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Research Article

A Proximal Point Method Involving Two Resolvent Operators

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We construct a sequence of proximal iterates that converges strongly (under minimal assumptions) to a common zero of two maximal monotone operators in a Hilbert space. The algorithm introduced in this paper puts together several proximal point algorithms under one frame work. Therefore, the results presented here generalize and improve many results related to the proximal point algorithm which were announced recently in the literature.

1. Introduction

Let K_1 and K_2 be nonempty, closed, and convex subsets of a real Hilbert space H with nonempty intersection and consider the following problem:

$$\text{find an } x \in H \text{ such that } x \in K_1 \cap K_2. \tag{1.1}$$

In his 1933 paper, von Neumann showed that if K_1 and K_2 are subspaces of H, then the method of alternating projections, defined by

$$H \ni x_0 \longmapsto x_1 = P_{K_1} x_0 \longmapsto x_2 = P_{K_2} x_1 \longmapsto x_3 = P_{K_1} x_2 \longmapsto x_4 = P_{K_2} x_3 \longmapsto \dots, \tag{1.2}$$

converges strongly to a point in $K_1 \cap K_2$ which is closest to the starting point x_0. The proof of this classical result can be found for example in [1, 2]. Ever since von Neumann announced his result, many researchers have dedicated their time to study the convex feasibility problem (1.1). In his paper, Bregman [3] showed that if K_1 and K_2 are two arbitrary nonempty, closed, and convex subsets in H with nonempty intersection, then the sequence (x_n) generated from the method of alternating projections converges weakly to a point in $K_1 \cap K_2$. The work of Hundal [4] revealed that the method of alternating projections fails in general to converge strongly, see also [5].

Recall that the projection operator coincides with the resolvent of a normal cone. Thus, the method of alternating projections can be extended in a natural way as follows: Given $x_0 \in H$, define a sequence (x_n) iteratively by

$$\begin{aligned} x_{2n+1} &= J_{\beta_n}^{A}(x_{2n} + e_n) \quad \text{for } n = 0, 1, \dots, \\ x_{2n} &= J_{\mu_n}^{B}(x_{2n-1} + e_n') \quad \text{for } n = 1, 2, \dots, \end{aligned} \tag{1.3}$$

for $\beta_n, \mu_n \in (0, \infty)$, and two maximal monotone operators A and B, where (e_n) and (e_n') are sequences of computational errors. Here $J_\mu^A := (I + \mu A)^{-1}$ is the resolvent of A. In this case, problem (1.1) can be restated as

$$\text{find an } x \in D(A) \cap D(B) \text{ such that } x \in A^{-1}(0) \cap B^{-1}(0). \tag{1.4}$$

For $e_n = 0 = e_n'$ and $\beta_n = \mu_n = \mu > 0$, Bauschke et al. [6] proved that sequences generated from the method of alternating resolvents (1.3) converges weakly to some point that solves problem (1.4). In fact, they showed that such a sequences converges weakly to a point in $\operatorname{Fix} J_\mu^A J_\mu^B$ provided that the fixed point set of the composition mapping $J_\mu^A J_\mu^B$ is nonempty. Note that strong convergence of this method fails in general, (the same counter example of Hundal [4] applies). For convergence analysis of algorithm (1.3) in the case when any of the sequences of real numbers (β_n) and (μ_n) is not a constant, and when the error sequences (e_n) and (e_n') are not zero for all $n \geq 1$, we refer the reader to [7].

There are other papers in the literature that address strong convergence of a given iterative process to solutions of (1.4). For example, several authors have discussed strong convergence of an iterative process of the Halpern type to common solutions of a finite family of maximal monotone operators in Hilbert spaces (or even m-accretive operators in Banach spaces). Among the most recent works in this direction is due to Hu and Liu [8]. They showed that under appropriate conditions, an iterative process of Halpern type defined by

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n S_{r_n} x_n, \quad n \geq 0, \tag{1.5}$$

where $\alpha_n, \delta_n, \gamma_n \in (0,1)$ with $\alpha_n + \delta_n + \gamma_n = 1$ for all $n \geq 0$, $u, x_0 \in H$ are given, $S_{r_n} := a_0 I + a_1 J_{r_n}^1 + a_2 J_{r_n}^2 + \cdots + a_l J_{r_n}^l$ with $J_r^i = (I + rA_i)^{-1}$ for $a_i \in (0,1)$, $i = 0, 1, \dots, l$ and $\sum_{i=0}^{l} a_i = 1$, converges strongly to a point in $\bigcap_{i=1}^{n} A_i^{-1}(0)$ nearest to u.

