \infty} \left[G(x_{k(j)}, x_{k(j)}, x_{l(j)}) + G(y_{k(j)}, y_{k(j)}, y_{l(j)})\right] = \epsilon. \tag{3.17}$$

Again, by using rectangle inequality, we obtain

$$\begin{aligned}
\alpha_j &= G(x_{k(j)}, x_{k(j)}, x_{l(j)}) + G(y_{k(j)}, y_{k(j)}, y_{l(j)}) \\
&\leq G(x_{k(j)}, x_{k(j)}, x_{k(j)+1}) + G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1}) + G(x_{l(j)+1}, x_{l(j)+1}, x_{l(j)}) \\
&\quad + G(y_{k(j)}, y_{k(j)}, y_{k(j)+1}) + G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1}) + G(y_{l(j)+1}, y_{l(j)+1}, y_{l(j)}) \\
&= s_{l(j)} + G(x_{k(j)}, x_{k(j)}, x_{k(j)+1}) + G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1}) \\
&\quad + G(y_{k(j)}, y_{k(j)}, y_{k(j)+1}) + G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1}).
\end{aligned} \tag{3.18}$$

By using Lemma 2.4, the above inequality becomes

$$\begin{aligned}
\alpha_j &\leq s_{l(j)} + 2G(x_{k(j)+1}, x_{k(j)+1}, x_{k(j)}) + 2G(y_{k(j)+1}, y_{k(j)+1}, y_{k(j)}) \\
&\quad + G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1}) + G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1});
\end{aligned} \tag{3.19}$$

this implies that

$$\alpha_j \leq s_{l(j)} + 2s_{k(j)} + G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1}) + G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1}). \tag{3.20}$$

Operating φ on both sides of the above inequality,

$$\begin{aligned}
\varphi(\alpha_j) &\leq \varphi(s_{l(j)} + 2s_{k(j)} + G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1}) + G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1})) \\
&= \varphi(s_{l(j)} + 2s_{k(j)}) + \varphi(G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1})) + \varphi(G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1})).
\end{aligned} \tag{3.21}$$

Now we find the expressions $\varphi(G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1}))$ and $\varphi(G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1}))$ in terms of φ and ψ by using (3.1) and (3.2); that is,

$$\begin{aligned}
\varphi(G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1})) &= \varphi(G(F(x_{k(j)}, y_{k(j)}), F(x_{k(j)}, y_{k(j)}), F(x_{l(j)}, y_{l(j)}))) \\
&\leq \frac{1}{2}\varphi(G(x_{k(j)}, x_{k(j)}, x_{l(j)}) + G(y_{k(j)}, y_{k(j)}, y_{l(j)})) \\
&\quad - \psi\left(\frac{G(x_{k(j)}, x_{k(j)}, x_{l(j)}) + G(y_{k(j)}, y_{k(j)}, y_{l(j)})}{2}\right),
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
\varphi(G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1})) &= \varphi(G(F(y_{k(j)}, x_{k(j)}), F(y_{k(j)}, x_{k(j)}), F(y_{l(j)}, x_{l(j)}))) \\
&\leq \frac{1}{2}\varphi(G(y_{k(j)}, y_{k(j)}, y_{l(j)}) + G(x_{k(j)}, x_{k(j)}, x_{l(j)})) \\
&\quad - \psi\left(\frac{G(y_{k(j)}, y_{k(j)}, y_{l(j)}) + G(x_{k(j)}, x_{k(j)}, x_{l(j)})}{2}\right).
\end{aligned} \tag{3.23}$$

Adding (3.22) and (3.23), we get

$$\varphi(G(x_{k(j)+1}, x_{k(j)+1}, x_{l(j)+1})) + \varphi(G(y_{k(j)+1}, y_{k(j)+1}, y_{l(j)+1})) \leq \varphi(\alpha_j) - 2\psi\left(\frac{\alpha_j}{2}\right). \tag{3.24}$$

From (3.21) and (3.24), we obtain

$$\varphi(\alpha_j) \leq \varphi(s_{l(j)} + 2s_{k(j)}) + \varphi(\alpha_j) - 2\psi\left(\frac{\alpha_j}{2}\right). \tag{3.25}$$

