Research Article

Constrained C^0 Finite Element Methods for Biharmonic Problem

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This paper presents some constrained C^0 finite element approximation methods for the biharmonic problem, which include the C^0 symmetric interior penalty method, the C^0 nonsymmetric interior penalty method, and the C^0 nonsymmetric superpenalty method. In the finite element spaces, the C^1 continuity across the interelement boundaries is obtained weakly by the constrained condition. For the C^0 symmetric interior penalty method, the optimal error estimates in the broken H^2 norm and in the L^2 norm are derived. However, for the C^0 nonsymmetric interior penalty method, the error estimate in the broken H^2 norm is optimal and the error estimate in the L^2 norm is suboptimal because of the lack of adjoint consistency. To obtain the optimal L^2 error estimate, the C^0 nonsymmetric superpenalty method is introduced and the optimal L^2 error estimate is derived.

1. Introduction

The discontinuous Galerkin methods (DGMs) have become a popular method to deal with the partial differential equations, especially for nonlinear hyperbolic problem, which exists the discontinuous solution even when the data is well smooth, and the convection-dominated diffusion problem, and the advection-diffusion problem. For the second-order elliptic problem, according to the different numerical fluxes, there exist different discontinuous Galerkin methods, such as the interior penalty method (IP), the nonsymmetric interior penalty method (NIPG), and local discontinuous Galerkin method (LDG). A unified analysis of discontinuous Galerkin methods for the second-order elliptic problem is studied by Arnold et al. in [1].

The DGM for the fourth-order elliptic problem can be traced back to 1970s. Baker in [2] used the IP method to study the biharmonic problem and obtained the optimal error estimates. Moreover, for IP method, the C^0 and C^1 continuity can be achieved weakly by the interior penalty. Recently, using IP method and NIPG method, Süli and Mozolevski in [3–5] studied the *hp*-version DGM for the biharmonic problem, where the error estimates

are optimal with respect to the mesh size h and are suboptimal with respect to the degree of the piecewise polynomial approximation p. However, we observe that the bilinear forms and the norms corresponding to the IP method in [3–5] are much complicated. A method to simplify the bilinear forms and the norms is using C^0 interior penalty method. C^0 interior penalty method for the biharmonic problem was introduced by Babuška and Zlámal in [6], where they used the nonconforming element and considered the inconsistent formulation and obtained the suboptimal error estimate. Motivated by the Engel and his collaborators' work [7], Brenner and Sung in [8] studied the C^0 interior penalty method for fourth-order problem on polygonal domains. They used the C^0 finite element solution to approximate C^1 solution by a postprocessing procedure, and the C^1 continuity can be achieved weakly by the penalty on the jump of the normal derivatives on the interelement boundaries.

In this paper, thanks to Rivière et al.'s idea in [9], we will study some constrained C^0 finite element approximation methods for the biharmonic problem. The C^1 continuity can be weakly achieved by a constrained condition that integrating the jump of the normal derivatives over the inter-element boundaries vanish. Under this constrained condition, we discuss three C^0 finite element methods which include the C^0 symmetric interior penalty method based on the symmetric bilinear form, the C^0 nonsymmetric interior penalty method, and C^0 nonsymmetric superpenalty method based on the nonsymmetric bilinear forms. First, we study the C^0 symmetric interior penalty method and obtain the optimal error estimates in the broken H^2 norm and in L^2 norm. However, for the C^0 nonsymmetric interior penalty method, the L^2 norm is suboptimal because of the lack of adjoint consistency. Finally, in order to improve the order of the L^2 error estimate, we give the C^0 nonsymmetric superpenalty method and show the optimal L^2 error estimates.

2. C^0 **Finite Element Approximation**

Let $\Omega \subset \mathbb{R}^2$ be a bounded and convex domain with boundary $\partial\Omega$. Consider the following biharmonic problem:

$$\Delta^2 u = f, \quad \text{in } \Omega,$$
$$u = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega, \tag{2.1}$$

where n denotes the unit external normal vector to $\partial\Omega$. We assume that f is sufficiently smooth such that the problem (2.1) admits a unique solution $u \in H^4(\Omega) \cap H_0^2(\Omega)$.

Let $\mathcal{T}_h$ be a family of nondegenerate triangular partition of Ω into triangles. The corresponding ordered triangles are denoted by $K_1, K_2, \ldots, K_N$. Let $h_i = \text{diam}(K_i)$, $i = 1, \ldots, N$, and $h = \max\{h_1, h_2, \ldots, h_N\}$. The nondegenerate requirement is that there exists $\rho > 0$ such that K_i contains a ball of radius ρh_i in its interior. Conventionally, the boundary of K_i is denoted by ∂K_i. We denote

$$e_{ij} = \partial K_i \cap \partial K_j, \qquad e_i = \partial K_i \cap \partial\Omega, \qquad h_{e_{ij}} = \text{diam}(e_{ij}), \qquad h_{e_i} = \text{diam}(e_i). \tag{2.2}$$

Assume that the partition $\mathcal{T}_h$ is quasiuniform; that is, there exists a positive constant v such that

$$h \leqslant v h_i, \quad i = 1, \ldots, N. \tag{2.3}$$

Let $\mathcal{E}_I$ and $\mathcal{E}_B$ be the set of interior edges and boundary edges of $\mathcal{T}_h$, respectively. Let $\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_B$. Denote by v^i the restriction of v to K_i. Let $e = e_{ij} \in \mathcal{E}_I$ with $i > j$. Then we denote the jump $[v]$ and the average $\{v\}$ of v on e by

$$[v]|_e = v^i\Big|_e - v^j\Big|_e, \qquad \{v\}|_e = \frac{1}{2}\left(v^i\Big|_e + v^j\Big|_e \right). \tag{2.4}$$

If $e = e_i \in \mathcal{E}_B$, we denote $[v]$ and $\{v\}$ of v on e by

$$[v]|_e = \{v\}|_e = v^i\Big|_e. \tag{2.5}$$

Define V by

$$V = \left\{ v \in H_0^1(\Omega), v|_K \in H^s(K), \forall K \in \mathcal{T}_h \right\}, \quad s \geqslant 3 \tag{2.6}$$

with broken H^s norm

$$\|\|v\|\|_s = \left(\sum_{K \in \mathcal{T}_h} \|v\|_{s,K}^2 \right)^{1/2}, \quad \forall v \in V, \tag{2.7}$$

where $\| \cdot \|_{s,K}$ is the standard Sobolev norm in $H^s(K)$. Define the broken H^2 norm by

$$\|v\|_h = \left(\sum_{K \in \mathcal{T}_h} |v|_{2,K}^2 \right)^{1/2}, \quad \forall v \in V, \tag{2.8}$$

where $| \cdot |_{2,K}$ is the seminorm in $H^2(K)$.

