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Research Article

On the Convergence of Mann and Ishikawa Iterative Processes for Asymptotically ϕ-Strongly Pseudocontractive Mappings

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We prove the equivalence and the strong convergence of (1) the modified Mann iterative process and (2) the modified Ishikawa iterative process for asymptotically ϕ-strongly pseudocontractive mappings in a uniformly smooth Banach space.

1. Introduction

Let X be a Banach space and X^* the dual space of X. Let J denote the normalized duality mapping from X into 2^{X^*} given by $J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$ for all $x \in X$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

In 1972, Goebel and Kirk [1] introduced the class of *asymptotically nonexpansive mappings* as follows.

Definition 1.1. Let K be a subset of a Banach space X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for each $x, y \in K$

$$\left\|T^n x - T^n y\right\| \le k_n \left\|x - y\right\|, \tag{1.1}$$

where $\{k_n\}_n \subset [1,\infty)$ is a sequence of real numbers converging to 1.

Their scope was to extend the well-known Browder fixed point theorem [2] to this class of mappings.

This class is really more general than the class of nonexpansive mappings.

Example 1.2 (see [1]). If B is the unit ball of l^2 and $T : B \to B$ is defined as

$$T(x_1, x_2, \ldots) = \left(0, x_1^2, a_2x_2, a_3x_3, \ldots\right), \tag{1.2}$$

where $\{a_i\}_{i\in\mathbb{N}} \subset (0,1)$ is such that $\prod_{i=2}^{\infty} a_i = 1/2$, it satisfies

$$\|Tx - Ty\| \le 2\|x - y\|, \qquad \|T^n x - T^n y\| \le 2\prod_{i=2}^{n} a_i \|x - y\|. \tag{1.3}$$

In 1973, the same authors [3] proved that the Browder result remains valid for the broader class of *uniformly L-Lipschitzian mappings* (i.e., $\|T^n x - T^n y\| \le L\|x - \mathrm{y}\|$, for every n): every uniformly L-Lipschitzian self-mapping $T : K \to K$ (with $L < \gamma$ and γ sufficiently close to 1) defined on a closed and convex subset of a uniformly convex Banach space has a fixed point.

Taking in to account these papers one can study (1) if a suitable iterative method converges to a fixed point of the mapping and (2) the equivalence of two (given) iterative methods, that is, the first one converges if and only if the second one converges.

In 1991, Schu [4] introduced the class of *asymptotically pseudocontractive mappings*.

Definition 1.3 (see [4]). Let X be a normed space, $K \subset X$, and $\{k_n\}_n \subset [1, \infty)$. A mapping $T : K \to K$ is said to be asymptotically pseudocontractive with the sequence $\{k_n\}_n$ if and only if $\lim_{n\to\infty} k_n = 1$, and for all $n \in \mathbb{N}$ and all $x, y \in K$ there exists $j(x - y) \in J(x - y)$ such that

$$\left\langle T^n x - T^n y, j(x - y)\right\rangle \le k_n\|x - y\|^2, \tag{1.4}$$

where J is the normalized duality mapping.

Obviously every asymptotically nonexpansive mapping is asymptotically pseudocontractive but the converse is not valid: it is well known that $T : [0,1] \to [0,1]$ defined by $Tx = (1 - x^{2/3})^{3/2}$ is not Lipschitzian but asymptotically pseudocontractive [5].

In [4], Schu proved the following.

Theorem 1.4 (see [4]). *Let H be a Hilbert space and $A \subset H$ closed and convex; $L > 0$; $T : A \to A$ completely continuous, uniformly L-Lipschitzian, and asymptotically pseudocontractive with sequence $\{k_n\}_n \in [1, \infty)$; $q_n := 2k_n - 1$ for all $n \in \mathbb{N}$; $\sum_n (q_n^2 - 1) < \infty$; $\{\alpha_n\}_n, \{\beta_n\}_n \in [0,1]$; $\epsilon \le \alpha_n \le \beta_n \le b$ for all $n \in \mathbb{N}$, some $\epsilon > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$; $x_1 \in A$; for all $n \in \mathbb{N}$ define*

$$\begin{aligned} z_n &:= \beta_n T^n(x_n) + (1 - \beta_n)x_n, \\ x_{n+1} &:= \alpha_n T^n(z_n) + (1 - \alpha_n)x_n. \end{aligned} \tag{1.5}$$

Then, $\{x_n\}_n$ converges strongly to some fixed point of T.

From 1991 to 2009 no fixed point theorem for asymptotically pseudocontractive mappings was proved. First Zhou in [6] completed this lack in the setting of Hilbert spaces proving: (1) a fixed point theorem for an asymptotically pseudocontractive mapping that is also uniformly L-Lipschitzian and uniformly asymptotically regular, (2) that the set of fixed points of T is closed and convex, and (3) the strong convergence of a CQ-iterative method. The literature on asymptotical-type mappings is very wide (see, [7–15]).

In 1967, Browder [16] and Kato [17], independently, introduced the accretive operators (see, for details, Chidume [18]). Their interest was connected with the existence of results in the theory of nonlinear equation of evolution in Banach spaces.

In 1974 Deimling [19], studying the zeros of accretive operators, introduced the class of *φ-strongly accretive operators*.

Definition 1.5. An operator A defined on a subset K of a Banach space X is said to be φ-strongly accretive if

$$\langle Ax - Ay, j(x-y)\rangle \geq \varphi(\|x-y\|)\|x-y\|, \tag{1.6}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\varphi(0) = 0$ and $j(x-y) \in J(x-y)$.

Note that in the special case in which $\varphi(t) = kt$, $k \in (0,1)$, we obtain a strongly accretive operator.

However, it is not difficult to prove (see Osilike [20]) that $Ax = x - x/(x+1)$ in $\mathbb{R}^+$ is φ-strongly accretive with $\varphi(s) = (s^2/(1+s))$ but not strongly accretive.

Since an operator A is a strongly accretive operator if and only if $(I-A)$ is a strongly pseudocontractive mapping (i.e., $\langle (I-A)x - (I-A)y, j(x-y)\rangle \leq k\|x-y\|^2,\ k<1$), taking into account Definition 1.5, it is natural to study the class of φ-pseudocontractive mappings, that is, the mappings such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \varphi(\|x-y\|)\|x-y\|, \tag{1.7}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\varphi(0) = 0$. Of course the set of fixed points for this mapping contains, at most, only one point.

Recently, in same papers the following has been introduced.

Definition 1.6. A mapping T is a ϕ-strongly pseudocontractive mapping if

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|), \tag{1.8}$$

where $j(x-y) \in J(x-y)$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\phi(0) = 0$. (In the literature this class is also known as *generalized ϕ-strongly pseudocontractive* according to (1.7). We prefer to not use the term *generalized* because this class is narrower than pseudocontractive mappings.)