Suppose that we want to find solutions to problem (1.4) iteratively. Then we observe that when using the iterative process (1.5), one has to calculate two resolvents of maximal monotone operators in order to find the next iterate. On the other hand, for algorithm (1.3), one needs to calculate only one resolvent operator at each step. This clearly shows that theoretically, algorithm (1.5) requires more computational time compared to algorithm (1.3). The only disadvantage with algorithm (1.3) is that it does not always converge strongly and the limit to which it converges to is not characterized. This is not the case with algorithm (1.5).

Since weak convergence is not good for an effective algorithm, our purpose in this paper is to modify algorithm (1.3) in such a way that strong convergence is guaranteed. More precisely, for any two maximal monotone operators A and B, we define an iterative process in the following way: For $x_0, u \in H$ given, a sequence (x_n) is generated using the rule

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n} x_{2n} + e_n \quad \text{for } n = 0, 1, \ldots, \tag{1.6}$$

$$x_{2n} = J^B_{\mu_n}\left(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n\right) \quad \text{for } n = 1, 2, \ldots, \tag{1.7}$$

where $\alpha_n, \delta_n, \gamma_n, \lambda_n \in [0,1]$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\beta_n, \mu_n \in (0, \infty)$. We will also show that algorithm (1.6), (1.7) contains several algorithms such as the prox-Tikhonov method, the Halpern-type proximal point algorithm, and the regularized proximal method as special cases. That is, with our algorithm, we are able to put several algorithms under one frame work. Therefore, our main results improve, generalize, and unify many related results announced recently in the literature.

2. Preliminary Results

In the sequel, H will be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We recall that a map $T : H \to H$ is called nonexpansive if for every $x, y \in H$ we have $\|Tx - Ty\| \le \|x - y\|$. We say that a map T is firmly nonexpansive if for every $x, y \in H$, we have

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2. \tag{2.1}$$

It is clear that firmly nonexpansive mappings are also nonexpansive. The converse need not be true. The excellent book by Goebel and Reich [9] is recommended to the reader who is interested in studying properties of firmly nonexpansive mappings. An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if

$$\langle x - x', y - y' \rangle \ge 0, \quad \forall (x, y), (x', y') \in G(A), \tag{2.2}$$

where $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ is the graph of A. In other words, an operator is monotone if its graph is a monotone subset of the product space $H \times H$. An operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. Note that if A is maximal monotone, then so is its inverse A^{-1}. For a maximal monotone operator A, the resolvent of A, defined by $J^A_\beta := (I + \beta A)^{-1}$, is well defined on the whole space H, is single-valued, and is firmly nonexpansive for every $\beta > 0$. It is known that the Yosida approximation of A, an operator defined by $A_\beta := \beta^{-1}(I - J^A_\beta)$ (where I is the identity operator) is maximal monotone for every $\beta > 0$. For the properties of maximal monotone operators discussed above, we refer the reader to [10].

Notations. given a sequence (x_n), we will use $x_n \to x$ to mean that (x_n) converges strongly to x whereas $x_n \rightharpoonup x$ will mean that (x_n) converges weakly to x. The weak ω-limit set of a sequence (x_n) will be denoted by $\omega_w((x_n))$. That is,

$$\omega_w((x_n)) = \left\{x \in H : x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k}) \text{ of } (x_n)\right\}. \tag{2.3}$$

The following lemmas will be useful in proving our main results. The first lemma is a basic property of norms in Hilbert spaces.

