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## Review Article

# Some Properties and Identities of Bernoulli and Euler Polynomials Associated with p-adic Integral on $\mathbb{Z}_p$

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We investigate some properties and identities of Bernoulli and Euler polynomials. Further, we give some formulae on Bernoulli and Euler polynomials by using p-adic integral on  $\mathbb{Z}_p$ .

#### 1. Introduction

Let p be a fixed odd prime number. Throughout this paper,  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of  $\mathbb{Q}_p$ . Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = 1/p$ .

For  $f \in UD(\mathbb{Z}_p)$ , the *p*-adic invariant integral on  $\mathbb{Z}_p$  in the bosonic sense is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x) d\mu(x)$$
 (1.1)

(see [1, 2]). The fermionic *p*-adic integral on  $\mathbb{Z}_p$  is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) (-1)^x$$
 (1.2)

(see [3]). As is well known, Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$
(1.3)

with the usual convention about replacing  $B^n(x)$  by  $B_n(x)$ , symbolically (see [1–19]). In the special case x = 0,  $B_n(0) = B_n$  is called the nth Bernoulli number.

The Euler polynomials are also defined by the generating function as follows:

$$\frac{2}{e^t + 1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$
(1.4)

with the usual convention about replacing  $E^n(x)$  by  $E_n(x)$ , symbolically (see [1–19]). In the special case x = 0,  $E_n(0) = E_n$  is called the n-th Euler number.

By (1.3) and (1.4), we easily see that

$$B_{n}(x) = (B+x)^{n} = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_{l},$$

$$E_{n}(x) = (E+x)^{n} = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l},$$
(1.5)

where  $\binom{n}{l} = n!/(n-l)! l! = n(n-1)(n-2)\cdots(n-l+1)/l!$  (see [14, 16, 19]).

The following properties of Bernoulli numbers and polynomials are well known (see [10, 11]).

For  $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ ,

$$\sum_{j=0}^{n} {n \choose j} y^{n-j} \frac{B_{j+1}(x)}{j+1} = \frac{B_{n+1}(x+y) - y^{n+1}}{n+1},$$
(1.6)

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} y^{n-2i} \frac{B_{2j+1}(x)}{2j+1} = \frac{B_{n+1}(x+y) + (-1)^n B_{n+1}(x-y)}{2n+2},$$
(1.7)

$$\sum_{j=1}^{\left[(n+1)/2\right]} {n \choose 2j-1} y^{n+1-2j} \frac{B_{2j}(x)}{2j} = \frac{B_{n+1}(x+y) + (-1)^{n-1} B_{n+1}(x-y) - 2y^{n+1}}{2n+2},$$
(1.8)

where  $[\cdot]$  is Gauss' symbol.

First, we investigate some identities of Euler polynomials corresponding to (1.6), (1.7) and (1.8). From those identities, we derive some interesting identities and properties by using p-adic integral on  $\mathbb{Z}_p$ .

### 2. Some Identities of Bernoulli and Euler Polynomials

By (1.4), we get

$$E_k(x+y) = \sum_{j=0}^k \binom{k}{j} y^{k-j} E_j(x), \quad \text{for } \in \mathbb{Z}_+.$$
 (2.1)

From (2.1), we note that

$$E_{k}(x+y) = \sum_{j=0}^{k} {k \choose j} y^{k-j} E_{j}(x)$$

$$= y^{k} + \sum_{j=1}^{k} \frac{k}{j} {k-1 \choose j-1} y^{k-j} E_{j}(x).$$
(2.2)

Thus, we have

$$\sum_{j=0}^{k-1} {k-1 \choose j} y^{k-1-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_k(x+y) - y^k}{k}.$$
 (2.3)

Replacing k by k + 1 in (2.3), we obtain the following proposition.