Choosing $\phi(t) = \varphi(t)t$, we obtain (1.7). In Xiang's paper [21], it was remarked that it is an open problem if every ϕ-strongly pseudocontractive mapping is φ-pseudocontractive mapping. In the same paper, Xiang obtained a fixed point theorem for continuous and ϕ-strongly pseudocontractive mappings in the setting of the Banach spaces.

In this paper our attention is on the class of the *asymptotically ϕ-strongly pseudocontractive* mappings defined as follows.

Definition 1.7. If X is a Banach space and K is a subset of X, a mapping $T : K \to K$ is said to be asymptotically ϕ-strongly pseudocontractive if

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2 - \phi(\|x-y\|), \tag{1.9}$$

where $j(x-y) \in J(x-y)$, $\{k_n\}_n \subset [1,\infty)$ is converging to one, and $\phi : [0,\infty) \to [0,\infty)$ is strictly increasing such that $\phi(0) = 0$.

One can note that if T has fixed points, then it is unique. In fact if x, z are fixed points for T, then, for every $n \in \mathbb{N}$,

$$\|x-z\|^2 = \langle T^n x - T^n z, j(x,z)\rangle \leq k_n \|x-y\|^2 - \phi(\|x-y\|), \tag{1.10}$$

so, passing n to $+\infty$ results in

$$\|x-z\|^2 \leq \|x-z\|^2 - \phi(\|x-y\|) \Longrightarrow -\phi(\|x-y\|) \geq 0. \tag{1.11}$$

Since $\phi : [0,\infty) \to [0,\infty)$ is strictly increasing and $\phi(0) = 0$ then, $x = z$.

We now give two examples.

Example 1.8. The mapping $Tx = x/(x+1)$, where $x \in [0,1]$, is asymptotically ϕ-strongly pseudocontractive with $k_n = 1$, for all $n \in \mathbb{N}$ and $\phi(t) = t^3/(1+t)$. However, T is not strongly pseudocontractive, see [20].

Example 1.9. The mapping $Tx = x/(1+0.01x)$, where $x \in [0,1]$, is asymptotically ϕ-strongly pseudocontractive with $k_n = (n+100)/n$, for all $n \in \mathbb{N}$ and $\phi(x) = x^3/(1+x)$. However T is not strongly pseudocontractive, nor ϕ-strongly Pseudocontractive.

Proof. First we prove that T is not strongly pseudocontractive. For arbitrary $k < 1$, there exist $x, y \in [0,1]$, such that

$$\frac{1}{(1+0.01x)(1+0.01y)} > k. \tag{1.12}$$

So we have

$$\langle Tx - Ty, j(x-y)\rangle = \frac{1}{(1+0.01x)(1+0.01y)}(x-y)^2 > k\|x-y\|^2. \tag{1.13}$$

Next we prove that T is not ϕ-strongly pseudocontractive. Taking $y = 0$, we have, for all $x \in [0,1]$,

$$\begin{aligned}\langle Tx - Ty, j(x-y)\rangle &= \frac{x^2}{(1+0.01x)},\\ \|x-y\|^2 - \phi(\|x-y\|) &= \frac{x^2}{1+x},\end{aligned} \tag{1.14}$$

therefore, T is not ϕ-strongly pseudocontractive. Finally, we prove that T is asymptotically ϕ-strongly Pseudocontractive.

For arbitrary $x, y \in [0,1]$, without loss of generality, let $x > y$. Then,

$$\begin{aligned}
\langle T^n x - T^n y, j(x-y)\rangle &= \frac{(x-y)^2}{(1+0.01nx)(1+0.01ny)}, \\
k_n\|x-y\|^2 - \phi(\|x-y\|) &= \frac{n+100}{n}(x-y)^2 - \frac{(x-y)^3}{1+(x-y)} \\
&= \left[1 + \frac{100}{n} - \frac{(x-y)}{1+(x-y)}\right](x-y)^2 \\
&= \frac{1+(100/n)+(100/n)(x-y)}{1+(x-y)}(x-y)^2.
\end{aligned} \tag{1.15}$$

We only need to prove that

$$1 + x - y \leq (1+0.01nx)(1+0.01ny)\left[1 + \frac{100}{n} + \frac{100}{n}(x-y)\right]. \tag{1.16}$$

Using $x > y$, this is easy. □

In this paper we study the equivalence between two kinds of iterative methods involving asymptotically ϕ-strongly Pseudocontractive mappings.

Moreover, we prove that, in opportune hypotheses, these methods are equivalent and strongly convergent to the unique fixed point of the asymptotically ϕ-strongly Pseudocontractive T.

2. Preliminaries

Throughout this paper, we will assume that X is a uniformly smooth Banach space. It is well known that if X is uniformly smooth, then the duality mapping J is single-valued and is norm-to-norm uniformly continuous on any bounded subset of X. In the sequel, we will denote the single valued duality mapping by j.

Let us introduce the modified Mann and Ishikawa iterative processes as follows.

For any given $x_0 \in X$, the sequence $\{x_n\}_n$ defined by

$$\begin{aligned}
y_n &= (1-\beta_n-\delta_n)x_n + \beta_n T^n x_n + \delta_n v_n, \\
x_{n+1} &= (1-\alpha_n-\gamma_n)x_n + \alpha_n T^n y_n + \gamma_n u_n, \quad n \geq 0,
\end{aligned} \tag{2.1}$$

is called the *modified Ishikawa iteration sequence*, where $\{\alpha_n\}_n$, $\{\gamma_n\}_n$, $\{\beta_n\}_n$, and $\{\delta_n\}_n$ are four sequences in $(0,1)$ satisfying the conditions $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 0$.

In particular, if $\beta_n = \delta_n = 0$ for all $n \geq 0$, we can define a sequence $\{z_n\}_n$ by

$$\begin{gathered} z_0 \in X, \\ z_{n+1} = (1 - \alpha_n - \gamma_n) z_n + \alpha_n T^n z_n + \gamma_n w_n, \quad n \geq 0, \end{gathered} \tag{2.2}$$

called *the modified Mann iteration sequence*. In [22, 23] the methods are also called *modified Mann (Ishikawa resp.) iterative processes with errors.*

These kind of iterative processes with errors were studied, as an example, in [23–26]. Equivalence theorems for Mann and Ishikawa methods are studied, among others, in [27, 28].

Huang in [29] had established the equivalence theorems of the convergence between the modified Mann iteration process (2.2) and the modified Ishikawa iteration process (2.1) for strongly successively ϕ-pseudocontractive mappings in uniformly smooth Banach space.

In next section, we prove that, in the setting of the uniformly smooth Banach space, if T is asymptotically ϕ-strongly Pseudocontractive, then (2.1) and (2.2) are equivalent. Moreover we prove also that (2.1) and (2.2) strongly converge to the unique fixed point of T if it exists.