Lemma 2.1. *For all $x, y \in H$, one has*

$$\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle. \tag{2.4}$$

The next lemma is well known, it can be found for example in [10, page 20].

Lemma 2.2. *Any maximal monotone operator $A : D(A) \subset H \to 2^H$ satisfies the demiclosedness principle. In other words, given any two sequences (x_n) and (y_n) satisfying $x_n \to x$ and $y_n \rightharpoonup y$ with $(x_n, y_n) \in G(A)$, then $(x, y) \in G(A)$.*

Lemma 2.3 (Xu [11]). *For any $x \in H$ and $\mu \geq \beta > 0$,*

$$\left\|x - J_\beta^A x\right\| \leq 2\left\|x - J_\mu^A x\right\|, \tag{2.5}$$

where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator.

We end this section with the following key lemmas.

Lemma 2.4 (Boikanyo and Moroşanu [12]). *Let (s_n) be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)(1-\lambda_n)s_n + \alpha_n b_n + \lambda_n c_n + d_n, \qquad n \geq 0, \tag{2.6}$$

where (α_n), (λ_n), (b_n), (c_n), and (d_n) satisfy the conditions: (i) *$\alpha_n, \lambda_n \in [0,1]$, with $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$,* (ii) *$\limsup_{n\to\infty} b_n \leq 0$,* (iii) *$\limsup_{n\to\infty} c_n \leq 0$, and* (iv) *$d_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^{\infty} d_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.*

Remark 2.5. Note that if $\lim_{n\to\infty}\alpha_n = 0$, then $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$ if and only if $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Lemma 2.6 (Maingé [13]). *Let (s_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} < s_{n_j+1}$ for all $j \geq 0$. Define an integer sequence $(\tau(n))_{n\geq n_0}$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}. \tag{2.7}$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}. \tag{2.8}$$

3. Main Results

We first begin by giving a strong convergence result associated with the exact iterative process

$$v_{2n+1} = \alpha_n u + \delta_n v_{2n} + \gamma_n J^A_{\beta_n} v_{2n} \quad \text{for } n = 0, 1, \dots, \tag{3.1}$$

$$v_{2n} = J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)v_{2n-1}) \quad \text{for } n = 1, 2, \dots, \tag{3.2}$$

where $\alpha_n, \delta_n, \gamma_n, \lambda_n \in [0,1]$ with $\alpha_n + \delta_n + \gamma_n = 1$, $\beta_n, \mu_n \in (0,\infty)$ and $v_0, u \in H$ are given. The proof of the following theorem makes use of some ideas of the papers [12–15].

Theorem 3.1. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $v_0, u \in H$, let (v_n) be the sequence generated by* (3.1), (3.2), *where $\alpha_n, \delta_n, \gamma_n, \lambda_n \in [0,1]$ with $\alpha_n + \delta_n + \gamma_n = 1$, and $\beta_n, \mu_n \in (0,\infty)$. Assume that* (i) $\lim_{n\to\infty} \alpha_n = 0$, *$\gamma_n \geq \gamma$ for some $\gamma > 0$ and* $\lim_{n\to\infty} \lambda_n = 0$, (ii) *either* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *or* $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and* (iii) *$\beta_n \geq \beta$ and $\mu_n \geq \mu$ for some $\beta, \mu > 0$. Then (v_n) converges strongly to the point of F nearest to u.*

Proof. Let $p \in F$. Then from (3.2) and the fact that the resolvent operator of a maximal monotone operator B is nonexpansive, we have

$$\begin{aligned} \|v_{2n} - p\| &\leq \|\lambda_n(u-p) + (1-\lambda_n)(v_{2n-1} - p)\| \\ &\leq \lambda_n\|u-p\| + (1-\lambda_n)\|v_{2n-1} - p\|. \end{aligned} \tag{3.3}$$