**Proposition 2.1.** *For*  $k \in \mathbb{Z}_+$ *, one has* 

$$\sum_{j=0}^{k} {k \choose j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x+y) - y^{k+1}}{k+1}.$$
 (2.4)

Let us replace y by -y in Proposition 2.1. Then we have

$$\sum_{j=0}^{k} {k \choose j} (-1)^{k-j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x-y) - (-1)^{k+1} y^{k+1}}{k+1}.$$
 (2.5)

Thus, we see that

$$\sum_{j=0}^{k} {k \choose j} (-1)^j y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{(-1)^k E_{k+1}(x-y) + y^{k+1}}{k+1}.$$
 (2.6)

Therefore, adding (2.4) and (2.6), we obtain the following proposition.

**Proposition 2.2.** *For*  $k \in \mathbb{Z}_+$ *, one has* 

$$\sum_{j=0}^{[k/2]} {k \choose 2j} y^{k-2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{k+1}(x+y) + (-1)^k E_{k+1}(x-y)}{2k+2}.$$
 (2.7)

From (2.2), we note that

$$\sum_{j=1}^{k} \frac{1}{j} {k-1 \choose j-1} y^{k-j} (-1)^{j} E_{j}(x) = \frac{(-1)^{k} E_{k}(x-y) - y^{k}}{k}.$$
 (2.8)

By (2.3) and (2.8), we get

$$\sum_{j=1}^{\lfloor k/2 \rfloor} \frac{1}{2j} \binom{k-1}{2j-1} y^{k-2j} E_{2j}(x) = \frac{E_k(x+y) + (-1)^k E_k(x-y) - 2y^k}{2k}.$$
 (2.9)

Therefore, replacing k by k + 1, we obtain the following proposition.

**Proposition 2.3.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=1}^{\left[(k+1)/2\right]} {k \choose 2j-1} y^{k+1-2j} \frac{E_{2j}(x)}{2j} = \frac{E_{k+1}(x+y) + (-1)^{k+1} E_{k+1}(x-y) - 2y^{k+1}}{2k+2}.$$
 (2.10)

Letting y = 1 in Proposition 2.1, we have

$$\sum_{j=0}^{k} {k \choose j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x+1) - 1}{k+1},$$
(2.11)

$$E_{k+1}(x+1) = \sum_{l=0}^{k+1} {k+1 \choose l} (E+1)^l x^{k+1-l}$$

$$= (2-E_0)x^{k+1} - \sum_{l=1}^{k+1} {k+1 \choose l} E_l x^{k+1-l}$$

$$= 2x^{k+1} - \sum_{l=0}^{k+1} {k+1 \choose l} E_l x^{k+1-l} = 2x^{k+1} - E_{k+1}(x).$$
(2.12)

Therefore, by (2.11) and (2.12), we obtain the following corollary.

**Corollary 2.4.** *For*  $k \in \mathbb{Z}_+$ *, one has* 

$$\sum_{j=0}^{k} {k \choose j} \frac{E_{j+1}(x)}{j+1} = -\frac{E_{k+1}(x)}{k+1} + \frac{2x^{k+1} - 1}{k+1}.$$
 (2.13)

Replacing y by 1 and k by 2k in Proposition 2.2, we have

$$\sum_{j=0}^{k} {2k \choose 2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{2k+1}(x+1) + E_{2k+1}(x-1)}{4k+2} 
= \frac{E_{2k+1}(x+1) + E_{2k+1}(x) + E_{2k+1}(x) + E_{2k+1}(x-1)}{4k+2} - \frac{2E_{2k+1}(x)}{4k+2} 
= \frac{2x^{2k+1} + 2(x-1)^{2k+1}}{4k+2} - \frac{E_{2k+1}(x)}{2k+1}.$$
(2.14)

Therefore, by (2.14), we obtain the following corollary.