For every $K \in \mathcal{T}_h$ and any $v \in V$, we apply the integration by parts formula to obtain

$$\begin{aligned}
\int_K \left(\Delta^2 u \right) v \, dx &= -\int_K \nabla(\Delta u) \cdot \nabla v \, dx + \int_{\partial K} \frac{\partial \Delta u}{\partial n} v \, ds \\
&= \int_K \Delta u \Delta v \, dx - \int_{\partial K} \Delta u \frac{\partial v}{\partial n} \, ds + \int_{\partial K} \frac{\partial \Delta u}{\partial n} v \, ds.
\end{aligned} \tag{2.9}$$

Summing all $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
&\sum_{K \in \mathcal{T}_h} \int_K \Delta u \Delta v \, dx - \sum_{e \in \mathcal{E}_I} \int_e \{\Delta u\} \left[\frac{\partial v}{\partial n} \right] + [\Delta u] \left\{ \frac{\partial v}{\partial n} \right\} ds + \sum_{e \in \mathcal{E}_I} \int_e \left[\frac{\partial \Delta u}{\partial n} \right] \{v\} \\
&+ \left\{ \frac{\partial \Delta u}{\partial n} \right\} [v] ds - \int_{\partial \Omega} \Delta u \frac{\partial v}{\partial n} \, ds = \int_\Omega f v \, dx.
\end{aligned} \tag{2.10}$$

Since $u \in H^4(\Omega) \cap H_0^2(\Omega)$ and $v \in V$, then $[\Delta u] = [\partial \Delta u/\partial n] = [v] = 0$ on $e \in \mathcal{E}_I$. Thus, the previous identity can be simplified as follows:

$$\sum_{K \in \mathcal{T}_h} \int_K \Delta u \Delta v \, dx - \sum_{e \in \mathcal{E}_I} \int_e \{\Delta u\} \left[\frac{\partial v}{\partial n}\right] ds - \int_{\partial \Omega} \Delta u \frac{\partial v}{\partial n} ds = \int_\Omega f v \, dx. \tag{2.11}$$

Now, we introduce the following two bilinear forms:

$$\begin{aligned}
a_S(u,v) &= \sum_{K \in \mathcal{T}_h} \int_K \Delta u \Delta v \, dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta u\} \left[\frac{\partial v}{\partial n}\right] ds - \sum_{e \in \mathcal{E}} \int_e \{\Delta v\} \left[\frac{\partial u}{\partial n}\right] ds \\
&\quad + \sum_{e \in \mathcal{E}} \frac{\beta}{h_e} \int_e \left[\frac{\partial u}{\partial n}\right] \left[\frac{\partial v}{\partial n}\right] ds, \\
a_{\mathrm{NS}}(u,v) &= \sum_{K \in \mathcal{T}_h} \int_K \Delta u \Delta v \, dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta u\} \left[\frac{\partial v}{\partial n}\right] ds + \sum_{e \in \mathcal{E}} \int_e \{\Delta v\} \left[\frac{\partial u}{\partial n}\right] ds \\
&\quad + \sum_{e \in \mathcal{E}} \frac{1}{h_e} \int_e \left[\frac{\partial u}{\partial n}\right] \left[\frac{\partial v}{\partial n}\right] ds.
\end{aligned} \tag{2.12}$$

It is clear that $a_S(\cdot,\cdot)$ is a symmetric bilinear form and $a_{\mathrm{NS}}(\cdot,\cdot)$ is a nonsymmetric bilinear form. In terms of (2.11) and

$$\sum_{e \in \mathcal{E}_I} \int_e \{\Delta v\} \left[\frac{\partial u}{\partial n}\right] ds + \int_{\partial \Omega} \Delta v \frac{\partial u}{\partial n} ds = 0, \tag{2.13}$$

the solution u to problem (2.1) satisfies the following variational problems:

$$a_S(u,v) = (f,v), \quad \forall v \in V, \tag{SP}$$

$$a_{NS}(u,v) = (f,v), \quad \forall v \in V. \tag{NSP}$$

Let $P_r(K)$ denote the space of the polynomials on K of degree at most r. Define the following constrained C^0 finite element space:

$$V_h = \left\{ v_h \in H_0^1(\Omega), v_h|_K \in P_r(K), r \geqslant 4, \forall K \in \mathcal{T}_h, \int_e \left[\frac{\partial v_h}{\partial n}\right] ds = 0, \forall e \in \mathcal{E} \right\}, \tag{2.14}$$

from which we note that the C^1 continuity of $v_h \in V_h$ can be weakly achieved by the constrained condition $\int_e [\partial v_h/\partial n] ds = 0$ for all $e \in \mathcal{E}$. Next, we define the degrees of freedom for this finite element space. To this end, for any $K \in \mathcal{T}_h$, denote by p_i $(i = 1,2,3)$ the three

vertices of K. Recall that the degrees of freedom of Lagrange element on K are $v(p)$, for all $p \in C'$ with (cf. [10])

$$C' := \left\{ p = \sum_{j=1}^{3} \lambda_j p_j; \ \sum_{j=1}^{3} \lambda_j = 1, \ \lambda_j \in \left\{ 0, \frac{1}{r}, \dots, \frac{r-1}{r}, 1 \right\}, \ 1 \leqslant j \leqslant 3 \right\}. \tag{2.15}$$

Then we modify the degrees of freedom of Lagrange element to suit the constraint of normal derivatives over the edges in V_h. Specifically speaking, the degrees of freedom of V_h are given by

$$v(p), \quad \forall p \in C \text{ with } C := C' \setminus \left\{ \frac{1}{r}(p_1 + p_2 + p_3) + \frac{r-3}{r} p_i, \ i = 1,2,3 \right\},$$

$$\int_e \frac{\partial v}{\partial n} ds \quad \text{for each edge } e \text{ of } K. \tag{2.16}$$

