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Research Article

Alternative Forms of Compound Fractional Poisson Processes

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We study here different fractional versions of the compound Poisson process. The fractionality is introduced in the counting process representing the number of jumps as well as in the density of the jumps themselves. The corresponding distributions are obtained explicitly and proved to be solution of fractional equations of order less than one. Only in the final case treated in this paper, where the number of jumps is given by the fractional-difference Poisson process defined in Orsingher and Polito (2012), we have a fractional driving equation, with respect to the time argument, with order greater than one. Moreover, in this case, the compound Poisson process is Markovian and this is also true for the corresponding limiting process. All the processes considered here are proved to be compositions of continuous time random walks with stable processes (or inverse stable subordinators). These subordinating relationships hold, not only in the limit, but also in the finite domain. In some cases the densities satisfy master equations which are the fractional analogues of the well-known Kolmogorov one.

1. Introduction and Preliminary Results

The fractional Poisson process (FPP), which we will denote by $\mathcal{N}_\beta(t)$, $t > 0$, $\beta \in (0,1]$, has been introduced in [1], by replacing, in the differential equation governing the Poisson process, the time derivative with a fractional one. Later, in [2, 3], it was proved to be a renewal process with Mittag-Leffler distributed waiting times (and therefore with infinite mean). In [4] it has been expressed as the composition $N(\mathcal{T}_\beta(t))$ of a standard Poisson process N with the fractional diffusion $\mathcal{T}_\beta$, independent of N. A full characterization of $\mathcal{N}_\beta$ in terms of its finite multidimensional distributions can be found in [5]. In [6] the coincidence between $\mathcal{N}_\beta$ and the fractal time Poisson process (FTPP) defined as $N(\mathcal{L}_\beta(t))$ has been proved, where

$\mathcal{L}_\beta(t)$, $t \geq 0$ is the inverse of the stable subordinator $\mathcal{A}_\beta(t)$ of index β (with parameters $\mu = 0$, $\theta = 1$, $\sigma = (t\cos\pi\beta/2)^{1/\beta}$, in the notation of [7], that we will adopt hereafter). Thus, the process $\mathcal{A}_\beta$ is characterized by the following Laplace pairs:

$$\begin{gathered}\mathbb{E}e^{-k\mathcal{A}_\beta(t)} = e^{-k^\beta t}, \quad k,t>0,\\ \int_0^{+\infty} e^{-st}h_\beta(x,t)dt = x^{\beta-1}E_{\beta,\beta}\left(-sx^\beta\right), \quad s,x>0,\end{gathered} \tag{1.1}$$

where $E_{\beta,\delta}$ is the Mittag-Leffler function of parameters β, δ and $h_\beta(x,t)$ is the density of $\mathcal{A}_\beta(t)$. The inverse stable subordinator $\mathcal{L}_\beta$ is defined by the following relation:

$$\mathcal{L}_\beta(t) := \inf\{s : \mathcal{A}_\beta(s) = t\}, \quad z,t>0, \tag{1.2}$$

and therefore we get

$$\begin{gathered}\mathbb{E}e^{-k\mathcal{L}_\beta(t)} = E_{\beta,1}\left(-kt^\beta\right), \quad k,t>0,\\ \int_0^{+\infty} e^{-st}l_\beta(x,t)dt = s^{\beta-1}e^{-xs^\beta}, \quad s,x>0,\end{gathered} \tag{1.3}$$

where $l_\beta(x,t)$ is the density of $\mathcal{L}_\beta(t)$.

We will make use also of different forms of FPP such as the alternative fractional Poisson process in [8] and the fractional-difference Poisson process in [9].

In this paper we study several fractional compound Poisson processes and, to help the reader, we list the acronyms used throughout the paper by the end of the paper.

The first form of fractional compound Poisson process has been introduced in [10], in the form of a continuous time random walk with infinite-mean waiting times (see also [11]). This corresponds to the following random walk time changed via the FTPP, that is,

$$Y_\beta(t) = \sum_{j=1}^{N(\mathcal{L}_\beta(t))} X_j, \quad t \geq 0, \tag{1.4}$$

with X_j, $j = 1,2,\ldots$ are i.i.d. random variables, independent from N and $\mathcal{L}_\beta$. The last assumption (that we will adopt throughout the paper) corresponds to the so-called uncoupled case.