**Corollary 2.5.** *For*  $k \in \mathbb{Z}_+$ *, one has* 

$$\sum_{j=0}^{k} {2k \choose 2j} \frac{E_{2j+1}(x)}{2j+1} = -\frac{E_{2k+1}(x)}{2k+1} + \frac{x^{2k+1} + (x-1)^{2k+1}}{2k+1}.$$
 (2.15)

Replacing y by 1 and k by 2k in Proposition 2.3, we have

$$\sum_{j=1}^{k} {2k \choose 2j-1} \frac{E_{2j}(x)}{2j} = \frac{E_{2k+1}(x+1) - E_{2k+1}(x-1) - 2}{4k+2}$$

$$= \frac{(E_{2k+1}(x+1) + E_{2k+1}(x)) - (E_{2k+1}(x) + E_{2k+1}(x-1))}{4k+2} - \frac{1}{2k+1}$$

$$= \frac{2x^{2k+1} - 2(x-1)^{2k+1}}{4k+2} - \frac{1}{2k+1}$$

$$= \frac{x^{2k+1} - (x-1)^{2k+1}}{2k+1} - \frac{1}{2k+1}.$$
(2.16)

Therefore, by (2.16), we obtain the following corollary.

**Corollary 2.6.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=1}^{k} {2k \choose 2j-1} \frac{E_{2j}(x)}{2j} = \frac{x^{2k+1} - (x-1)^{2k+1}}{2k+1} - \frac{1}{2k+1}.$$
 (2.17)

Replacing y by 1/2 and k by 2k in Proposition 2.3, we get

$$\sum_{j=1}^{k} {2k \choose 2j-1} \left(\frac{1}{2}\right)^{2k+1-2j} \frac{E_{2j}(x)}{2j} = \frac{E_{2k+1}(x+1/2) - E_{2k+1}(x-1/2) - 2^{-2k}}{4k+2}.$$
 (2.18)

Thus, we have

$$\sum_{j=1}^{k} {2k \choose 2j-1} 2^{2j} \frac{E_{2j}(x)}{2j} = \frac{2^{2k} (E_{2k+1}(x+1/2) - E_{2k+1}(x-1/2)) - 1}{2k+1}, \tag{2.19}$$

$$E_{2k+1}\left(x+\frac{1}{2}\right) = E_{2k+1}\left(x-\frac{1}{2}+1\right) = \sum_{l=0}^{2k+1} {2k+1 \choose l} \left(x-\frac{1}{2}\right)^{2k+1-l} (E+1)^{l}$$

$$= 2\left(x-\frac{1}{2}\right)^{2k+1} - \sum_{l=0}^{2k+1} {2k+1 \choose l} \left(x-\frac{1}{2}\right)^{2k+1-l} E_{l}$$

$$= 2\left(x-\frac{1}{2}\right)^{2k+1} - E_{2k+1}\left(x-\frac{1}{2}\right).$$

$$(2.20)$$

Therefore, by (2.19) and (2.20), we obtain the following corollary.

**Corollary 2.7.** *For*  $k \in \mathbb{N}$ *, we have* 

$$\sum_{j=1}^{k} {2k \choose 2j-1} 2^{2j} \frac{E_{2j}(x)}{2j} = -\frac{2^{2k+1}E_{2k+1}(x-1/2)}{2k+1} + \frac{2^{2k+1}(x-1/2)^{2k+1}}{2k+1} - \frac{1}{2k+1}.$$
 (2.21)

Replacing y by 1 and k by 2k + 1 in Proposition 2.2, we get

$$\sum_{j=0}^{k} {2k+1 \choose 2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{2k+2}(x+1) - E_{2k+2}(x-1)}{4k+4}$$

$$= \frac{(E_{2k+2}(x+1) + E_{2k+2}(x)) - (E_{2k+2}(x) + E_{2k+2}(x-1))}{4k+4}$$

$$= \frac{2x^{2k+2} - 2(x-1)^{2k+2}}{4k+4} = \frac{x^{2k+2} - (x-1)^{2k+2}}{2k+2}.$$
(2.22)

Therefore, by (2.22), we obtain the following corollary.