Based on the symmetric bilinear form $a_S(\cdot, \cdot)$, the C^0 symmetric interior penalty finite element approximation of (2.1) is

$$\text{find } u_h^S \in V_h \text{ such that,}$$

$$a_S\left(u_h^S, v_h\right) = (f, v_h), \quad \forall v_h \in V_h. \tag{2.17}$$

Based on the nonsymmetric bilinear form $a_{NS}(\cdot, \cdot)$, the C^0 nonsymmetric interior penalty finite element approximation of (2.1) is

$$\text{find } u_h^{NS} \in V_h \text{ such that,}$$

$$a_{NS}\left(u_h^{NS}, v_h\right) = (f, v_h), \quad \forall v_h \in V_h. \tag{2.18}$$

Moreover, the following orthogonal equations hold:

$$a_S\left(u_h^S - u, v_h\right) = 0, \quad \forall v_h \in V_h, \tag{2.19}$$

$$a_{NS}\left(u_h^{NS} - u, v_h\right) = 0, \quad \forall v_h \in V_h. \tag{2.20}$$

In order to introduce a global interpolation operator, we first define $\phi_h^i \in P_r(K_i)$ for $\phi \in H^s(K_i)$ and $K_i \in \mathcal{T}_h$ according to the degrees of freedom of V_h by

$$\phi_h^i(p) = \phi(p), \quad \text{if } p \in C \setminus \partial\Omega, \qquad \phi_h^i(p) = 0, \quad \text{if } p \in C \cap \partial\Omega,$$

$$\int_e \frac{\partial \phi_h^i}{\partial n} ds = \int_e \frac{\partial \phi}{\partial n} ds, \quad \text{if edge } e \subset \partial K_i \setminus \partial\Omega, \qquad \int_e \frac{\partial \phi_h^i}{\partial n} ds = 0, \quad \text{if edge } e \subset \partial K_i \cap \partial\Omega.$$

$$\tag{2.21}$$

Due to standard scaling argument and Sobolev embedding theorem (cf. [10]), we have that for every $K_i \in \mathcal{T}_h$, $e \in \partial K_i$, $\phi \in H^s(K_i)$ with $s \geqslant 2$, $i = 1, \ldots, N$,

$$\left\| \phi - \phi_h^i \right\|_{q, K_i} \leqslant ch^{\mu - q} \|\phi\|_{s, K_i}, \quad 0 \leqslant q \leqslant \mu,$$

$$\left\| \phi - \phi_h^i \right\|_{m, e} \leqslant ch_e^{\mu - 1/2 - m} \|\phi\|_{s, K_i}, \quad m = 0, 1, 2, \tag{2.22}$$

$$\left\| \left[\frac{\partial (\phi - \phi_h^i)}{\partial n} \right] \right\|_{0, e} \leqslant ch_e^{\mu - 3/2} \|\phi\|_{s, K_i},$$

where $\mu = \min\{s, r + 1\}$ and $c > 0$ is independent of h. We also suppose that the following inverse inequalities hold:

$$\left\| \phi_h^i \right\|_{0, e} \leqslant ch_e^{-1/2} \left\| \phi_h^i \right\|_{0, K_i}, \qquad \left\| \phi_h^i \right\|_{1, e} \leqslant ch_e^{-1/2} \left\| \phi_h^i \right\|_{1, K_i}, \qquad \left\| \phi_h^i \right\|_{1, K_i} \leqslant ch^{-1} \left\| \phi_h^i \right\|_{0, K_i}, \tag{2.23}$$

where $c > 0$ is independent of h. Then for every $\phi \in V$, we define the global interpolation operator $P_h : V \to V_h$ by $P_h \phi|_{K_i} = \phi_h^i$. Moreover, from (2.22) there holds

$$\left\| \phi - P_h \phi \right\|_h \leqslant ch^{\mu - 2} \|\|\phi\|\|_s, \tag{2.24}$$

where $c > 0$ is independent of h.

The following lemma is useful to establish the existence and uniqueness of the finite element approximation solution.

Lemma 2.1. *There exists some constant $\kappa_0 > 0$ independent of h such that*

$$\kappa_0 \|v_h\|_h^2 \leqslant \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0, K}^2 + \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0, e}^2, \quad \forall v_h \in V_h. \tag{2.25}$$

Proof. Introduce a H_0^2 conforming finite element space Z_h thanks to Guzmán and Neilan [11]. The advantage of Z_h is that the degrees of freedom depend only on the values of functions and their first derivatives. Denote by L_h the interpolation operator from V_h to Z_h. Then there holds

$$\sum_{K \in \mathcal{T}_h} |v_h - L_h v_h|_{2, K}^2 \leqslant C_1 \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0, e}^2, \quad \forall v_h \in V_h, \tag{2.26}$$

where $C_1 > 0$ is independent of h. Thus, we have

$$
\begin{aligned}
\|v_h\|_h^2 &\leqslant \sum_{K \in \mathcal{T}_h} |v_h - L_h v_h|_{2,K}^2 + \sum_{K \in \mathcal{T}_h} |L_h v_h|_{2,K}^2 \leqslant C_1 \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 + C_2 \sum_{K \in \mathcal{T}_h} \|\Delta(L_h v_h)\|_{0,K}^2 \\
&\leqslant C_1 \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 + C_2 \sum_{K \in \mathcal{T}_h} \|\Delta(L_h v_h - v_h)\|_{0,K}^2 + C_2 \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 \\
&\leqslant C_1 \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 + C_2 \sum_{K \in \mathcal{T}_h} |L_h v_h - v_h|_{2,K}^2 + C_2 \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 \\
&\leqslant C_1(1 + C_2) \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 + C_2 \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2,
\end{aligned}
\tag{2.27}
$$

which completes the proof of (2.25) with $\kappa_0 = 1/\max\{C_1(1 + C_2), C_2\}$. $\qquad\square$

3. C^0 Symmetric Interior Penalty Method

In this section, we will show the optimal error estimates in the broken H^2 norm and in the L^2 norm between the solution u to problem (2.1) and the solution u_h^S to the problem (2.17). First, concerning the symmetric $a_S(\cdot, \cdot)$, we have the following coercive property in V_h.