Again using the fact that the resolvent operator $J^A_{\beta_n} : H \to H$ is nonexpansive, we have from (3.1)

$$\begin{aligned} \|v_{2n+1} - p\| &\leq \alpha_n\|u-p\| + \delta_n\|v_{2n} - p\| + \gamma_n\left\|J^A_{\beta_n} v_{2n} - p\right\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\|v_{2n} - p\| \\ &\leq [\alpha_n + (1-\alpha_n)\lambda_n]\|u-p\| + (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - p\| \\ &= [1 - (1-\alpha_n)(1-\lambda_n)]\|u-p\| + (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - p\|, \end{aligned} \tag{3.4}$$

where the last inequality follows from (3.3). Using a simple induction argument, we get

$$\|v_{2n+1} - p\| \leq \left[1 - \prod_{k=1}^{n}(1-\alpha_k)(1-\lambda_k)\right]\|u-p\| + \|v_1 - p\|\prod_{k=1}^{n}(1-\alpha_k)(1-\lambda_k). \tag{3.5}$$

This shows that the subsequence (v_{2n+1}) of (v_n) is bounded. In view of (3.3), the subsequence (v_{2n}) is also bounded. Hence the sequence (v_n) must be bounded.

Now from the firmly nonexpansive property of $J^A_{\beta_n} : H \to H$, we have for any $p \in F$

$$\left\|J^A_{\beta_n} v_{2n} - p\right\|^2 \leq \|v_{2n} - p\|^2 - \left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2, \tag{3.6}$$

which in turn gives

$$
\begin{aligned}
2\left\langle v_{2n} - p, J_{\beta_n}^A v_{2n} - p \right\rangle &= \|v_{2n} - p\|^2 + \left\| J_{\beta_n}^A v_{2n} - p \right\|^2 - \left\| v_{2n} - J_{\beta_n}^A v_{2n} \right\|^2 \\
&\leq 2\left(\|v_{2n} - p\|^2 - \left\| v_{2n} - J_{\beta_n}^A v_{2n} \right\|^2 \right).
\end{aligned} \tag{3.7}
$$

Again by using the firmly nonexpansive property of the resolvent $J_{\beta_n}^A : H \to H$, we see that

$$
\begin{aligned}
&\left\| \delta_n (v_{2n} - p) + \gamma_n \left(J_{\beta_n}^A v_{2n} - p \right) \right\|^2 \\
&\quad = \delta_n^2 \|v_{2n} - p\|^2 + \gamma_n^2 \left\| J_{\beta_n}^A v_{2n} - p \right\|^2 + 2\gamma_n \delta_n \left\langle v_{2n} - p, J_{\beta_n}^A v_{2n} - p \right\rangle \\
&\quad \leq (1 - \alpha_n)^2 \|v_{2n} - p\|^2 - \gamma_n (\gamma_n + 2\delta_n) \left\| v_{2n} - J_{\beta_n}^A v_{2n} \right\|^2.
\end{aligned} \tag{3.8}
$$

Now from (3.1) and Lemma 2.1, we have

$$
\begin{aligned}
\|v_{2n+1} - p\|^2 &\leq \left\| \delta_n (v_{2n} - p) + \gamma_n \left(J_{\beta_n}^A v_{2n} - p \right) \right\|^2 + 2\alpha_n \langle u - p, v_{2n+1} - p \rangle \\
&\leq (1 - \alpha_n) \|v_{2n} - p\|^2 - \varepsilon \left\| v_{2n} - J_{\beta_n}^A v_{2n} \right\|^2 + 2\alpha_n \langle u - p, v_{2n+1} - p \rangle,
\end{aligned} \tag{3.9}
$$

where $\varepsilon > 0$ is such that $\gamma_n(\gamma_n + 2\delta_n) \geq \varepsilon$. On the other hand, we observe that (3.2) is equivalent to

$$
v_{2n} - p + \mu_n B v_{2n} \ni \lambda_n (u - p) + (1 - \lambda_n)(v_{2n-1} - p). \tag{3.10}
$$