For the sake of completeness, we recall some definitions and conclusions.

Definition 2.1. X is said to be a uniformly smooth Banach space if the smooth module of X

$$\rho_X(t) = \sup\left\{ \frac{1}{2}(\|x - y\| + \|x + y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\} \tag{2.3}$$

satisfies $\lim_{t \to 0}(\rho_X(t)/t) = 0$.

Lemma 2.2 (see [30]). *Let X be a Banach space, and let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$ one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{2.4}$$

The following lemma is a key of our proofs.

Lemma 2.3 (see [29]). *Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\phi(0) = 0$, and let $\{a_n\}_n$, $\{b_n\}_n$, $\{c_n\}_n$, and $\{e_n\}_n$ be nonnegative real sequences such that*

$$\lim_{n \to \infty} b_n = 0, \qquad c_n = o(b_n), \qquad \sum_{n=1}^{\infty} b_n = \infty, \qquad \lim_{n \to \infty} e_n = 0. \tag{2.5}$$

Suppose that there exists an integer $N_1 > 0$ such that

$$a_{n+1}^2 \leq a_n^2 - 2b_n\phi(|a_{n+1} - e_n|) + c_n, \quad \forall n \geq N_1, \tag{2.6}$$

then $\lim_{n \to \infty} a_n = 0$.

Proof. The proof is the same as that in [29] but change in (2.6), $(a_{n+1} - e_n)$ with $|a_{n+1} - e_n|$. □

Lemma 2.4 (see [31]). *Let $\{s_n\}_n$, $\{c_n\}_n \subset \mathbb{R}_+$, $\{a_n\}_n \subset (0,1)$, and $\{b_n\}_n \subset \mathbb{R}$ be sequences such that*

$$s_{n+1} \leq (1 - a_n)s_n + b_n + c_n, \tag{2.7}$$

for all $n \geq 0$. Assume that $\sum_n |c_n| < \infty$. Then, the following results hold

(1) *If $b_n \leq \beta a_n$ (where $\beta \geq 0$), then $\{s_n\}_n$ is a bounded sequence.*

(2) *If one has $\sum_n a_n = \infty$ and $\limsup_n (b_n/a_n) \leq 0$ then, $s_n \to 0$ as $n \to \infty$.*

Remark 2.5. If in Lemma 2.3 choosing $e_n = 0$, for all n, $\phi(t) = kt^2$ $(k < 1)$ then, the inequality (2.6) becomes,

$$\begin{aligned} a_{n+1}^2 &\leq a_n^2 - 2b_n k a_{n+1}^2 + c_n \Longrightarrow \\ a_{n+1}^2 &\leq \frac{1}{1+2b_nk} a_n^2 + \frac{c_n}{1+2b_nk} \\ &\leq \left(1 - \frac{2b_nk}{1+2b_nk}\right) a_n^2 + \frac{c_n}{1+2b_nk}, \end{aligned} \tag{2.8}$$

where $\alpha_n := 2b_nk/(1+2b_nk)$ and $\beta_n := c_n/(1+2b_nk)$. In the hypotheses of Lemma 2.3, $\alpha_n \to 0$ as $n \to \infty$, $\sum_n \alpha_n = \infty$, and $\limsup_n(\beta_n/\alpha_n) = 0$. So we reobtain Lemma 2.4 in the case $c_n = 0$.

3. Main Results

The ideas of the proofs of our main Theorems take into account those contained in the papers of Chang and Chidume et al. [22, 32, 33].

Theorem 3.1. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be an asymptotically ϕ-strongly Pseudocontractive mapping with fixed point x^* and bounded range.*

Let $\{x_n\}$ and $\{z_n\}$ be the sequences defined by (2.1) *and* (2.2)*, respectively, where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$, and $\{\delta_n\} \subset [0,1]$ satisfy*

(H1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\delta_n = 0$ *and* $\gamma_n = o(\alpha_n)$,

(H2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

and the sequences $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are bounded in X. Then, for any initial point $z_0, x_0 \in X$, the following two assertions are equivalent.

(i) *The modified Ishikawa iteration sequence* (2.1) *converges to x^*.*

(ii) *The modified Mann iteration sequence* (2.2) *converges to x^*.*

Proof. First of all we note that by boundedness of the range of T and of the sequences $\{w_n\}$, $\{u_n\}$, $\{v_n\}$ and by Lemma 2.4, it and follows that $\{z_n\}$ and $\{x_n\}$ are bounded sequences. So we can set

$$M = \sup_n \left\{ \begin{array}{c} \|T^n z_n - T^n y_n\|, \|T^n x_n - x_n\|, \\ \|T^n y_n - x_n\|, \|T^n z_n - z_n\|, \\ \|z_n - x_n\|, \|u_n - x_n\|, \|v_n - x_n\|, \\ \|w_n - z_n\|, \|w_n - u_n\|. \end{array} \right\}. \tag{3.1}$$

By Lemma 2.2, we have

$$\begin{aligned} \|z_{n+1} - x_{n+1}\|^2 &\le \left\|(1-\alpha_n-\gamma_n)(z_n - x_n) + \alpha_n(T^n z_n - T^n y_n) + \gamma_n(w_n - u_n)\right\|^2 \\ &\le (1-\alpha_n-\gamma_n)^2\|z_n - x_n\|^2 + 2\langle \alpha_n(T^n z_n - T^n y_n) + \gamma_n(w_n - u_n), j(z_{n+1} - x_{n+1})\rangle \\ &\le (1-\alpha_n)^2\|z_n - x_n\|^2 + 2\alpha_n\langle T^n z_n - T^n y_n, j(z_n - y_n)\rangle \\ &\quad + 2\alpha_n\langle T^n z_n - T^n y_n, j(z_{n+1} - x_{n+1}) - j(z_n - y_n)\rangle + 2\gamma_n\langle w_n - u_n, j(z_{n+1} - x_{n+1})\rangle \\ &\le (1-\alpha_n)^2\|z_n - x_n\|^2 + 2\alpha_n k_n\|z_n - y_n\|^2 - 2\alpha_n\phi(\|z_n - y_n\|) \\ &\quad + 2\alpha_n\|T^n z_n - T^n y_n\|\|j(z_{n+1} - x_{n+1}) - j(z_n - y_n)\| + 2\gamma_n\|w_n - u_n\|\|z_{n+1} - x_{n+1}\| \\ &\le (1-\alpha_n)^2\|z_n - x_n\|^2 + 2\alpha_n k_n\|z_n - y_n\|^2 - 2\alpha_n\phi(\|z_n - y_n\|) \\ &\quad + 2\alpha_n\sigma_n M + 2\gamma_n M^2, \end{aligned} \tag{3.2}$$

where $\sigma_n = \|j(z_{n+1} - x_{n+1}) - j(z_n - y_n)\|$. Using (2.1) and (2.2), we have

$$\begin{aligned} \|(z_{n+1} - x_{n+1}) - (z_n - y_n)\| &\le \|x_{n+1} - y_n\| + \|z_{n+1} - z_n\| \\ &= \|\alpha_n(T^n y_n - x_n) + \gamma_n(u_n - x_n) - \beta_n(T^n x_n - x_n) - \delta_n(v_n - x_n)\| \\ &\quad + \|\alpha_n(T^n z_n - z_n) + \gamma_n(w_n - z_n)\| \\ &\le 2M(\alpha_n + \gamma_n + \beta_n + \delta_n) \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{3.3}$$