Lemma 3.1. *For sufficiently large β, there exists some constant $\kappa_1 > 0$ such that*

$$
a_S(v_h, v_h) \geqslant \kappa_1 \|v_h\|_h^2, \quad \forall v_h \in V_h.
\tag{3.1}
$$

Proof. According to the definition of $a_S(\cdot, \cdot)$, we have

$$
a_S(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K |\Delta v_h|^2 dx - 2 \sum_{e \in \mathcal{E}} \int_e \{\Delta v_h\} \left[\frac{\partial v_h}{\partial n} \right] ds + \sum_{e \in \mathcal{E}} \frac{\beta}{h_e} \int_e \left| \left[\frac{\partial v_h}{\partial n} \right] \right|^2 ds.
\tag{3.2}
$$

Using the Hölder's inequality and the Young's inequality, we have

$$
\begin{aligned}
\sum_{e \in \mathcal{E}} \int_e \{\Delta v_h\} \left[\frac{\partial v_h}{\partial n} \right] ds &\leqslant \left(\sum_{e \in \mathcal{E}} h_e \|\{\Delta v_h\}\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} h_e^{-1} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2} \\
&\leqslant c \left(\sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} h_e^{-1} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2} \\
&\leqslant \varepsilon \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 + c\varepsilon^{-1} \sum_{e \in \mathcal{E}} h_e^{-1} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2,
\end{aligned}
\tag{3.3}
$$

where $c > 0$ is independent of h and $\varepsilon > 0$ is a sufficiently small constant. Thus

$$a_S(v_h, v_h) \geqslant \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 - 2\varepsilon \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 - c\varepsilon^{-1} \sum_{e \in \mathcal{E}} h_e^{-1} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2$$

$$+ \sum_{e \in \mathcal{E}} \frac{\beta}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 ds. \tag{3.4}$$

Taking $\varepsilon = 1/4$, then, for sufficiently large β such that $\beta > 4c$, using (2.25) we have

$$a_S(v_h, v_h) \geqslant \frac{1}{2} \sum_{K \in \mathcal{T}_h} \|\Delta v_h\|_{0,K}^2 + (\beta - 4c) \sum_{e \in \mathcal{E}} h_e^{-1} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \geqslant \kappa_1 \|v_h\|_h^2, \tag{3.5}$$

with $\kappa_1 = \min\{1/2, \beta - 4c\}\kappa_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A direct result of Lemma 3.1 is that the discretized problem (2.17) admits a unique solution $u_h^S \in V_h$ for sufficiently large β.

Lemma 3.2. *For all $\phi \in V$, there holds*

$$a_S(\phi - P_h\phi, v_h) \leqslant ch^{\mu-2} \|v_h\|_h \||\phi\||_s, \quad \forall v_h \in V_h, \tag{3.6}$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Proof. For all $\phi \in V$ and $v_h \in V_h$, we have

$$a_S(\phi - P_h\phi, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \Delta(\phi - P_h\phi) \Delta v_h dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta(\phi - P_h\phi)\} \left[\frac{\partial v_h}{\partial n} \right] ds$$

$$- \sum_{e \in \mathcal{E}} \int_e \{\Delta v_h\} \left[\frac{\partial(\phi - P_h\phi)}{\partial n} \right] ds + \sum_{e \in \mathcal{E}} \frac{\beta}{h_e} \int_e \left[\frac{\partial(\phi - P_h\phi)}{\partial n} \right] \left[\frac{\partial v_h}{\partial n} \right] ds$$

$$\leqslant \|\phi - P_h\phi\|_h \|v_h\|_h + \left(\sum_{e \in \mathcal{E}} \|\{\Delta(\phi - P_h\phi)\}\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2}$$

$$+ \left(\sum_{e \in \mathcal{E}} h_e \|\{\Delta v_h\}\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} h_e^{-1} \left\| \left[\frac{\partial(\phi - P_h\phi)}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2}$$

$$+ c \left(\sum_{e \in \mathcal{E}} h_e^{-2} \left\| \left[\frac{\partial(\phi - P_h\phi)}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2},$$

$$\tag{3.7}$$

where $c > 0$ is independent of h. Note that $\int_e [(\partial v_h)/\partial n] ds = 0$ for $e \in \mathcal{E}$. Thus, for some constant C^e depending on e there holds

$$\int_e \left|\left[\frac{\partial v_h}{\partial n}\right]\right|^2 ds = \int_e \left[\frac{\partial v_h}{\partial n}\right]\left[\frac{\partial v_h}{\partial n} - C^e\right] ds \leqslant \left(\int_e \left|\left[\frac{\partial v_h}{\partial n}\right]\right|^2 ds\right)^{1/2}\left(\int_e \left|\left[\frac{\partial v_h}{\partial n} - C^e\right]\right|^2 ds\right)^{1/2}.$$
(3.8)

That is

$$\int_e \left|\left[\frac{\partial v_h}{\partial n}\right]\right|^2 ds \leqslant \int_e \left|\left[\frac{\partial v_h}{\partial n} - C^e\right]\right|^2 ds.$$
(3.9)

From [9], we have

$$\left\|\left[\frac{\partial v_h}{\partial n} - C^e\right]\right\|_{0,e} \leqslant ch^{1/2}|v_h|_{2,K},$$
(3.10)

where $c > 0$ is independent of h. Thus

$$\left(\sum_{e \in \mathcal{E}}\left\|\left[\frac{\partial v_h}{\partial n}\right]\right\|_{0,e}^2\right)^{1/2} \leqslant ch^{1/2}\|v_h\|_h.$$
(3.11)

Substituting (3.11) into (3.7) and using (2.22)-(2.23) give

$$a_S(\phi - P_h\phi, v_h) \leqslant \|\phi - P_h\phi\|_h\|v_h\|_h + ch^{1/2}\left(\sum_{e \in \mathcal{E}}\|\{\Delta(\phi - P_h\phi)\}\|_{0,e}^2\right)^{1/2}\|v_h\|_h$$

$$+ c\left(\sum_{e \in \mathcal{E}}h_e^{-1}\left\|\left[\frac{\partial(\phi - P_h\phi)}{\partial n}\right]\right\|_{0,e}^2\right)^{1/2}\|v_h\|_h$$

$$+ ch^{1/2}\left(\sum_{e \in \mathcal{E}}h_e^{-2}\left\|\left[\frac{\partial(\phi - P_h\phi)}{\partial n}\right]\right\|_{0,e}^2\right)^{1/2}\|v_h\|_h \leqslant ch^{\mu-2}\||\phi\||_s\|v_h\|_h.$$
(3.12)

$\square$

Theorem 3.3. *Suppose that $u \in V$ and $u_h^S \in V_h$ are the solutions to problems (SP) and (2.17), respectively; then the following optimal broken H^2 error estimate holds:*

$$\left\|u - u_h^S\right\|_h \leqslant ch^{\mu-2}\||u\||_s,$$
(3.13)