Multiplying this inclusion scalarly by $2(v_{2n} - p)$ and using the monotonicity of B, we obtain

$$
\begin{aligned}
2\|v_{2n} - p\|^2 &\leq 2\lambda_n \langle u - p, v_{2n} - p \rangle + 2(1 - \lambda_n) \langle v_{2n-1} - p, v_{2n} - p \rangle \\
&= (1 - \lambda_n) \left[\|v_{2n-1} - p\|^2 + \|v_{2n} - p\|^2 - \|v_{2n} - v_{2n-1}\|^2 \right] + 2\lambda_n \langle u - p, v_{2n} - p \rangle,
\end{aligned} \tag{3.11}
$$

which implies that

$$
\|v_{2n} - p\|^2 \leq (1 - \lambda_n) \left[\|v_{2n-1} - p\|^2 - \|v_{2n} - v_{2n-1}\|^2 \right] + 2\lambda_n \langle u - p, v_{2n} - p \rangle. \tag{3.12}
$$

Using this inequality in (3.9), we get

$$\begin{aligned}\left\|v_{2n+1}-p\right\|^2 \leq{} & (1-\alpha_n)(1-\lambda_n)\left\|v_{2n-1}-p\right\|^2-\varepsilon\left\|v_{2n}-J_{\beta_n}^{A}v_{2n}\right\|^2 \\ & +2\alpha_n\langle u-p,v_{2n+1}-p\rangle-(1-\alpha_n)(1-\lambda_n)\|v_{2n}-v_{2n-1}\|^2 \\ & +2\lambda_n(1-\alpha_n)\langle u-p,v_{2n}-p\rangle.\end{aligned} \tag{3.13}$$

If we denote $s_n := \|v_{2n-1}-P_F u\|^2$, then we have for some positive constant M

$$s_{n+1}-s_n+\|v_{2n}-v_{2n-1}\|^2+\varepsilon\left\|v_{2n}-J_{\beta_n}^{A}v_{2n}\right\|^2 \leq (\alpha_n+\lambda_n)M. \tag{3.14}$$

We now show that (s_n) converges to zero strongly. For this purpose, we consider two possible cases on the sequence (s_n).

Case 1. (s_n) is eventually decreasing (i.e., there exists $N \geq 0$ such that (s_n) is decreasing for all $n \geq N$). In this case, (s_n) is convergent. Letting $n \to \infty$ in (3.14), we get

$$\lim_{n\to\infty}\|v_{2n}-v_{2n-1}\| = 0 = \lim_{n\to\infty}\left\|v_{2n}-J_{\beta_n}^{A}v_{2n}\right\|. \tag{3.15}$$

Now using the second part of (3.15) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we get

$$\|v_{2n+1}-v_{2n}\| \leq \alpha_n\left\|u-J_{\beta_n}^{A}v_{2n}\right\|+\gamma_n\left\|J_{\beta_n}^{A}v_{2n}-v_{2n}\right\| \longrightarrow 0, \tag{3.16}$$

as $n \to \infty$. Also, we have the following from Lemma 2.3 and the first part of (3.15)

$$\left\|v_{2n}-J_{\beta}^{A}v_{2n}\right\| \leq 2\left\|v_{2n}-J_{\beta_n}^{A}v_{2n}\right\| \longrightarrow 0, \tag{3.17}$$

as $n \to \infty$. Since A_β^{-1}, where A_β denotes the Yosida approximation of A, is demiclosed, it follows that $\omega_w((v_{2n})) \subset A^{-1}(0)$. On the other hand, from the nonexpansive property of the resolvent operator of B, we get

$$\begin{aligned}\left\|v_{2n}-J_{\mu}^{B}v_{2n}\right\| &\leq 2\left\|v_{2n}-J_{\mu_n}^{B}v_{2n}\right\| \\ &\leq 2(\lambda_n\|u-v_{2n-1}\|+\|v_{2n-1}-v_{2n}\|),\end{aligned} \tag{3.18}$$

where the first inequality follows from Lemma 2.3. Since B_μ^{-1} is demiclosed, passing to the limit in the above inequality yields $\omega_w((v_{2n})) \subset B^{-1}(0)$, showing that