**Corollary 2.8.** *For*  $k \in \mathbb{Z}_+$ *, one has* 

$$\sum_{j=0}^{k} {2k+1 \choose 2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{x^{2k+2} - (x-1)^{2k+2}}{2k+2}.$$
 (2.23)

Replacing k by 2k + 1 and y by 1 in Proposition 2.3, we get

$$\sum_{j=1}^{k+1} {2k+1 \choose 2j-1} \frac{E_{2j}(x)}{2j} = \frac{E_{2k+2}(x+1) + E_{2k+2}(x-1) - 2}{4k+4}$$

$$= \frac{(E_{2k+2}(x+1) + E_{2k+2}(x)) + (E_{2k+2}(x) + E_{2k+2}(x-1))}{4k+4} - \frac{E_{2k+2}(x) + 1}{2k+2}$$

$$= \frac{x^{2k+2} + (x-1)^{2k+2}}{2k+2} - \frac{E_{2k+2}(x) + 1}{2k+2}.$$
(2.24)

Therefore, by (2.24), we obtain the following corollary.

**Corollary 2.9.** *For*  $k \in \mathbb{Z}_+$ *, we have* 

$$\sum_{j=1}^{k+1} {2k+1 \choose 2j-1} \frac{E_{2j}(x)}{2j} = \frac{x^{2k+2} + (x-1)^{2k+2}}{2k+2} - \frac{E_{2k+2}(x) + 1}{2k+2}.$$
 (2.25)

Replacing k by 2k + 1 and y by 1/2 in Proposition 2.2, we have

$$\sum_{j=0}^{k} {2k+1 \choose 2j} \left(\frac{1}{2}\right)^{2k+1-2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{2k+2}(x+1/2) - E_{2k+2}(x-1/2)}{4k+4}.$$
 (2.26)

Thus, by multipling  $2^{2k+1}$  on both sides, we get

$$\sum_{j=0}^{k} {2k+1 \choose 2j} 2^{2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{2^{2k+1} \{ E_{2k+2}(x+1/2) - E_{2k+2}(x-1/2) \}}{4k+4}.$$
 (2.27)

By (2.20) and (2.27), we see that

$$\sum_{j=0}^{k} {2k+1 \choose 2j} 2^{2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{2^{2k} \left( 2(x-1/2)^{2k+2} - 2E_{2k+2}(x-1/2) \right)}{2k+2} \\
= \frac{2^{2k} (x-1/2)^{2k+2} - 2^{2k} E_{2k+2}(x-1/2)}{k+1}.$$
(2.28)

Therefore, by (2.28), we obtain the following corollary.

**Corollary 2.10.** *For*  $k \in \mathbb{Z}_+$ *, we have* 

$$\sum_{j=0}^{k} {2k+1 \choose 2j} 2^{2j} \frac{E_{2j+1}(x)}{2j+1} = -\frac{2^{2k} E_{2k+2}(x-1/2)}{k+1} + \frac{2^{2k} (x-1/2)^{2k+2}}{k+1}.$$
 (2.29)

From (1.6), we can derive the following equation:

$$\sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1}(x)}{j+1} = x^k - \frac{1}{k+1}, \quad \text{for } k \in \mathbb{N}.$$
 (2.30)

Let us take the *p*-adic integral on both sides in (2.30) as follows: for  $k \in \mathbb{N}$ ,

$$I_{1} = \sum_{j=0}^{k-1} {k \choose j} \int_{\mathbb{Z}_{p}} \frac{B_{j+1}(x)}{j+1} d\mu(x) = \sum_{j=0}^{k-1} {k \choose j} \frac{1}{j+1} \sum_{l=0}^{j+1} {j+1 \choose l} B_{j+1-l} \int_{\mathbb{Z}_{p}} x^{l} d\mu(x)$$

$$= \sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} {k \choose j} {j+1 \choose l} B_{j+1-l} B_{l}.$$
(2.31)

On the other hand,

$$I_1 = \int_{\mathbb{Z}_p} x^k d\mu(x) - \frac{1}{k+1} \int_{\mathbb{Z}_p} d\mu(x) = B_k - \frac{1}{k+1}.$$
 (2.32)

Therefore, by (2.31) and (2.32), we obtain the following theorem.