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Proof. According to Lemma 3.1, we have

$$
\kappa_1 \left\| P_h u - u_h^S \right\|_h^2 \leqslant a_S\left(P_h u - u_h^S, P_h u - u_h^S \right) = a_S\left(P_h u - u, P_h u - u_h^S \right) + a_S\left(u - u_h^S, P_h u - u_h^S \right)
$$

$$
= a_S\left(P_h u - u, P_h u - u_h^S \right)
$$

$$
\leqslant ch^{\mu-2} \|\|u\|\|_s \left\| P_h u - u_h^S \right\|_h,
$$

(3.14)

where we use the orthogonal equation (2.19) and Lemma 3.2. The previous estimate implies

$$
\left\| P_h u - u_h^S \right\|_h \leqslant ch^{\mu-2} \|\|u\|\|_s.
$$

(3.15)

Finally, the triangular inequality and (2.24) yield

$$
\left\| u - u_h^S \right\|_h \leqslant \| u - P_h u \|_h + \left\| P_h u - u_h^S \right\|_h \leqslant ch^{\mu-2} \|\|u\|\|_s.
$$

(3.16)

□

Next, we will show the optimal L^2 error estimate in terms of the duality technique. Suppose $g \in L^2(\Omega)$ and consider the following biharmonic problem:

$$
\Delta^2 w = g, \quad \text{in } \Omega,
$$
$$
w = \frac{\partial w}{\partial n} = 0, \quad \text{on } \partial\Omega.
$$

(3.17)

Suppose that problem (3.17) admits a unique solution $w \in H_0^2(\Omega) \cap H^4(\Omega)$ such that

$$
\|w\|_4 \leqslant c \|g\|,
$$

(3.18)

where $\| \cdot \|_4$ denotes the H^4 norm in Ω and $\| \cdot \|$ denotes the L^2 norm in Ω and $c > 0$ is independent of h.

Denote by Π_h the C^1 continuous interpolate operator from V to $H_0^2(\Omega) \cap V_h$, and Π_h satisfies the approximation property (2.22). Then for the solution $w \in H^4(\Omega) \cap H_0^2(\Omega)$ to problem (3.17), there hold

$$
\|w - \Pi_h w\|_2 \leqslant ch^2 \|w\|_4 \leqslant ch^2 \|g\|,
$$
$$
\|w - \Pi_h w\|_{2,e} \leqslant ch_e^{3/2} \|w\|_4 \leqslant ch_e^{3/2} \|g\|, \quad \forall e \in \mathcal{E},
$$

(3.19)

where $c > 0$ is independent of h.

Theorem 3.4. *Suppose that $u \in V$ and $u_h^S \in V_h$ are the solutions to problems (SP) and (2.17), respectively; then the following optimal L^2 error estimate holds:*

$$\left\| u - u_h^S \right\| \leqslant ch^\mu \|\|u\|\|_s, \tag{3.20}$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Proof. Setting $g = u - u_h^S$ in (3.17), multiplying (3.17) by $u - u_h^S$, and integrating over Ω, we have

$$
\begin{aligned}
\left\| u - u_h^S \right\|^2 &= \int_\Omega \left(\Delta^2 w\right)\left(u - u_h^S\right) dx = a_S\left(w, u - u_h^S\right) = a_S\left(w - \Pi_h w, u - u_h^S\right) \\
&= a_S\left(w - \Pi_h w, P_h u - u_h^S\right) + a_S(w - \Pi_h w, u - P_h u),
\end{aligned}
\tag{3.21}
$$

where we use the orthogonal equation (2.19). We estimate two terms in the right-hand side of (3.21) as follows:

$$
\begin{aligned}
&a_S\left(w - \Pi_h w, P_h u - u_h^S\right) \\
&\quad = \sum_{K \in \mathcal{T}_h} \int_K \Delta(w - \Pi_h w)\Delta\left(P_h u - u_h^S\right) dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta(w - \Pi_h w)\}\left[\frac{\partial(P_h u - u_h^S)}{\partial n}\right] ds \\
&\qquad - \sum_{e \in \mathcal{E}} \int_e \left\{\Delta\left(P_h u - u_h^S\right)\right\}\left[\frac{\partial(w - \Pi_h w)}{\partial n}\right] ds + \sum_{e \in \mathcal{E}} \frac{\beta}{h_e} \int_e \left[\frac{\partial(w - \Pi_h w)}{\partial n}\right]\left[\frac{\partial(P_h u - u_h^S)}{\partial n}\right] ds \\
&\quad = \sum_{K \in \mathcal{T}_h} \int_K \Delta(w - \Pi_h w)\Delta\left(P_h u - u_h^S\right) dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta(w - \Pi_h w)\}\left[\frac{\partial(P_h u - u_h^S)}{\partial n}\right] ds \\
&\quad \leqslant \|w - \Pi_h w\|_2 \left\| P_h u - u_h^S \right\|_h + \left(\sum_{e \in \mathcal{E}} \|w - \Pi_h w\|_{2,e}^2\right)^{1/2}\left(\sum_{e \in \mathcal{E}} \left\| P_h u - u_h^S \right\|_{1,e}^2\right)^{1/2} \\
&\quad \leqslant ch^2 \left\| P_h u - u_h^S \right\|_h \left\| u - u_h^S \right\| \leqslant ch^\mu \|\|u\|\|_s \left\| u - u_h^S \right\|,
\end{aligned}
\tag{3.22}
$$

where we use the estimate (3.15). In terms of the inequalities (2.22)-(2.23), we have

$$
\begin{aligned}
&a_S(w - \Pi_h w, u - P_h u) \\
&\quad = \sum_{K \in \mathcal{T}_h} \int_K \Delta(w - \Pi_h w)\Delta(u - P_h u) dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta(w - \Pi_h w)\}\left[\frac{\partial(u - P_h u)}{\partial n}\right] ds \\
&\qquad - \sum_{e \in \mathcal{E}} \int_e \{\Delta(u - P_h u)\}\left[\frac{\partial(w - \Pi_h w)}{\partial n}\right] ds + \sum_{e \in \mathcal{E}} \frac{\beta}{h_e} \int_e \left[\frac{\partial(w - \Pi_h w)}{\partial n}\right]\left[\frac{\partial(u - P_h u)}{\partial n}\right] ds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{K \in \tau_h} \int_K \Delta(w - \Pi_h w)\Delta(u - P_h u)dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta(w - \Pi_h w)\}\left[\frac{\partial(u - P_h u)}{\partial n}\right]ds \\
&\leqslant \|w - P_h w\|_2 \|u - P_h u\|_h + \left(\sum_{e \in \mathcal{E}}\|w - \Pi_h w\|_{2,e}^2\right)^{1/2}\left(\sum_{e \in \mathcal{E}}\left\|\left[\frac{\partial(u - P_h u)}{\partial n}\right]\right\|_{0,e}^2\right)^{1/2} \\
&\leqslant ch^\mu \||u\||_s \left\|u - u_h^S\right\|.
\end{aligned}
\tag{3.23}
$$