In view of the uniform continuity of j we obtain that $\sigma_n \to 0$ as $n \to \infty$. Furthermore, it follows from the definition of $\{y_n\}$ that for all $n \ge 0$

$$\begin{aligned} \|z_n - y_n\|^2 &= \|z_n - x_n + \beta_n(-T^n x_n + x_n) + \delta_n(-v_n + x_n)\|^2 \\ &\le [\|z_n - x_n\| + \beta_n\|T^n x_n - x_n\| + \delta_n\|v_n - x_n\|]^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left[\|z_n - x_n\| + (\beta_n + \delta_n) M\right]^2 \\
&= \|z_n - x_n\|^2 + (\beta_n + \delta_n)\left(2\|z_n - x_n\| M + (\beta_n + \delta_n) M^2\right) \\
&\leq \|z_n - x_n\|^2 + 3(\beta_n + \delta_n) M^2,
\end{aligned} \tag{3.4}$$

$$\left\|y_n - x_n\right\| \leq \beta_n \|T^n x_n - x_n\| + \delta_n \|v_n - x_n\| \leq (\beta_n + \delta_n) M \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{3.5}$$

and so

$$\begin{aligned}
\|z_{n+1} - x_{n+1}\| &= \left\|z_n - x_n - (\alpha_n + \gamma_n)(z_n - x_n) + \alpha_n (T^n z_n - T^n y_n) + \gamma_n (w_n - u_n)\right\| \\
&\leq \|z_n - x_n\| + (\alpha_n + \gamma_n) \|z_n - x_n\| + \alpha_n \left\|T^n z_n - T^n y_n\right\| + \gamma_n \|w_n - u_n\| \\
&\leq \left\|z_n - y_n\right\| + \left\|y_n - x_n\right\| + 2(\alpha_n + \gamma_n) M. \\
&\leq \left\|z_n - y_n\right\| + (\beta_n + \delta_n) M + 2(\alpha_n + \gamma_n) M.
\end{aligned} \tag{3.6}$$

Therefore, we have

$$\left\|z_n - y_n\right\| \geq \|z_{n+1} - x_{n+1}\| - e_n, \tag{3.7}$$

where $e_n = (\beta_n + \delta_n)M + 2(\alpha_n + \gamma_n)M$. In view of (H1), we have that $e_n \rightarrow 0$ as $n \rightarrow \infty$.

If $\|z_{n+1} - x_{n+1}\| - e_n \leq 0$ for an infinite number of indexes we can extract a subsequence such that $\|z_{n_k} - x_{n_k}\| - e_{n_k - 1} \leq 0$. For this subsequence $\|z_{n_k} - x_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$.

We can prove that, in this case, $\|z_n - x_n\| \rightarrow 0$, that is, the thesis.

Firstly we note that substituting (3.4) into (3.2) and simplifying, we have

$$\begin{aligned}
\|z_{n+1} - x_{n+1}\|^2 &\leq \|z_n - x_n\|^2 + \alpha_n^2 \|z_n - x_n\|^2 + 2\alpha_n (k_n - 1) \|z_n - x_n\|^2 \\
&\quad + 6\alpha_n k_n (\beta_n + \delta_n) M^2 - 2\alpha_n \phi(\left\|z_n - y_n\right\|) + 2\alpha_n \sigma_n M + 2\gamma_n M^2 \\
&\leq \|z_n - x_n\|^2 - 2\alpha_n \phi(\left\|z_n - y_n\right\|) + \alpha_n^2 M^2 + 2\alpha_n (k_n - 1) M^2 \\
&\quad + 6\alpha_n \overline{k} (\beta_n + \delta_n) M^2 + 2\alpha_n \sigma_n M + 2\gamma_n M^2 \\
&= \|z_n - x_n\|^2 - \alpha_n \phi(\left\|z_n - y_n\right\|) + 2\gamma_n M^2 \\
&\quad - \alpha_n \left[\phi(\left\|z_n - y_n\right\|) - \alpha_n M^2 - 2(k_n - 1) M^2 - 6\overline{k} (\beta_n + \delta_n) M^2 - 2\sigma_n M\right],
\end{aligned} \tag{3.8}$$

where $\overline{k} := \sup_n (k_n)$.

Moreover, we observe that

$$\left\|z_{n_j} - y_{n_j}\right\| \leq \left\|z_{n_j} - x_{n_j}\right\| + \left\|y_{n_j} - x_{n_j}\right\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.9}$$

Thus, for every fixed $\epsilon > 0$, there exists j_1 such that, for all $j > j_1$,

$$\left\|z_{n_j} - y_{n_j}\right\| < 2\epsilon, \qquad \left\|z_{n_j} - x_{n_j}\right\| < \epsilon. \tag{3.10}$$

Since $\{\alpha_n\}_n$, $\{(k_n-1)\}_n$, $\{(\beta_n+\delta_n)\}_n$, $\{\sigma_n\}_n$, and $\{\gamma_n\}_n$ are real null sequences (and in particular $\gamma_n = o(\alpha_n)$), for the previous fixed $\epsilon > 0$, there exist an index N such that, for all $n > N$,

$$\begin{gathered}
|\alpha_n| < \min\left\{\frac{\epsilon}{16M}, \frac{\phi(\epsilon/2)}{8M^2}\right\}, \\
|\gamma_n| < \frac{\epsilon}{16M}, \qquad \left|\frac{\gamma_n}{\alpha_n}\right| < \frac{\phi(\epsilon/2)}{4M^2}, \\
|k_n - 1| < \frac{\phi(\epsilon/2)}{16M^2}, \\
|\beta_n + \delta_n| < \min\left\{\frac{\epsilon}{4M}, \frac{\phi(\epsilon/2)}{48\overline{k}M^2}\right\}, \\
|\sigma_n| < \frac{\phi(\epsilon/2)}{16M},
\end{gathered} \tag{3.11}$$

for all $n > N$.

Take $n^* > \max\{N, n_{j_1}\}$ such that $n^* = n_k$ for a certain k.

We prove, by induction, that $\|z_{n^*+i} - x_{n^*+i}\| < \epsilon$, for every $i \in \mathbb{N}$. Let $i = 1$.