Taking limit as $j \to \infty$ on both sides of the above inequality, we get

$$\varphi(\epsilon) \leq \varphi(0) + \varphi(\epsilon) - 2\lim_{j\to\infty}\psi\left(\frac{\alpha_j}{2}\right) = \varphi(\epsilon) - 2\lim_{\alpha_j\to\epsilon}\psi\left(\frac{\alpha_j}{2}\right) < \varphi(\epsilon), \tag{3.26}$$

which is a contradiction, and hence (x_n) and (y_n) are Cauchy sequences in the G-metric space (X, G). Since (X, G) is complete G-metric space, hence (x_n) and (y_n) are G-convergent. Then, there exist $x, y \in X$ such that (x_n) and (y_n) are G-convergent to x and y, respectively. Suppose that condition (a) holds. Letting $n \to \infty$ in (3.2), we get $x = F(x, y)$ and $y = F(y, x)$. Lastly, suppose that assumption (b) holds. Since a sequence (x_n) is nondecreasing and G-convergent to x and also (y_n) is nonincreasing sequence and G-convergent to y, by assumption (b), we have $x_n \preceq x$ and $y_n \succeq y$ for all n. Using the rectangle inequality, write

$$\begin{aligned} G(x, x, F(x, y)) &\leq G(x, x, x_{n+1}) + G(x_{n+1}, x_{n+1}, F(x, y)) \\ &= G(x, x, x_{n+1}) + G(F(x_n, y_n), F(x_n, y_n), F(x, y)). \end{aligned} \tag{3.27}$$

Applying the function φ on both sides of the above equation and using (3.1), we have

$$\begin{aligned} \varphi(G(x, x, F(x, y))) &\leq \varphi(G(x, x, x_{n+1})) + \varphi(G(F(x_n, y_n), F(x_n, y_n), F(x, y))) \\ &\leq \varphi(G(x, x, x_{n+1})) + \frac{1}{2}\varphi(G(x_n, x_n, x) + G(y_n, y_n, y)) \\ &\quad - \psi\left(\frac{G(x_n, x_n, x) + G(y_n, y_n, y)}{2}\right). \end{aligned} \tag{3.28}$$

Letting $n \to \infty$, we get $G(x, x, F(x, y)) = 0$. Hence $x = F(x, y)$. Similarly we obtain $y = F(y, x)$. Thus, we conclude that F has a coupled fixed point. □

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set, and let G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F : X \times X \to X$ is a mapping having mixed monotone property. Assume that there exists $\psi \in \Psi$ such that for all $x, y, u, v, s, t \in X$,*

$$G(F(x, y), F(u, v), F(s, t)) \leq \frac{G(x, u, s) + G(y, v, t)}{2} - \psi\left(\frac{G(x, u, s) + G(y, v, t)}{2}\right), \tag{3.29}$$

with $x \succeq u \succeq s$ and $y \preceq v \preceq t$ where either $u \neq s$ or $v \neq t$. Suppose that either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence (x_n) is G-convergent to x, then $x_n \preceq x$, for all n,*

(ii) *if a nonincreasing sequence (y_n) is G-convergent to y, then $y_n \succeq y$, for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$; that is, F has a coupled point in X.

Proof. Taking $\varphi(t) = t$ in Theorem 3.1 and proceeding the same lines as in this theorem, we get the desired result. □

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set, and let G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose $F : X \times X \to X$ is a mapping having mixed monotone property and assume that there exists $k \in [0, 1)$ such that*

$$G(F(x, y), F(u, v), F(s, t)) \leq \frac{k}{2}\left[G(x, u, s) + G(y, v, t)\right], \tag{3.30}$$

for all $x, y, u, v, s, t \in X$ with $x \succeq u \succeq s$ and $y \preceq v \preceq t$ where either $u \neq s$ or $v \neq t$. Suppose that either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence (x_n) is G-convergent to x, then $x_n \preceq x$, for all n,*

(ii) *if a nonincreasing sequence (y_n) is G-convergent to y, then $y_n \succeq y$, for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled point in X.