Substituting the estimates (3.22)-(3.23) into (3.21) yields

$$
\left\|u - u_h^S\right\| \leqslant ch^\mu \||u\||_s.
\tag{3.24}
$$

$\square$

4. C^0 Nonsymmetric Interior Penalty Method

In this section, we will show the error estimates in the broken H^2 norm and in L^2 norm between the solution u to problem (2.1) and the solution u_h^{NS} to the problem (2.18). The optimal broken H^2 error estimate is derived. However, the L^2 error estimate is suboptimal because of the lack of adjoint consistency. According to Lemma 2.1, we have

$$
a_{\mathrm{NS}}(v_h, v_h) \geqslant \kappa_0 \|v_h\|_h^2, \quad \forall v_h \in V_h.
\tag{4.1}
$$

Moreover, for the nonsymmetric bilinear form $a_{\mathrm{NS}}(\cdot, \cdot)$, proceeding as in the proof of Lemma 3.2, we have the following lemma.

Lemma 4.1. *For all $\phi \in V$, there holds*

$$
a_{\mathrm{NS}}(\phi - P_h \phi, v_h) \leqslant ch^{\mu-2}\|v_h\|_h \||\phi\||_s, \quad \forall v_h \in V_h,
\tag{4.2}
$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Theorem 4.2. *Suppose that $u \in V$ and $u_h^{\mathrm{NS}} \in V_h$ are the solutions to problems (NSP) and (2.18), respectively; then there holds*

$$
\left\|u - u_h^{\mathrm{NS}}\right\|_h \leqslant ch^{\mu-2}\||u\||_s,
\tag{4.3}
$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Proof. According to (2.20), (4.1), and Lemma 4.1, we have

$$
\begin{aligned}
\kappa_0 \left\| P_h u - u_h^{\mathrm{NS}} \right\|_h^2 &\leqslant a_{\mathrm{NS}}\left(P_h u - u_h^{\mathrm{NS}}, P_h u - u_h^{\mathrm{NS}}\right) \\
&= a_{\mathrm{NS}}\left(P_h u - u, P_h u - u_h^{\mathrm{NS}}\right) + a_{\mathrm{NS}}\left(u - u_h^{\mathrm{NS}}, P_h u - u_h^{\mathrm{NS}}\right) \qquad (4.4) \\
&= a_{\mathrm{NS}}\left(P_h u - u, P_h u - u_h^{\mathrm{NS}}\right) \leqslant c h^{\mu-2} \|\|u\|\|_s \left\| P_h u - u_h^{\mathrm{NS}} \right\|_h,
\end{aligned}
$$

where $c > 0$ is independent of h. That is

$$
\left\| P_h u - u_h^{\mathrm{NS}} \right\|_h \leqslant c h^{\mu-2} \|\|u\|\|_s. \qquad (4.5)
$$

Using the triangular inequality yields

$$
\left\| u - u_h^{\mathrm{NS}} \right\|_h \leqslant \|u - P_h u\|_h + \left\| P_h u - u_h^{\mathrm{NS}} \right\|_h \leqslant c h^{\mu-2} \|\|u\|\|_s. \qquad (4.6)
$$

$\square$

Theorem 4.3. *Suppose that $u \in V$ and $u_h^{\mathrm{NS}} \in V_h$ are the solutions to problems (NSP) and (2.18), respectively; then there holds*

$$
\left\| u - u_h^{\mathrm{NS}} \right\| \leqslant c h^{\mu-1} \|\|u\|\|_s, \qquad (4.7)
$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Proof. Setting $g = u - u_h^{\mathrm{NS}}$ in (3.17), multiplying (3.17) by $u - u_h^{\mathrm{NS}}$, and integrating over Ω, we have

$$
\begin{aligned}
\left\| u - u_h^{\mathrm{NS}} \right\|^2 &= \int_\Omega \left(\Delta^2 w\right)\left(u - u_h^{\mathrm{NS}}\right) dx = a_{\mathrm{NS}}\left(w, u - u_h^{\mathrm{NS}}\right) \\
&= a_{\mathrm{NS}}\left(w - \Pi_h w, u - u_h^{\mathrm{NS}}\right) + 2\sum_{e \in \mathcal{E}} \int_e \{\Delta \Pi_h w\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n} \right] ds \\
&\quad - 2\sum_{e \in \mathcal{E}} \int_e \left\{ \Delta\left(u - u_h^{\mathrm{NS}}\right) \right\} \left[\frac{\partial \Pi_h w}{\partial n} \right] ds \\
&= a_{\mathrm{NS}}\left(w - \Pi_h w, u - u_h^{\mathrm{NS}}\right) + 2\sum_{e \in \mathcal{E}} \int_e \{\Delta \Pi_h w\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n} \right] ds
\end{aligned}
$$

$$
= \sum_{K \in \tau_h} \int_K \Delta(w - \Pi_h w) \Delta\left(u - u_h^{\mathrm{NS}}\right) dx
$$

$$
+ 3 \sum_{e \in \mathcal{E}} \int_e \{\Delta(\Pi_h w - w)\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds + 2 \sum_{e \in \mathcal{E}} \int_e \{\Delta w\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds
$$

$$
= I_1 + I_2 + I_3,
$$

$$(4.8)$$

where we use $\Pi_h w \in H_0^2(\Omega)$. We estimate I_1 as follows:

$$
I_1 = \sum_{K \in \tau_h} \int_K \Delta(w - \Pi_h w) \Delta\left(u - u_h^{\mathrm{NS}}\right) dx \leqslant \left\|u - u_h^{\mathrm{NS}}\right\|_h \|w - \Pi_h w\|_2
$$

$$
\leqslant c h^\mu \|\|u\|\|_s \left\|u - u_h^{\mathrm{NS}}\right\|.
$$

$$(4.9)$$

We estimate I_2 as follows:

$$
I_2 = 3 \sum_{e \in \mathcal{E}} \int_e \{\Delta(\Pi_h w - w)\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds
$$