**Theorem 2.11.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} B_l = B_k - \frac{1}{k+1}.$$
 (2.33)

In (2.30), let us take the fermionic *p*-adic integral on both sides as follows:

$$I_{2} = \sum_{j=0}^{k-1} {k \choose j} \frac{1}{j+1} \int_{\mathbb{Z}_{p}} B_{j+1}(x) d\mu_{-1}(x)$$

$$= \sum_{j=0}^{k-1} \frac{{k \choose j}}{j+1} \sum_{l=0}^{j+1} {j+1 \choose l} B_{j+1-l} \int_{\mathbb{Z}_{p}} x^{l} d\mu_{-1}(x)$$

$$= \sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} {k \choose j} {j+1 \choose l} B_{j+1-l} E_{l}.$$

$$(2.34)$$

On the other hand

$$I_2 = \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) - \frac{1}{k+1} \int_{\mathbb{Z}_p} d\mu_{-1}(x) = E_k - \frac{1}{k+1}.$$
 (2.35)

Therefore, by (2.34) and (2.35), we obtain the following theorem.

**Theorem 2.12.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} E_l = E_k - \frac{1}{k+1}.$$
 (2.36)

From (1.7), we can easily derive the following equation:

$$\sum_{j=0}^{k-1} {2k \choose 2j} \frac{B_{2j+1}(x)}{2j+1} = \frac{x^{2k} - (x-1)^{2k}}{2}.$$
 (2.37)

Let us take  $\int_{\mathbb{Z}_p} d\mu(x)$  on both sides in (2.37). Then we have

$$I_{3} = \sum_{j=0}^{k-1} {2k \choose 2j} \frac{1}{2j+1} \int_{\mathbb{Z}_{p}} B_{2j+1}(x) d\mu(x)$$

$$= \sum_{j=0}^{k-1} {2k \choose 2j} \frac{1}{2j+1} \sum_{l=0}^{2j+1} {2j+1 \choose l} B_{2j+1-l} \int_{\mathbb{Z}_{p}} x^{l} d\mu(x)$$

$$= \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} {2k \choose 2j} {2j+1 \choose l} B_{2j+1-l} B_{l}.$$
(2.38)

On the other hand,

$$I_{3} = \frac{1}{2} \left( \int_{\mathbb{Z}_{p}} x^{2k} d\mu(x) - \int_{\mathbb{Z}_{p}} (x - 1)^{2k} d\mu(x) \right)$$

$$= \frac{1}{2} (B_{2k} - B_{2k}(-1)) = \frac{1}{2} (B_{2k} - B_{2k}(2))$$

$$= \frac{1}{2} (B_{2k} - (2k + \delta_{1,2k} + B_{2k})),$$
(2.39)

where  $\delta_{n,k}$  is a Kronecker symbol.

Therefore, by (2.38) and (2.39), we obtain the following theorem.

**Theorem 2.13.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} B_l = -k.$$
 (2.40)

Taking  $\int_{\mathbb{Z}_n} d\mu_{-1}(x)$  on both sides in (2.37), we get

$$I_{4} = \sum_{j=0}^{k-1} {2k \choose 2j} \frac{1}{2j+1} \sum_{l=0}^{2j+1} {2j+1 \choose l} B_{2j+1-l} \int_{\mathbb{Z}_{p}} x^{l} d\mu_{-1}(x)$$

$$= \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} {2k \choose 2j} {2j+1 \choose l} B_{2j+1-l} E_{l}.$$
(2.41)

On the other hand

$$I_{4} = \frac{1}{2} \left( \int_{\mathbb{Z}_{p}} x^{k} d\mu_{-1}(x) - \int_{\mathbb{Z}_{p}} (x - 1)^{2k} d\mu_{-1}(x) \right)$$

$$= \frac{1}{2} \left( \int_{\mathbb{Z}_{p}} x^{2k} d\mu_{-1}(x) - \int_{\mathbb{Z}_{p}} (x + 2)^{2k} d\mu_{-1}(x) \right)$$

$$= \frac{1}{2} (E_{2k} - E_{2k}(2)) = \frac{1}{2} \{ E_{2k} - (2 + E_{2k} - 2\delta_{0,2k}) \}$$

$$= -1 + \delta_{0,k}.$$
(2.42)

Therefore, by (2.41) and (2.42), we obtain the following theorem.