Proof. Taking $\varphi(t) = t$ and $\psi(t) = ((1 - k)/2)t$ in Theorem 3.1 and proceeding the same lines as in this theorem, we get the desired result. □

Remark 3.4. To assure the uniqueness of a coupled fixed point, we shall consider the following condition. If $(Y, \preceq)$ is a partially ordered set, we endowed the product $Y \times Y$ with

$$(x, y) \preceq (u, v) \text{ iff } x \preceq u, y \succeq v, \tag{3.31}$$

for all $(x, y), (u, v) \in Y \times Y$.

Theorem 3.5. *In addition to the hypothesis of Theorem 3.1, suppose that for all $(x, y), (s, t) \in X \times X$, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(s, t), F(t, s))$. Then, F has a unique coupled fixed point.*

Proof. It follows from Theorem 3.1 that the set of coupled fixed points is nonempty. Suppose (x, y) and (s, t) are coupled fixed points of the mappings $F : X \times X \to X$; that is, $x = F(x, y)$, $y = F(y, x)$, and $s = F(s, t)$, $t = F(t, s)$. By assumption there exists (u, v) in $X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(s,t),F(t,s))$. Put $u = u_0$ and $v = v_0$ and choose $u_1, v_1 \in X$ such that $u_1 = F(u_1, v_1)$ and $v_1 = F(v_1, u_1)$. Thus, we can define two sequences (u_n) and (v_n) as

$$u_{n+1} = F(u_n, v_n), \qquad v_{n+1} = F(v_n, u_n). \tag{3.32}$$

Since (u,v) is comparable to (x,y), we can assume that $(x,y) \succeq (u,v) = (u_0, v_0)$. Then it is easy to show that (u_n, v_n) and (x,y) are comparable; that is, $(x,y) \succeq (u_n, v_n)$, for all n. Thus, from (3.1), we have

$$\begin{aligned}\varphi(G(u_{u+1},x,x)) &= \varphi(G(F(u_u,v_n),F(x,y),F(x,y)))\\ &\leq \frac{1}{2}\varphi(G(u_n,x,x)+G(v_n,y,y)) - \psi\left(\frac{G(u_n,x,x)+G(v_n,y,y)}{2}\right),\end{aligned} \tag{3.33}$$

$$\begin{aligned}\varphi(G(y,y,v_{u+1})) &= \varphi(G(F(y,x),F(y,x),F(v_u,u_n)))\\ &\leq \frac{1}{2}\varphi(G(y,y,v_n)+G(x,x,u_n)) - \psi\left(\frac{G(y,y,v_n)+G(x,x,u_n)}{2}\right).\end{aligned} \tag{3.34}$$

Using the property of φ and adding (3.33) and (3.34), we get

$$\begin{aligned}\varphi(G(u_{u+1},x,x)+G(v_{u+1},y,y)) \leq{}& \varphi(G(u_n,x,x)+G(v_n,y,y))\\ &- 2\psi\left(\frac{G(u_n,x,x)+G(v_n,y,y)}{2}\right),\end{aligned} \tag{3.35}$$

which implies that

$$\varphi(G(u_{u+1},x,x)+G(v_{u+1},y,y)) \leq \varphi(G(u_n,x,x)+G(v_n,y,y)). \tag{3.36}$$

Therefore,

$$G(u_{u+1},x,x)+G(v_{u+1},y,y) \leq G(u_n,x,x)+G(v_n,y,y). \tag{3.37}$$

We see that the sequence $(G(u_n,x,x)+G(v_n,y,y))$ is decreasing; there exists some $\xi \geq 0$ such that

$$\lim_{n\to\infty}\left[G(u_n,x,x)+G(v_n,y,y)\right] = \xi. \tag{3.38}$$

Now we have to show that $\xi = 0$. On the contrary, suppose that $\xi > 0$. Letting $n \to \infty$ in (3.35), we get

$$\varphi(\xi) \le \varphi(\xi) - 2\psi \lim_{n\to\infty}\left(\frac{G(u_n, x, x) + G(v_n, y, y)}{2}\right) < \varphi(\xi), \tag{3.39}$$

which is not possible. Hence $\xi = 0$. Therefore, (3.38) becomes

$$\lim_{n\to\infty}\left[G(u_n, x, x) + G(v_n, y, y)\right] = 0, \tag{3.40}$$

which implies

$$\lim_{n\to\infty} G(u_n, x, x) = 0 = \lim_{n\to\infty} G(v_n, y, y). \tag{3.41}$$