$$
= 3 \sum_{e \in \mathcal{E}} \int_e \{\Delta(\Pi_h w - w)\} \left[\frac{\partial(u - P_h u)}{\partial n}\right] ds + 3 \sum_{e \in \mathcal{E}} \int_e \{\Delta(\Pi_h w - w)\} \left[\frac{\partial\left(P_h u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds
$$

$$
\leqslant c \left(\sum_{e \in \mathcal{E}} \|w - \Pi_h w\|_{2,e}^2\right)^{1/2} \left(\sum_{e \in \mathcal{E}} \left\|\left[\frac{\partial(u - P_h u)}{\partial n}\right]\right\|_{0,e}^2\right)^{1/2}
$$

$$
+ \left(\sum_{e \in \mathcal{E}} \|w - \Pi_h w\|_{2,e}^2\right)^{1/2} \left(\sum_{e \in \mathcal{E}} \left\|\left[\frac{\partial\left(P_h u - u_h^{\mathrm{NS}}\right)}{\partial n}\right]\right\|_{0,e}^2\right)^{1/2}
$$

$$
\leqslant c h^{3/2} \left(h^{\mu - 3/2} \|\|u\|\|_s + c h^{1/2} \left\|P_h u - u_h^{\mathrm{NS}}\right\|_h\right) \left\|u - u_h^{\mathrm{NS}}\right\| \leqslant c h^\mu \|\|u\|\|_s \left\|u - u_h^{\mathrm{NS}}\right\|.
$$

$$(4.10)$$

We estimate I_3 as follows:

$$
I_3 = 2 \sum_{e \in \mathcal{E}} \int_e \{\Delta w\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds = 2 \sum_{e \in \mathcal{E}} \int_e \{\Delta w - C^e\} \left[\frac{\partial\left(u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds
$$

$$
= 2 \sum_{e \in \mathcal{E}} \int_e \{\Delta w - C^e\} \left[\frac{\partial(\mathrm{u} - P_h u)}{\partial n}\right] ds + 2 \sum_{e \in \mathcal{E}} \int_e \{\Delta w - C^e\} \left[\frac{\partial\left(P_h u - u_h^{\mathrm{NS}}\right)}{\partial n}\right] ds
$$

$$\leqslant c \left(\sum_{e \in \mathcal{E}} \|\Delta w - C^e\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} \|u - P_h u\|_{1,e}^2 \right)^{1/2}$$

$$+ \left(\sum_{e \in \mathcal{E}} \|\Delta w - C^e\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} \left\| \left[\!\!\left[\frac{\partial \left(P_h u - u_h^{\mathrm{NS}} \right)}{\partial n} \right]\!\!\right] \right\|_{0,e}^2 \right)^{1/2}$$

$$\leqslant c \left(\sum_{e \in \mathcal{E}} \|\Delta w - C^e\|_{0,e}^2 \right)^{1/2} \left(h^{\mu-3/2} \|\!|u|\!\|_s + c h^{1/2} \left\| P_h u - u_h^{\mathrm{NS}} \right\|_h \right)$$

$$\leqslant c h^{\mu-3/2} \left(\sum_{e \in \mathcal{E}} \|\Delta w - C^e\|_{0,e}^2 \right)^{1/2} \|\!|u|\!\|_s,$$

$$(4.11)$$

where C^e is some positive constant. Substituting the following estimate

$$\sum_{e \in \mathcal{E}} \|\Delta w - C^e\|_{0,e}^2 \leqslant c \sum_{e \in \mathcal{E}} h_e \|w\|_{3,e}^2 \leqslant c h \|w\|_4^2 \leqslant c h \left\| u - u_h^{\mathrm{NS}} \right\|^2 \qquad (4.12)$$

into (4.11) gives

$$I_3 \leqslant c h^{\mu-1} \|\!|u|\!\|_s \left\| u - u_h^{\mathrm{NS}} \right\|. \qquad (4.13)$$

Finally, substituting (4.9)–(4.13) into (4.8), we obtain

$$\left\| u - u_h^{\mathrm{NS}} \right\| \leqslant c h^{\mu-1} \|\!|u|\!\|_s. \qquad (4.14)$$

$\square$

5. C^0 Superpenalty Nonsymmetric Method

In order to obtain the optimal L^2 error estimate for the nonsymmetric method, in this section we will consider the C^0 superpenalty nonsymmetric method. First, we introduce a new nonsymmetric bilinear form:

$$a_{\mathrm{SNS}}(\mathrm{u}, v) = \sum_{K \in \mathcal{T}_h} \int_K \Delta u \Delta v \, dx - \sum_{e \in \mathcal{E}} \int_e \{\Delta u\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds + \sum_{e \in \mathcal{E}} \int_e \{\Delta v\} \left[\!\!\left[\frac{\partial u}{\partial n} \right]\!\!\right] ds$$

$$+ \sum_{e \in \mathcal{E}} \frac{1}{h_e^3} \int_e \left[\!\!\left[\frac{\partial u}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right].$$

$$(5.1)$$

The broken H^2 norm is modified to

$$\||v_h\||_h = \left(\sum_{K \in \tau_h} |v_h|_{2,K_i}^2 + \sum_{e \in \mathcal{E}} \frac{1}{h_e^3} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2}, \quad \forall v_h \in V_h. \tag{5.2}$$

From Lemma 2.1, it is easy to show that there exists some constant $\kappa_2 > 0$ such that

$$a_{\mathrm{SNS}}(v_h, v_h) \geqslant \kappa_2 \||v_h\||_h^2, \quad \forall v_h \in V_h. \tag{5.3}$$

In fact, we have

$$
\begin{aligned}
a_{\mathrm{SNS}}(v_h, v_h) &= \sum_{K \in \tau_h} \|\Delta v_h\|_{0,K}^2 + \sum_{e \in \mathcal{E}} \frac{1}{h_e^3} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \\
&\geqslant \frac{1}{2} \sum_{e \in \mathcal{E}} \frac{1}{h_e^3} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 + \frac{1}{2} \left(\sum_{K \in \tau_h} \|\Delta v_h\|_{0,K}^2 + \sum_{e \in \mathcal{E}} \frac{1}{h_e} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 \right) \\
&\geqslant \frac{1}{2} \sum_{e \in \mathcal{E}} \frac{1}{h_e^3} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_{0,e}^2 + \frac{\kappa_0}{2} \sum_{K \in \tau_h} |v_h|_{2,K}^2 \geqslant \kappa_2 \||v_h\||_h^2
\end{aligned}
\tag{5.4}
$$

for $\kappa_2 = \min\{1/2, \kappa_0/2\}$. Since the solution u to problem (2.1) belongs to $H^4(\Omega) \cap H_0^2(\Omega)$, then it satisfies

$$a_{\mathrm{SNS}}(u, v) = (f, v), \quad \forall v \in V. \tag{SNSP}$$

In this case, the the C^0 superpenalty nonsymmetric finite element approximation of (2.1) is

$$
\begin{aligned}
&\text{find } u_h^{\mathrm{SNS}} \in V_h \text{ such that,} \\
&a_{\mathrm{SNS}}\left(u_h^{\mathrm{SNS}}, v_h \right) = (f, v_h), \quad \forall v_h \in V_h.
\end{aligned}
\tag{5.5}
$$