**Theorem 2.14.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} E_l = -1.$$
 (2.43)

From (1.8), we can also derive the following equation:

$$\sum_{j=1}^{k} {2k \choose 2j-1} \frac{B_{2j}(x)}{2j} = \frac{x^{2k} + (x-1)^{2k}}{2} - \frac{1}{2k+1}.$$
 (2.44)

Let us take the bosonic *p*-adic integral on both sides in (2.44). Then we get

$$I_{5} = \sum_{j=1}^{k} {2k \choose 2j-1} \frac{1}{2j} \int_{\mathbb{Z}_{p}} B_{2j}(x) d\mu(x)$$

$$= \sum_{j=1}^{k} {2k \choose 2j-1} \frac{1}{2j} \sum_{l=0}^{2j} {2j \choose l} B_{2j-l} \int_{\mathbb{Z}_{p}} x^{l} d\mu(x)$$

$$= \sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} {2k \choose 2j-1} {2j \choose l} B_{2j-l} B_{l}.$$

$$(2.45)$$

On the other hand,

$$I_{5} = \frac{1}{2} \int_{\mathbb{Z}_{p}} \left( x^{2k} + (-1+x)^{2k} \right) d\mu(x) - \frac{1}{2k+1} \int_{\mathbb{Z}_{p}} d\mu(x)$$

$$= \frac{1}{2} \int_{\mathbb{Z}_{p}} \left( x^{2k} + (x+2)^{2k} \right) d\mu(x) - \frac{1}{2k+1}$$

$$= \frac{1}{2} (B_{2k} + B_{2k}(2)) - \frac{1}{2k+1}$$

$$= \frac{1}{2} (B_{2k} + 2k + B_{2k} + \delta_{1,2k}) - \frac{1}{2k+1}.$$
(2.46)

Therefore, by (2.45) and (2.46), we obtain the following theorem.

**Theorem 2.15.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} B_l = B_{2k} + k - \frac{1}{2k+1}. \tag{2.47}$$

Now, let us consider the fermionic *p*-adic integral on both sides in (2.44):

$$I_{6} = \sum_{j=1}^{k} {2k \choose 2j-1} \frac{1}{2j} \sum_{l=0}^{2j} {2j \choose l} B_{2j-l} \int_{\mathbb{Z}_{p}} x^{l} d\mu_{-1}(x)$$

$$= \sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} {2k \choose 2j-1} {2j \choose l} B_{2j-l} E_{l}.$$
(2.48)

On the other hand,

$$I_{6} = \frac{1}{2} \int_{\mathbb{Z}_{p}} \left( x^{2k} + (x-1)^{2k} \right) d\mu_{-1}(x) - \frac{1}{2k+1} \int_{\mathbb{Z}_{p}} d\mu_{-1}(x)$$

$$= \frac{1}{2} \int_{\mathbb{Z}_{p}} \left( x^{2k} + (x+2)^{2k} \right) d\mu_{-1}(x) - \frac{1}{2k+1}$$

$$= \frac{1}{2} (E_{2k} + E_{2k}(2)) - \frac{1}{2k+1}$$

$$= \frac{1}{2} (E_{2k} + (2 + E_{2k} - 2\delta_{0,2k})) - \frac{1}{2k+1}$$

$$= E_{2k} + 1 - \delta_{0,2k} - \frac{1}{2k+1} = \frac{2k}{2k+1}.$$

$$(2.49)$$

Therefore, by (2.48) and (2.49), we obtain the following theorem.

**Theorem 2.16.** *For*  $k \in \mathbb{N}$ *, one has* 

$$\sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} E_l = \frac{2k}{2k+1}. \tag{2.50}$$

## Acknowledgment

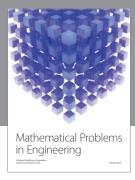
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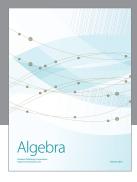
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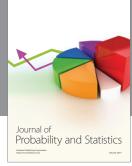
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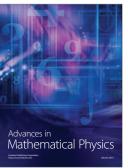




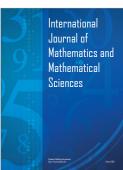


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