Let us suppose that $\|z_{n^*+1} - x_{n^*+1}\| \geq \epsilon$.

By (3.6), we have

$$\begin{aligned}
\epsilon \leq \|z_{n^*+1} - x_{n^*+1}\| &\leq \left\|z_{n^*} - y_{n^*}\right\| + (\beta_{n^*} + \delta_{n^*})M + 2(\alpha_{n^*} + \gamma_{n^*})M \\
&\leq \left\|z_{n^*} - y_{n^*}\right\| + M\frac{\epsilon}{4M} + 2M\left(\frac{\epsilon}{8M} + \frac{\epsilon}{8M}\right) \\
&= \left\|z_{n^*} - y_{n^*}\right\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \left\|z_{n^*} - y_{n^*}\right\| + \frac{\epsilon}{2}.
\end{aligned} \tag{3.12}$$

Thus, $\|z_{n^*} - y_{n^*}\| \geq \epsilon/2$. Since ϕ is strictly increasing, $\phi(\|z_{n^*} - y_{n^*}\|) \geq \phi(\epsilon/2)$.

From (3.8), we obtain that

$$\begin{aligned}
\|z_{n^*+1} - x_{n^*+1}\|^2 < \epsilon^2 - \alpha_n^*\left(\phi\left\|z_{n^*} - y_{n^*}\right\| - 2M^2\frac{\gamma_n^*}{\alpha_n^*}\right) \\
- \alpha_n^*\Big[\phi(\left\|z_{n^*} - y_{n^*}\right\|) - \alpha_n^* M^2 - 2(k_n^* - 1)M^2 - 6\overline{k}(\beta_n^* + \delta_n^*)M^2 - 2\sigma_{n^*}M\Big].
\end{aligned} \tag{3.13}$$

One can note that

$$\alpha_n^* M^2 + 2(k_n^* - 1)M^2 + 6\overline{k}(\beta_n^* + \delta_n^*)M^2 + 2\sigma_{n^*} M \leq \frac{\phi(\epsilon/2)}{8} + \frac{\phi(\epsilon/2)}{8} + \frac{\phi(\epsilon/2)}{8} + \frac{\phi(\epsilon/2)}{8}; \tag{3.14}$$

hence

$$\phi(\|z_{n^*} - y_{n^*}\|) - \alpha_n^* M^2 - 2(k_{n^*} - 1)M^2 - 6\overline{k}(\beta_{n^*} + \delta_{n^*})M^2 - 2\sigma_{n^*} M \geq \phi\left(\frac{\epsilon}{2}\right) - \frac{\phi(\epsilon/2)}{2} > 0. \tag{3.15}$$

In the same manner

$$\phi(\|z_{n^*} - y_{n^*}\|) - 2M^2 \frac{\gamma_{n^*}}{\alpha_{n^*}} > \phi\left(\frac{\epsilon}{2}\right) - \frac{\phi(\epsilon/2)}{2} > 0. \tag{3.16}$$

Thus,

$$\|z_{n^*+1} - x_{n^*+1}\|^2 < \epsilon^2. \tag{3.17}$$

So, we have that $\|z_{n^*+1} - x_{n^*+1}\| < \epsilon$ and it denies that $\|z_{n^*+1} - x_{n^*+1}\| \geq \epsilon$. By the same idea we can prove (by contradiction) that $\|z_{n^*+2} - x_{n^*+2}\| < \epsilon$ and then, by inductive step, $\|z_{n^*+i} - x_{n^*+i}\| \leq \epsilon$, for all i. This is enough to assure that $\|z_n - x_n\| \to 0$.

If there are only finite indexes for which $\|z_{n+1} - x_{n+1}\| - e_n \leq 0$, then definitively $\|z_{n+1} - x_{n+1}\| - e_n \geq 0$. By the strict increasing function ϕ, we have definitively

$$\phi(\|z_n - y_n\|) \geq \phi(\|z_{n+1} - x_{n+1}\| - e_n). \tag{3.18}$$

Again substituting (3.4) and (3.18) into (3.2) and simplifying, we have

$$\begin{aligned}
\|z_{n+1} - x_{n+1}\|^2 &\leq \|z_n - x_n\|^2 + \alpha_n^2 \|z_n - x_n\|^2 + 2\alpha_n (k_n - 1)\|z_n - x_n\|^2 \\
&\quad + 6\alpha_n k_n (\beta_n + \delta_n) M^2 - 2\alpha_n \phi(\|z_{n+1} - x_{n+1}\| - e_n) + 2\alpha_n \sigma_n M + 2\gamma_n M^2 \\
&\leq \|z_n - x_n\|^2 - 2\alpha_n \phi(\|z_{n+1} - x_{n+1}\| - e_n) + \alpha_n^2 M^2 + 2\alpha_n (k_n - 1) M^2 \\
&\quad + 6\alpha_n k_n (\beta_n + \delta_n) M^2 + 2\alpha_n \sigma_n M + 2\gamma_n M^2.
\end{aligned} \tag{3.19}$$

Suppose that $a_n = \|z_n - x_n\|$, and $b_n = \alpha_n, c_n = \alpha_n^2 M^2 + 2\alpha_n(k_n - 1)M^2 + 6\alpha_n k_n(\beta_n + \delta_n)M^2 + 2\alpha_n \sigma_n M + 2\gamma_n M^2$. Since $\lim_{n\to\infty}\sigma_n = 0$, $\lim_{n\to\infty} e_n = 0$, and $\lim_{n\to\infty} k_n = 1$ and by the hypotheses, we have that $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, $e_n \to 0$ as $n \to \infty$. In view of Lemma 2.3 we obtain that $\lim_{n\to\infty} a_n = 0$. Hence, $\lim_{n\to\infty}\|z_n - x_n\| = 0$. □

Theorem 3.2. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be an asymptotically ϕ-strongly Pseudocontractive mapping with fixed point x^* and bounded range.*

Let $\{v_n\}_n$ be the sequences defined by

$$v_0 \in X,$$
$$v_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n v_n, \quad n \geq 0, \tag{3.20}$$

where $\{\alpha_n\}_n \subset [0,1]$ satisfy

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, for any initial point $v_0 \in X$, the sequence $\{v_n\}_n$ strongly converges to x^*.

Proof. By the boundedness of the range of T and by Lemma 2.4, we have that $\{v_n\}_n$ is bounded.