Then, we have the following orthogonal equation:

$$a_{\mathrm{SNS}}\left(u_h^{\mathrm{SNS}} - u, v_h \right) = 0, \quad \forall v_h \in V_h. \tag{5.6}$$

Let Π_h be the C^1 continuous interpolated operator defined in Section 3. Observe that $[(\partial u - \Pi_h u)/\partial n] = 0$ on every $e \in \mathcal{E}$, and proceeding as in the proof of Lemmas 3.2 and 4.1, we have the following.

Lemma 5.1. *For all $\phi \in V$, there holds*

$$a_{\mathrm{NS}}(\phi - \Pi_h \phi, v_h) \leqslant ch^{\mu-2} \||v_h\||_h \||\phi\||_s, \quad \forall v_h \in V_h, \tag{5.7}$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Using Lemma 5.1, the following optimal broken H^2 error estimate holds.

Theorem 5.2. *Suppose that $u \in V$ and $u_h^{\mathrm{SNS}} \in V_h$ are the solutions to problems (SNSP) and (5.5), respectively; then there holds*

$$\left\|\left\|u - u_h^{\mathrm{SNS}}\right\|\right\|_h \leqslant ch^{\mu-2}\|\|u\|\|_s,\tag{5.8}$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

The main result in this section is the following optimal L^2 error estimate.

Theorem 5.3. *Suppose that $u \in V$ and $u_h^{\mathrm{SNS}} \in V_h$ are the solutions to problems (SNSP) and (5.5), respectively; then there holds*

$$\left\|u - u_h^{\mathrm{SNS}}\right\| \leqslant ch^{\mu}\|\|u\|\|_s,\tag{5.9}$$

where $\mu = \min\{s, r+1\}$ and $c > 0$ is independent of h.

Proof. Let $g = u - u_h^{\mathrm{SNS}}$. Multiplying (3.17) by $u - u_h^{\mathrm{SNS}}$ and integrating over Ω, we have

$$
\begin{aligned}
\left\|u - u_h^{\mathrm{SNS}}\right\|^2 &= \int_{\Omega} \left(\Delta^2 w\right)\left(u - u_h^{\mathrm{SNS}}\right) dx = a_{\mathrm{SNS}}\left(w, u - u_h^{\mathrm{SNS}}\right) \\[2mm]
&= a_{\mathrm{SNS}}\left(w - \Pi_h w, u - u_h^{\mathrm{SNS}}\right) + 2\sum_{e \in \mathcal{E}} \int_e \{\Delta \Pi_h w\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n}\right] ds \\[2mm]
&\quad - 2\sum_{e \in \mathcal{E}} \int_e \left\{\Delta\left(u - u_h^{\mathrm{SNS}}\right)\right\} \left[\frac{\partial \Pi_h w}{\partial n}\right] ds \\[2mm]
&= a_{\mathrm{SNS}}\left(w - \Pi_h w, u - u_h^{\mathrm{SNS}}\right) + 2\sum_{e \in \mathcal{E}} \int_e \{\Delta \Pi_h w\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n}\right] ds \\[2mm]
&= \sum_{K \in \mathcal{T}_h} \int_K \Delta(w - \Pi_h w) \Delta\left(u - u_h^{\mathrm{SNS}}\right) dx \\[2mm]
&\quad + 3\sum_{e \in \mathcal{E}} \int_e \{\Delta(\Pi_h w - w)\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n}\right] ds + 2\sum_{e \in \mathcal{E}} \int_e \{\Delta w\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n}\right] ds \\[2mm]
&= I_4 + I_5 + I_6,
\end{aligned}
\tag{5.10}
$$

where we use $\Pi_h w \in C^1(\Omega) \cap H_0^2(\Omega)$. Proceeding as in the proof of Theorem 4.2, we can estimate I_4 and I_5 as follows:

$$I_4 = \sum_{K \in \mathcal{T}_h} \int_K \Delta(w - \Pi_h w) \Delta\left(u - u_h^{\mathrm{SNS}}\right) dx \leqslant ch^\mu \interleave u \interleave_s \left\| u - u_h^{\mathrm{SNS}} \right\|, \tag{5.11}$$

$$I_5 = 3\sum_{e \in \mathcal{E}} \int_e \left\{ \Delta(\Pi_h w - w) \right\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n} \right] ds \leqslant ch^\mu \interleave u \interleave_s \left\| u - u_h^{\mathrm{SNS}} \right\|. \tag{5.12}$$

The different estimate compared to Theorem 4.3 is the estimate of I_6. Under the new norm $\interleave \cdot \interleave_h$, we have

$$
\begin{aligned}
I_6 &= 2\sum_{e \in \mathcal{E}} \int_e \left\{ \Delta w \right\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n} \right] ds = 2\sum_{e \in \mathcal{E}} \int_e \left\{ \Delta w - c \right\} \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n} \right] ds \\
&\leqslant c \left(\sum_{e \in \mathcal{E}} h_e^3 \| \Delta w - C \|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} \frac{1}{h_e^3} \left\| \left[\frac{\partial\left(u - u_h^{\mathrm{SNS}}\right)}{\partial n} \right] \right\|_{0,e}^2 \right)^{1/2} \\
&\leqslant ch^2 \left\| u - u_h^{\mathrm{SNS}} \right\| \cdot \interleave u - u_h^{\mathrm{SNS}} \interleave_h \leqslant ch^\mu \interleave u \interleave_s \left\| u - u_h^{\mathrm{SNS}} \right\|.
\end{aligned}
\tag{5.13}
$$

Substituting (5.11)–(5.13) into (5.10) yields the following optimal L^2 error estimate:

$$\left\| u - u_h^{\mathrm{SNS}} \right\| \leqslant ch^\mu \interleave u \interleave_s. \tag{5.14}$$

$\square$

Acknowledgments

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