$\omega_w((v_{2n})) \subset F := A^{-1}(0) \cap B^{-1}(0)$. Therefore, there is a subsequence (v_{2n_k}) of (v_{2n}) converging weakly to some $z \in F$ such that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - P_F u, v_{2n} - P_F u\rangle &= \limsup_{k\to\infty}\langle u - P_F u, v_{2n_k} - P_F u\rangle \\ &= \langle u - P_F u, z - P_F u\rangle \leq 0,\end{aligned} \tag{3.19}$$

where the above inequality follows from the characterization of the projection operator. Note that by virtue of (3.16), we have

$$\limsup_{n\to\infty}\langle u - P_F u, v_{2n+1} - P_F u\rangle \leq 0 \tag{3.20}$$

as well. Now, we derive from (3.13)

$$\begin{aligned}\|v_{2n+1} - P_F u\|^2 \leq{}& (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + 2\alpha_n\langle u - P_F u, v_{2n+1} - P_F u\rangle \\ &+ 2\lambda_n(1-\alpha_n)\langle u - P_F u, v_{2n} - P_F u\rangle.\end{aligned} \tag{3.21}$$

Using Lemma 2.4 we get $\|v_{2n+1} - P_F u\| \to 0$ as $n \to \infty$. Passing to the limit in (3.12), we also get $\|v_{2n} - P_F u\| \to 0$ as $n \to \infty$. Therefore, we derive $\|v_n - P_F u\| \to 0$ as $n \to \infty$. This proves the result for the case when (s_n) is eventually decreasing.

Case 2. (s_n) is not eventually decreasing, that is, there is a subsequence (s_{n_j}) of (s_n) such that $s_{n_j} < s_{n_j+1}$ for all $j \geq 0$. We then define an integer sequence $(\tau(n))_{n\geq n_0}$ as in Lemma 2.6 so that $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$. Then from (3.14), it follows that

$$\lim_{n\to\infty}\left\|v_{2\tau(n)} - v_{2\tau(n)-1}\right\| = 0 = \lim_{n\to\infty}\left\|v_{2\tau(n)} - J^A_{\beta_{\tau(n)}} v_{2\tau(n)}\right\|. \tag{3.22}$$

We also derive from (3.1)

$$\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\| \leq \alpha_{\tau(n)}\left\|u - J^A_{\beta_{\tau(n)}} v_{2\tau(n)}\right\| + \gamma_{\tau(n)}\left\|J^A_{\beta_{\tau(n)}} v_{2\tau(n)} - v_{2\tau(n)}\right\| \longrightarrow 0, \tag{3.23}$$

as $n \to \infty$. In a similar way as in Case 1, we derive $\omega_w((v_{2\tau(n)})) \subset F$. Consequently,

$$\limsup_{n\to\infty}\left\langle u - P_F u, v_{2\tau(n)} - P_F u\right\rangle \leq 0. \tag{3.24}$$

Note that from (3.21) we have, for some positive constant K,

$$\begin{aligned}\|v_{2n+1} - P_F u\|^2 \leq{}& (1-\alpha_n)(1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + \alpha_n K\|v_{2n+1} - v_{2n}\| \\ &+ 2(\lambda_n(1-\alpha_n) + \alpha_n)\langle u - P_F u, v_{2n} - P_F u\rangle.\end{aligned} \tag{3.25}$$

Therefore, for all $n \geq n_0$, we have

$$\begin{aligned} s_{\tau(n)+1} \leq{}& (1-\alpha_{\tau(n)})(1-\lambda_{\tau(n)})s_{\tau(n)} + \alpha_{\tau(n)}K\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\| \\ &+ 2\left(\lambda_{\tau(n)}(1-\alpha_{\tau(n)}) + \alpha_{\tau(n)}\right)\left\langle u - P_F u, v_{2\tau(n)} - P_F u\right\rangle. \end{aligned} \tag{3.26}$$

Since $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$, we have

$$\begin{aligned} s_{\tau(n)+1} &\leq 2\left\langle u - P_F u, v_{2\tau(n)} - P_F u\right\rangle + \frac{\alpha_{\tau(n)}K\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\|}{\lambda_{\tau(n)}(1-\alpha_{\tau(n)}) + \alpha_{\tau(n)}} \\ &\leq 2\left\langle u - P_F u, v_{2\tau(n)} - P_F u\right\rangle + K\left\|v_{2\tau(n)+1} - v_{2\tau(n)}\right\|. \end{aligned} \tag{3.27}$$