Similarly, we can show that $\lim_{n\to\infty} G(u_n, s, s) = 0 = \lim_{n\to\infty} G(v_n, t, t)$. We conclude that $x = s$ and $y = t$. Thus, F has a unique coupled fixed point. □

Theorem 3.6. *In addition to the hypothesis of Theorem 3.1, suppose that x_0 and y_0 are comparable. Then, F has a unique fixed point.*

Proof. Proceeding the same lines as in the proof of Theorem 3.1, we know that a mapping $F : X \times X \to X$ has a coupled fixed point (x, y). Now we need to show that $x = y$. Since x_0 and y_0 are comparable, we can assume that $x_0 \ge y_0$. It is easy to show that $x_n \ge y_n$ for all $n \ge 0$, where $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Suppose that $G(x, x, y) > 0$, for all $x, y \in X$ with $x \ne y$. Using the rectangle inequality, write

$$\begin{aligned} G(x, x, y) &\le G(x, x, x_{n+1}) + G(x_{n+1}, x_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+1}, y) \\ &= G(x, x, x_{n+1}) + G(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) + G(y_{n+1}, y_{n+1}, y). \end{aligned} \tag{3.42}$$

Operating φ on both sides of the above inequality, we get

$$\varphi(G(x, x, y)) \le \varphi(G(x, x, x_{n+1}) + G(y_{n+1}, y_{n+1}, y)) + \varphi(G(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n))). \tag{3.43}$$

From (3.1), we have

$$\begin{aligned} \varphi(G(x, x, y)) &\le \varphi(G(x, x, x_{n+1}) + G(y_{n+1}, y_{n+1}, y)) \\ &\quad + \frac{1}{2}\varphi(G(x_n, x_n, y_n) + G(y_n, y_n, x_n)) - \psi\left(\frac{G(x_n, x_n, y_n) + G(y_n, y_n, x_n)}{2}\right). \end{aligned} \tag{3.44}$$

Similarly, we obtain

$$\varphi(G(y,y,x)) \le \varphi(G(y,y,y_{n+1}) + G(x_{n+1},x_{n+1},x)) + \frac{1}{2}\varphi(G(y_n,y_n,x_n) + G(x_n,x_n,y_n)) - \psi\left(\frac{G(y_n,y_n,x_n) + G(x_n,x_n,y_n)}{2}\right). \tag{3.45}$$

Adding (3.44) and (3.45) and using the property of φ, we get

$$\begin{aligned}\varphi(G(x,x,y) + G(y,y,x)) &\le \varphi(G(x,x,x_{n+1}) + G(y_{n+1},y_{n+1},y)) + \varphi(G(y,y,y_{n+1}) + G(x_{n+1},x_{n+1},x)) \\ &\quad + \varphi(G(x_n,x_n,y_n) + G(y_n,y_n,x_n)) - 2\psi\left(\frac{G(x_n,x_n,y_n) + G(y_n,y_n,x_n)}{2}\right).\end{aligned} \tag{3.46}$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned}\varphi(G(x,x,y) + G(y,y,x)) &\le \varphi(0) + \varphi(0) + \varphi(G(x,x,y) + G(y,y,x)) \\ &\quad - 2\lim_{n\to\infty}\psi\left(\frac{G(x_n,x_n,y_n) + G(y_n,y_n,x_n)}{2}\right),\end{aligned} \tag{3.47}$$

which implies that

$$\varphi(G(x,x,y) + G(y,y,x)) < \varphi(G(x,x,y) + G(y,y,x)), \tag{3.48}$$

which is not possible, and hence $G(x,x,y) = 0$. Thus $x = y$, whence the result. □