By Lemma 2.2 we observe that

$$\begin{aligned}
\|v_{n+1}-x^*\|^2 &\leq (1-\alpha_n)^2\|v_n-x^*\|^2 + 2\alpha_n\left\langle T^n v_n - x^*, j(v_{n+1}-x^*)\right\rangle \\
&\leq (1-\alpha_n)^2\|v_n-x^*\|^2 + 2\alpha_n\left\langle T_n v_n - x^*, j(v_{n+1}-x^*) - j(v_n-x^*)\right\rangle \\
&\quad + 2\alpha_n\left\langle T_n v_n - x^*, j(v_n-x^*)\right\rangle \\
&\leq (1-\alpha_n)^2\|v_n-x^*\|^2 + 2\alpha_n\left\langle T_n v_n - x^*, j(v_{n+1}-x^*) - j(v_n-x^*)\right\rangle \\
&\quad + 2\alpha_n k_n\|v_n-x^*\|^2 - 2\alpha_n\phi(\|v_n-x^*\|) \\
&= \left(1+\alpha_n^2-2\alpha_n\right)\|v_n-x^*\|^2 + 2\alpha_n\left\langle T_n v_n - x^*, j(v_{n+1}-x^*) - j(v_n-x^*)\right\rangle \\
&\quad + 2\alpha_n k_n\|v_n-x^*\|^2 - 2\alpha_n\phi(\|v_n-x^*\|) \\
&= \|v_n-x^*\|^2 + \left(\alpha_n^2 - 2\alpha_n + 2\alpha_n k_n\right)\|v_n-x^*\|^2 - 2\alpha_n\phi(\|v_n-x^*\|) + 2\alpha_n\mu_n,
\end{aligned} \tag{3.21}$$

where $\mu_n := \langle T_n v_n - x^*, j(v_{n+1}-x^*) - j(v_n-x^*)\rangle$. Let

$$M := \max\left\{\sup_n\|v_n-x^*\|, \sup_n\|T^n v_n - x^*\|\right\}. \tag{3.22}$$

We have

$$\begin{aligned}
\|v_{n+1}-x^*\|^2 &\leq \|v_n-x^*\|^2 + \left(\alpha_n^2 + 2\alpha_n(k_n-1)\right)M - 2\alpha_n\phi(\|v_n-x^*\|) + 2\alpha_n\mu_n \\
&= \|v_n-x^*\|^2 - \alpha_n\phi(\|v_n-x^*\|) \\
&\quad - \alpha_n\left[\phi(\|v_n-x^*\|) - 2\mu_n - (\alpha_n + 2(k_n-1))M\right],
\end{aligned} \tag{3.23}$$

and so we can observe that

(1) $\mu_n \to 0$ as $n \to \infty$, in fact from the inequality

$$|\|v_{n+1}-x^*\|-\|v_n-x^*\||\leq\|v_{n+1}-v_n\|\leq\alpha_n M\longrightarrow 0 \quad \text{as } n\longrightarrow\infty \tag{3.24}$$

and since j is norm-to-norm uniformly continuous, then $j(\|v_{n+1}-x^*\|)-j(\|v_n-x^*\|)\to 0$ as $n\to\infty$,

(2) $\inf_n(\|v_n-x^*\|)=0$, in fact, if we suppose that $\sigma:=\inf_n(\|v_n-x^*\|)>0$, by the monotonicity of ϕ,

$$\phi(\|v_n-x^*\|)\geq\phi(\sigma)>0. \tag{3.25}$$

Thus, by (1) and by the hypotheses on α_n and k_n, the value $-\alpha_n[\phi(\|v_n-x^*\|)-2\mu_n-(\alpha_n+2(k_n-1))M]$ is definitively negative. In this case we conclude that there exists $N>0$ such that for every $n>N$

$$\begin{aligned}\|v_{n+1}-x^*\|^2&\leq\|v_n-x^*\|^2-\alpha_n\phi(\|v_n-x^*\|)\\&\leq\|v_n-x^*\|^2-\alpha_n\phi(\sigma),\end{aligned} \tag{3.26}$$

and so

$$\alpha_n\phi(\sigma)\leq\|v_n-x^*\|^2-\|v_{n+1}-x^*\|^2 \quad \forall n>N. \tag{3.27}$$

In the same manner we obtain that

$$\begin{aligned}\phi(\sigma)\sum_{i=N}^{m}\alpha_i&\leq\sum_{i=N}^{m}\left[\|v_i-x^*\|^2-\|v_{i+1}-x^*\|^2\right]\\&=\|v_N-x^*\|^2-\|v_m-x^*\|^2.\end{aligned} \tag{3.28}$$

By the hypothesis $\sum_n\alpha_n=\infty$, the previous is a contradiction and it follows that $\inf_n(\|v_n-x^*\|)=0$.

Then, there exists a subsequence $\{v_{n_k}\}_k$ of $\{v_n\}_n$ that strongly converges to x^*. This implies that for every $\epsilon>0$ there exists an index $n_{k(\epsilon)}$ such that, for all $j\geq n_{k(\epsilon)}$, $\|v_{n_j}-x^*\|<\epsilon$.

Now we will prove that the entire sequence $\{v_n\}_n$ converges to x^*. Since the sequences in (3.37) are null sequences but $\sum_n\alpha_n=\infty$, then for every $\epsilon>0$ there exists an index $\overline{n}(\epsilon)$ such that for all $n\geq\overline{n}(\epsilon)$ it following that:

$$|\alpha_n|<\frac{1}{2M}\min\left\{\epsilon,\frac{\phi(\epsilon/2)}{2}\right\}, \qquad |k_n-1|<\frac{\phi(\epsilon/2)}{8M}, \qquad |\mu_n|<\frac{\phi(\epsilon/2)}{8}. \tag{3.29}$$

So, fixing $\epsilon > 0$, let $n^* > \max(n_{k(\epsilon)}, \overline{n}(\epsilon))$ with $n^* = n_j$ for a certain n_j. We will prove, by induction, that $\|v_{n^*+i} - x^*\| < \epsilon$ for every $i \in \mathbb{N}$. Let $i = 1$. If not, it follows that $\|v_{n^*+1} - x^*\| \geq \epsilon$. Thus

$$\begin{aligned} \epsilon \leq \|v_{n^*+1} - x^*\| &\leq \|v_{n^*} - x^*\| + \alpha_{n^*} \\ &< \|v_{n^*} - x^*\| + \frac{\epsilon}{2M} M = \|v_{n^*} - x^*\| + \frac{\epsilon}{2}, \end{aligned} \tag{3.30}$$

that is, $\|v_{n^*} - x^*\| > (\epsilon/2)$. By the strict increasing of ϕ, $\phi(\|v_{n^*} - x^*\|) > \phi(\epsilon/2)$. By (3.37), it follows that

$$\begin{aligned} \|v_{n^*+1} - x^*\|^2 < \epsilon^2 &- \alpha_{n^*}\phi(\|v_{n^*} - x^*\|) \\ &- \alpha_{n^*}\left[\phi(\|v_{n^*} - x^*\|) - 2\mu_{n^*} - (\alpha_{n^*} + 2(k_{n^*} - 1))M\right]. \end{aligned} \tag{3.31}$$

We can note that

$$2\mu_{n^*} + (\alpha_{n^*} + 2(k_{n^*} - 1))M \leq \frac{\phi(\epsilon/2)}{4} + \left(\frac{\phi(\epsilon/2)}{4M} + \frac{\phi(\epsilon/2)}{4M}\right)M, \tag{3.32}$$

and so

$$\phi(\|v_{n^*} - x^*\|) - 2\mu_{n^*} - (\alpha_{n^*} + 2(k_{n^*} - 1))M \geq \phi\left(\frac{\epsilon}{2}\right) - \frac{3\phi(\epsilon/2)}{4} > 0. \tag{3.33}$$

Moreover, $\phi(\|v_{n^*} - x^*\|) > \phi(\epsilon/2)/2 > 0$, and it follows that

$$\|v_{n^*+1} - x^*\|^2 \leq \epsilon^2. \tag{3.34}$$

This is absurd. Thus $\|v_{n^*+1} - x^*\| < \epsilon$.