Letting $n \to \infty$ in the above inequality, we see that $s_{\tau(n)+1} \to 0$. Hence from (2.8) it follows that $s_n \to 0$ as $n \to \infty$. That is, $v_{2n+1} \to P_F u$ as $n \to \infty$. Furthermore, for some positive constant C, we have from (3.12)

$$\|v_{2n} - P_F u\|^2 \leq (1-\lambda_n)\|v_{2n-1} - P_F u\|^2 + \lambda_n C, \tag{3.28}$$

which implies that $v_{2n} \to P_F u$ as $n \to \infty$. Hence, we have $v_n \to P_F u$ as $n \to \infty$. This completes the proof of the theorem. □

We are now in a position to give a strong convergence result for the inexact iteration process (1.6), (1.7). For the error sequence, we will use the 14 conditions established in [12].

Theorem 3.2. *Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be maximal monotone operators with $A^{-1}(0) \cap B^{-1}(0) =: F \neq \emptyset$. For arbitrary but fixed vectors $x_0, u \in H$, let (x_n) be the sequence generated by* (1.6), (1.7)*, where $\alpha_n, \delta_n, \gamma_n, \lambda_n \in [0,1]$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\beta_n, \mu_n \in (0,\infty)$. Assume that $\lim_{n\to\infty} \alpha_n = 0$, $\gamma_n \geq \gamma$ for some $\gamma > 0$ and $\lim_{n\to\infty} \lambda_n = 0$, either $\sum_{n=0}^{\infty} \alpha_n = \infty$ or $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\beta_n \geq \beta$ and $\mu_n \geq \mu$ for some $\beta, \mu > 0$. Then (x_n) converges strongly to the point of F nearest to u, provided that any of the following conditions is satisfied:*

(a) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;

(b) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_n \to 0$;

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e'_n\|/\lambda_n \to 0$;

(d) $\|e_n\|/\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;

(e) $\|e_n\|/\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;

(f) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$;

(g) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;

(h) $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_n \to 0$;

(i) $\|e_n\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;

(j) $\|e_n\|/\alpha_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$;

(k) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$;

(l) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\|e'_n\|/\lambda_n \to 0$;

(m) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e'_n\|/\alpha_{n-1} \to 0$;

(n) $\|e_{n-1}\|/\lambda_n \to 0$ *and* $\sum_{n=1}^{\infty} \|e'_n\| < \infty$.

Proof. Taking note of Theorem 3.1, it suffices to show that $\|x_n - v_n\| \to 0$ as $n \to \infty$. Since the resolvent of B is nonexpansive, we derive from (1.7) and (3.2) the following:

$$\|x_{2n} - v_{2n}\| \leq (1 - \lambda_n)\|x_{2n-1} - v_{2n-1}\| + \left\|e'_n\right\|. \tag{3.29}$$

Similarly, from (1.6) and (3.1), we have

$$\begin{aligned} \|x_{2n+1} - v_{2n+1}\| &\leq \delta_n\|x_{2n} - v_{2n}\| + \gamma_n\left\|J^A_{\beta_n}x_{2n} - J^A_{\beta_n}v_{2n}\right\| + \|e_n\| \\ &\leq (1 - \alpha_n)\|x_{2n} - v_{2n}\| + \|e_n\|. \end{aligned} \tag{3.30}$$

Substituting (3.29) into (3.30) yields

$$\|x_{2n+1} - v_{2n+1}\| \leq (1 - \alpha_n)(1 - \lambda_n)\|x_{2n-1} - v_{2n-1}\| + \|e_n\| + \left\|e'_n\right\|. \tag{3.31}$$