Theorem 3.7. *Let $(X,\preceq)$ be a partially ordered set, and let G be a G-metric on X such that (X,G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping such that F has a mixed monotone property and $F(x,y) \preceq F(y,x)$ whenever $x \preceq y$. Suppose that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $x,y,u,v,s,t \in X$, (3.1) holds with $x \succeq u \succeq s$, $y \preceq v \preceq t$ and $x \prec y$ where either $u \neq s$ or $v \neq t$. Assume that either*

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence (x_n) is G-convergent to x, then $x_n \preceq x$, for all n,*

(ii) *if a nonincreasing sequence (y_n) is G-convergent to y, then $y_n \succeq y$, for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq y_0$, $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$, then F has a coupled point; that is, there exist $x,y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We can choose $x_1, y_1 \in X$ such that $x_1 = F(x_0, y_0) \succeq x_0$ and $y_1 = F(y_0, x_0) \preceq y_0$. Since $x_0 \preceq y_0$, we have $F(x_0, y_0) \preceq F(y_0, x_0)$. Accordingly,

$$x_0 \preceq x_1 = F(x_0, y_0) \preceq F(y_0, x_0) = y_1 \preceq y_0. \tag{3.49}$$

Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that

$$x_n \preceq F(x_n, y_n) = x_{n+1} \preceq y_{n+1} = F(y_n, x_n) \preceq y_n, \tag{3.50}$$

for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \cdots \preceq y_{n+1} \preceq y_n \preceq \cdots \preceq y_2 \preceq y_1 \preceq y_0. \tag{3.51}$$

The rest of the proof can be done on the same lines as in Theorem 3.1. □

4. Example and the Concluding Remark

In the following, we construct an example of a G-metric space involving the idea of coupled fixed point to see the applicability of our results.

Let $X = \mathbb{N} \cup \{0\}$. Define a mapping G from X^3 to $\mathbb{R}$ by

$$G(x, y, z) = \begin{cases} x + y + z, & \text{if } x, y, z \text{ are all distinct and different from zero,} \\ x + z, & \text{if } x = y \neq z \text{ and all are different from zero,} \\ y + z + 1, & \text{if } x = 0,\ y \neq z \text{ and } y, z \text{ are different from zero,} \\ y + 2, & \text{if } x = 0,\ y = z \neq 0, \\ z + 1, & \text{if } x = 0 = y,\ z \neq 0, \\ 0, & \text{if } x = y = z. \end{cases} \tag{4.1}$$

Then (X, G) is a complete G-metric space [10]. Let us consider a partial order $\preceq$ on X be such that $x \preceq y$ holds if $x > y$, 3 divides $(x - y)$, and $3 \preceq 1$ and $0 \preceq 1$ hold, for all $x, y \in X$. Consider a mapping $F : X \times X \to X$ defined by

$$F(x, y) = \begin{cases} 1, & \text{if } x \prec y, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Suppose that $s \preceq u \preceq x \prec y \preceq v \preceq t$ holds. Therefore, we have $F(x, y) = 1 = F(u, v) = F(s, t)$. It follows that $G(F(x, y), F(u, v), F(s, t)) = 0$.

Now, applying the function φ to this equality and then using the hypothesis of this function, we see that (3.1) is satisfied since the left-hand side of (3.1) becomes 0. For $x_0 = 81$ and $y_0 = 0$, Theorem 3.7 is applicable. In this case, the coupled fixed point is not unique. Hence $(0, 0)$ and $(1, 0)$ are two coupled fixed points of F.

We remark that inequality (3.1) is not satisfied when $s = u = x = y = 3$, $v = 0$, and $t = 1$, and hence Theorem 3.1 does not work for this example. We know that a G-metric naturally induces a metric d_G given by $d_G(x, y) = G(x, y, y) + G(x, x, y)$ [2], but inequality (3.1) does not reduce to any metric inequality with the metric d_G due to the condition that either $u \neq s$ or $v \neq t$. Hence our theorems are more general, different from the classical results, and do not reduce to fixed point problems in the corresponding metric space (X, d_G).

Acknowledgment

The authors have benefited from the reports of the anonymous referees and they are thankful for their valuable comments on the first draft of this paper which improved the presentation and readability.

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