In the same manner, by induction, one obtains that, for every $i \geq 1$, $\|v_{n^*+i} - x^*\| < \epsilon$. So $\|v_n - x^*\| \to 0$. □

Corollary 3.3. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be an asymptotically ϕ-strongly Pseudocontractive mapping with fixed point x^* and bounded range.*

Let $\{z_n\}_n$ be the sequences defined by (2.2)*, where $\{\alpha_n\}_n$ and $\{\gamma_n\}_n \subset [0,1]$ satisfy*

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\gamma_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$,

and the sequence $\{w_n\}_n$ is bounded on X. Then, for any initial point $z_0 \in X$, the sequence $\{z_n\}_n$ strongly converges to x^*.

Proof. Following the idea of the proof of Theorem 3.2, by the boundedness of the range of T and by the boundedness of the sequence $\{w_n\}_n$ and by Lemma 2.4, we have that $\{z_n\}_n$ is bounded.

By Lemma 2.2 we observe that

$$\begin{aligned}
\|z_{n+1}-x^*\|^2 &\le (1-\alpha_n-\gamma_n)^2\|z_n-x^*\|^2+2\alpha_n\langle T^n z_n-x^*, j(z_{n+1}-x^*)\rangle \\
&\quad+2\gamma_n\langle w_n-x^*, j(z_{n+1}-x^*)\rangle \\
&\le (1-\alpha_n)^2\|z_n-x^*\|^2+2\alpha_n\langle T_n z_n-x^*, j(z_{n+1}-x^*)-j(z_n-x^*)\rangle \\
&\quad+2\alpha_n\langle T_n z_n-x^*, j(z_n-x^*)\rangle+2\gamma_n\langle w_n-x^*, j(z_{n+1}-x^*)\rangle \\
&\le (1-\alpha_n)^2\|z_n-x^*\|^2+2\alpha_n\langle T_n z_n-x^*, j(z_{n+1}-x^*)-j(z_n-x^*)\rangle \\
&\quad+2\alpha_n k_n\|z_n-x^*\|^2-2\alpha_n\phi(\|z_n-x^*\|) \\
&\quad+2\gamma_n\langle w_n-x^*, j(z_{n+1}-x^*)\rangle \\
&= \left(1+\alpha_n^2-2\alpha_n\right)\|z_n-x^*\|^2+2\alpha_n\langle T_n z_n-x^*, j(z_{n+1}-x^*)-j(z_n-x^*)\rangle \\
&\quad+2\alpha_n k_n\|z_n-x^*\|^2-2\alpha_n\phi(\|z_n-x^*\|) \\
&\quad+2\gamma_n\langle w_n-x^*, j(z_{n+1}-x^*)\rangle \\
&= \|z_n-x^*\|^2+\left(\alpha_n^2-2\alpha_n+2\alpha_n k_n\right)\|z_n-x^*\|^2-2\alpha_n\phi(\|z_n-x^*\|)+2\alpha_n\mu_n \\
&\quad+2\gamma_n\langle w_n-x^*, j(z_{n+1}-x^*)\rangle,
\end{aligned} \tag{3.35}$$

where $\mu_n := \langle T_n z_n - x^*, j(z_{n+1}-x^*) - j(z_n-x^*)\rangle$. Let

$$M := \max\left\{\sup_n\|z_n-x^*\|, \sup_n\|T^n z_n-x^*\|, \sup_n\|w_n-x^*\|, \sup_n\langle w_n-x^*, j(z_{n+1}-x^*)\rangle\right\}. \tag{3.36}$$

We have

$$\begin{aligned}
\|z_{n+1}-x^*\|^2 &\le \|z_n-x^*\|^2+\left(\alpha_n^2+2\alpha_n(k_n-1)\right)M-2\alpha_n\phi(\|z_n-x^*\|)+2\alpha_n\mu_n+2\gamma_n M \\
&= \|z_n-x^*\|^2-\alpha_n\phi(\|z_n-x^*\|)+2\gamma_n M \\
&\quad-\alpha_n\left[\phi(\|z_n-x^*\|)-2\mu_n-(\alpha_n+2(k_n-1))M\right],
\end{aligned} \tag{3.37}$$

and so we can observe that

(1) $\mu_n \to 0$ as $n \to \infty$, in fact from the inequality

$$\big|\|z_{n+1}-x^*\|-\|z_n-x^*\|\big| \le \|z_{n+1}-z_n\|(\alpha_n+\gamma_n) \quad M \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{3.38}$$

and since j is norm-to-norm uniformly continuous, then $j(\|z_{n+1}-x^*\|) - j(\|z_n - x^*\|) \to 0$ as $n \to \infty$,

(2) $\inf_n(\|z_n - x^*\|) = 0$, in fact, if we suppose that $\sigma := \inf_n(\|z_n - x^*\|) > 0$, by the monotonicity of ϕ,

$$\phi(\|z_n - x^*\|) \geq \phi(\sigma) > 0. \tag{3.39}$$

Thus, by (1) and by the hypotheses on α_n and k_n, the value $-\alpha_n[\phi(\|z_n - x^*\|) - 2\mu_n - (\alpha_n + 2(k_n - 1))M]$ is definitively negative. In this case, we conclude that there exists $N > 0$ such that for every $n > N$

$$\begin{aligned} \|z_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \alpha_n \phi(\|z_n - x^*\|) + 2\gamma_n M \\ &\leq \|z_n - x^*\|^2 - \alpha_n \phi(\sigma) + 2\gamma_n M, \end{aligned} \tag{3.40}$$

and so

$$\alpha_n \phi(\sigma) \leq \|z_n - x^*\|^2 - \|z_{n+1} - x^*\|^2 + 2\gamma_n M \quad \forall n > N. \tag{3.41}$$

In the same manner we obtain that

$$\begin{aligned} \phi(\sigma)\sum_{i=N}^{m}\alpha_i &\leq \sum_{i=N}^{m}\left[\|z_i - x^*\|^2 - \|z_{i+1} - x^*\|^2\right] + 2\sum_{i=N}^{m}\gamma_i M \\ &= \|z_N - x^*\|^2 - \|z_m - x^*\|^2 + 2M\sum_{i=N}^{m}\beta_i. \end{aligned} \tag{3.42}$$

By the hypotheses $\sum_n \gamma_n < \infty$ and $\sum_n \alpha_n = \infty$, the previous is a contradiction and it follows that $\inf_n(\|z_n - x^*\|) = 0$.