Therefore, if the error sequence satisfy any of the conditions (a)–(i), then it readily follows from Lemma 2.4 that $\|x_{2n+1} - v_{2n+1}\| \to 0$ as $n \to \infty$. Passing to the limit in (3.29), we derive $\|x_{2n} - v_{2n}\| \to 0$ as well. If the error sequence satisfy any of the conditions (j)–(n), then from (3.29) and (3.30), we have

$$\|x_{2n} - v_{2n}\| \leq (1 - \alpha_{n-1})(1 - \lambda_n)\|x_{2n-2} - v_{2n-2}\| + \|e_{n-1}\| + \left\|e'_n\right\|. \tag{3.32}$$

Then Lemma 2.4 guarantees that $\|x_{2n} - v_{2n}\| \to 0$ as $n \to \infty$. Passing to the limit in (3.30), we derive $\|x_{2n+1} - v_{2n+1}\| \to 0$ as well. This completes the proof of the theorem. □

Note that when $B = \partial I_H$ where ∂I_H is the subdifferential of the indicator function of H and $\lambda_n = 0 = e'_n$ for all $n \geq 1$, then algorithm (1.6), (1.7) is reduced to the contraction proximal point method which was introduced by Yao and Noor in 2008 [16]. Such a method is given by

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n J^A_{\beta_n}x_n + e_n \quad \text{for } n = 1, 2, \ldots, \tag{3.33}$$

where we have used the notation $x_n := x_{2n-1}$. Here (β_n) is a sequence in $(0, \infty)$ and $\alpha_n, \delta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \delta_n + \gamma_n = 1$. For this method, we have the following strong convergence result.

Corollary 3.3. *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $A^{-1}(0) =: S \neq \emptyset$. For arbitrary but fixed vectors $x_1, u \in H$, let (x_n) be the sequence generated by* (3.33) *where $\alpha_n, \delta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \delta_n + \gamma_n = 1$ and $\beta_n \in (0, \infty)$. Assume that* $\lim_{n\to\infty} \alpha_n = 0$ *with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\gamma_n \geq \gamma$ for some $\gamma > 0$ and $\beta_n \geq \beta$ for some $\beta > 0$. If either $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \to 0$, then (x_n) converges strongly to the point of S nearest to u.*

Corollary 3.3 generalizes and unifies many results announced recently in the literature such as [7, Theorem 4], [16, Theorem 3.3], [17, Theorem 2], and [18, Theorem 3.1]. We also recover [15, Theorem 1].

Remark 3.4. We refer the reader to the paper [12] for another generalization of the method (3.3).

In the case when $A = \partial I_H$ where ∂I_H is the subdifferential of the indicator function of H and $\alpha_n = 0 = e_n$ for all $n \geq 1$, then algorithm (1.6), (1.7) reduces to the regularization method

$$x_{n+1} = J_{\mu_n}^B \left(\lambda_n u + (1 - \lambda_n) x_n + e'_n \right) \quad \text{for } n = 1, 2, \ldots, \tag{3.34}$$

where we have used the notation $x_n := x_{2n}$. In this case, we have the following strong convergence result which improves results given in the papers [11, 19–21].

Corollary 3.5. *Let $B : D(B) \subset H \to 2^H$ be a maximal monotone operator with $B^{-1}(0) =: S \neq \emptyset$. For arbitrary but fixed vectors $x_1, u \in H$, let (x_n) be the sequence generated by* (3.34) *where $\lambda_n \in (0,1)$ and $\mu_n \in (0, \infty)$. Assume that* $\lim_{n\to\infty} \lambda_n = 0$ *with $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\mu_n \geq \mu$ for some $\mu > 0$. If either $\sum_{n=1}^{\infty} \|e'_n\| < \infty$ or $\|e'_n\| / \lambda_n \to 0$, then (x_n) converges strongly to the point of S nearest to u.*

It is worth mentioning that the regularization method is a generalization of the prox-Tikhonov method introduced by Lehdili and Moudafi [22], see [11]. We also mention that for $\lambda_n \to 0$ and $e'_n \to 0$, the regularization method (3.34) is equivalent to the inexact Halpern type proximal point algorithm, see [23]. Therefore Corollary 3.5 also improves many results given in the papers [15, 19, 22, 24–26] related to the inexact Halpern type proximal point algorithm.

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