Then, there exists a subsequence $\{z_{n_k}\}_k$ of $\{z_n\}_n$ that strongly converges to x^*. This implies that for every $\epsilon > 0$ there exists an index $n_{k(\epsilon)}$ such that, for all $j \geq n_{k(\epsilon)}$, $\|z_{n_j} - x^*\| < \epsilon$.

Now we will prove that the entire sequence $\{z_n\}_n$ converges to x^*. Since the sequences in (3.37) are null sequences and $\sum_n \gamma_n < \infty$ but $\sum_n \alpha_n = \infty$, then for every $\epsilon > 0$ there exists an index $\overline{n}(\epsilon)$ such that for all $n \geq \overline{n}(\epsilon)$ it follows that:

$$\begin{gathered} |\alpha_n| < \frac{1}{4M}\min\left\{\epsilon, \phi\left(\frac{\epsilon}{2}\right)\right\}, \\ |\gamma_n| < \frac{\epsilon}{4M}, \qquad \left|\frac{\gamma_n}{\alpha_n}\right| < \frac{\phi(\epsilon/2)}{4M}, \\ |k_n - 1| < \frac{\phi(\epsilon/2)}{8M}, \\ |\mu_n| < \frac{\phi(\epsilon/2)}{8}. \end{gathered} \tag{3.43}$$

So, fixing $\epsilon > 0$, let $n^* > \max(n_{k(\epsilon)}, \overline{n}(\epsilon))$ with $n^* = n_j$ for a certain n_j. We will prove, by induction, that $\|z_{n^*+i} - x^*\| < \epsilon$ for every $i \in \mathbb{N}$. Let $i = 1$. If not, it follows that $\|z_{n^*+1} - x^*\| \geq \epsilon$. Thus,

$$\begin{aligned}\epsilon \leq \|z_{n^*+1} - x^*\| &\leq \|z_{n^*} - x^*\| + \alpha_{n^*} M + \gamma_{n^*} M \\ &< \|z_{n^*} - x^*\| + \frac{\epsilon}{4M} M + \frac{\epsilon}{4M} M = \|z_{n^*} - x^*\| + \frac{\epsilon}{2},\end{aligned} \tag{3.44}$$

that is, $\|z_{n^*} - x^*\| > \epsilon/2$. By the strict increasing of ϕ, $\phi(\|z_{n^*} - x^*\|) > \phi(\epsilon/2)$.

By (3.37), it follows that

$$\begin{aligned}\|z_{n^*+1} - x^*\|^2 < \epsilon^2 &- \alpha_{n^*} \left(\phi(\|z_{n^*} - x^*\|) - 2M \frac{\gamma_{n^*}}{\alpha_{n^*}} \right) \\ &- \alpha_{n^*} \left[\phi(\|z_{n^*} - x^*\|) - 2\mu_{n^*} - (\alpha_{n^*} + 2(k_{n^*} - 1))M \right].\end{aligned} \tag{3.45}$$

We can note that

$$2\mu_{n^*} + (\alpha_{n^*} + 2(k_{n^*} - 1))M \leq \frac{\phi(\epsilon/2)}{4} + \left(\frac{\phi(\epsilon/2)}{4M} + \frac{\phi(\epsilon/2)}{4M} \right) M, \tag{3.46}$$

and so

$$\phi(\|z_{n^*} - x^*\|) - 2\mu_{n^*} - (\alpha_{n^*} + 2(k_{n^*} - 1))M \geq \phi\left(\frac{\epsilon}{2}\right) - \frac{3\phi(\epsilon/2)}{4} > 0. \tag{3.47}$$

Moreover, $\phi(\|z_{n^*} - x^*\|) - 2M(\gamma_{n^*}/\alpha_{n^*}) > \phi(\epsilon/2)/2 > 0$ so it follows that

$$\|z_{n^*+1} - x^*\|^2 \leq \epsilon^2. \tag{3.48}$$

This is absurd. Thus, $\|z_{n^*+1} - x^*\| < \epsilon$.

In the same manner, by induction, one obtains that, for every $i \geq 1$, $\|z_{n^*+i} - x^*\| < \epsilon$. So $\|z_n - x^*\| \to 0$. □

Corollary 3.4. *Let X be a uniformly smooth Banach space, and let $T : X \to X$ be asymptotically ϕ-strongly Pseudocontractive mapping with bounded range and fixed point x^*. The sequences $\{x_n\}_n$, $\{z_n\}_n$, and $\{z'_n\}_n$ defined by* (2.1) *and* (2.2), *respectively, where the sequences $\{\alpha_n\}_n$, $\{\beta_n\}_n$, $\{\gamma_n\}_n$, and $\{\delta_n\}_n \subset [0,1]$ satisfy*

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\delta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$,

and the sequences $\{u_n\}_n$, $\{v_n\}_n$, and $\{w_n\}_n$, are bounded in X. Then, for any initial point $x_0, z_0 \in X$, the following two assertions are equivalents, and true.

(i) *The modified Ishikawa iteration sequence* (2.1) *converges to the fixed point x^*.*

(ii) *The modified Mann iteration sequence* (2.2) *converges to the fixed point x^*.*

Example 3.5. Let us consider the mapping in Example 1.9, $Tx = x/(1+0.01x)$, where $x \in [0,1]$, that is asymptotically ϕ-strongly Pseudocontractive with $k_n = (n+100)/n$, for all $n \in \mathbb{N}$ and $\phi(x) = x^3/(1+x)$. The unique fixed point of T is $x = 0$. Let $z_0 \in (0,1]$ and, for all $n \in \mathbb{N}$, $\alpha_n = 1/n$, and $\gamma_n = 1/n^2$. Then the sequence generated by the scheme

$$z_{n+1} = \left(1 - \frac{1}{n} - \frac{1}{n^2}\right) z_n + \frac{1}{n} T^n z_n + \frac{1}{n^2} w_n, \quad n \geq 0, \tag{3.49}$$

converges to 0.

Remark 3.6. Our results are similar to those of Schu's [4, Theorem 1.4] However, our results hold in a more general setting of uniformly smooth Banach spaces, while Schu's result holds for completely continuous, uniformly Lipschitzian mappings, which are asymptotically pseudocontractive.

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