In [6] it is proved that subordinating random walk to the fractional Poisson process $\mathcal{N}_\beta(t)$, $t \geq 0$, produces the same one-dimensional distribution. The (generalized) density function of $Y_\beta(t)$ can be expressed as

$$g_{Y_\beta}(y,t) := E_{\beta,1}\left(-\lambda t^\beta\right)\delta(y) + f_{Y_\beta}(y,t), \quad y,t \geq 0, \tag{1.5}$$

where the first term refers to the probability mass concentrated in the origin, $\delta(y)$ denotes the Dirac delta function, and f_{Y_β} denotes the density of the absolutely continuous component. The function g_{Y_β} given in (1.5) satisfies the following fractional master equation, that is,

$$\frac{\partial^\beta}{\partial t^\beta} g_{Y_\beta}(y,t) = -\lambda g_{Y_\beta}(y,t) + \lambda \int_{-\infty}^{+\infty} g_{Y_\beta}(y-x,t) f_X(x) dx, \tag{1.6}$$

where $\partial^\beta/\partial t^\beta$ is the Caputo fractional derivative of order $\beta \in (0,1]$ (see, for example, [12]) and the random variables X_j, $j = 1,2,\ldots$ have continuous density f_X.

We also recall the following result proved in [13] for the rescaled version of the time-fractional compound Poisson process (hereafter TFCPP): if the random variables $X_j, j = 1,2,\ldots$ are centered and have finite variance, then

$$c^{-\beta/2} Y_\beta(ct) \Longrightarrow W(\mathcal{L}_\beta(t)), \quad c \longrightarrow \infty, \tag{1.7}$$

where W is a standard Brownian motion and $\Rightarrow$ denotes weak convergence.

A detailed exposition of the theory of TFCPP and continuous time random walks can be found in [14, 15], where the density f_{Y_β} is expressed in terms of successive derivatives of the Mittag-Leffler function as follows:

$$f_{Y_\beta}(y,t) = \sum_{n=1}^{\infty} f_X^{*n}(y) \Pr\{\mathcal{N}_\beta(t) = n\} = \sum_{n=1}^{\infty} f_X^{*n}(y) \frac{(\lambda t^\beta)^n}{n!} \frac{\partial^n}{\partial x^n} E_{\beta,1}(x)\bigg|_{x=-\lambda t^\beta}, \quad t, y \geq 0, \tag{1.8}$$

where f_X^{*n} is the nth convolution of the density f_X of the r.v.'s X_j.

A further asymptotic result has been proved in [15], under the assumption that the density of the jump variables (which we will denote, in this special case, as X_j^*) behaves asymptotically as

$$\widehat{f}_{X^*}(h\kappa) := \int_{-\infty}^{+\infty} e^{i\kappa h x} \widehat{f}_{X^*}(x) dx \simeq 1 - h^\alpha |\kappa|^\alpha, \quad h \longrightarrow 0, \ \alpha \in (0,1], \tag{1.9}$$

where $\widehat{\cdot}$ denotes the Fourier transform. In this case the TFCPP is defined as $Y_\beta^*(t) = \sum_{j=1}^{\mathcal{N}_\beta(t)} X_j^*$ and the rescaled version displays the following weak convergence:

$$h Y_\beta^*\left(\frac{t}{r}\right) \Longrightarrow Z(t), \tag{1.10}$$

for $h, r \rightarrow 0$, s.t. $h^\alpha / r^\beta \rightarrow 1$. The characteristic function of the limiting process $Z(t)$ is given by

$$E_{\beta,1}\left(-\lambda t^\beta |\kappa|^\alpha\right) \tag{1.11}$$

and thus it can be represented as $\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$, where $\mathcal{S}_\alpha$ is a symmetric α-stable process with parameters $\mu = 0$, $\theta = 0$, $\sigma = (t \cos \pi\alpha/2)^{1/\alpha}$. For $\beta < 1$, the inverse stable subordinator

$\mathcal{L}_\beta(t)$ is not Markovian as well as not Lévy (see [10]) and the same is true for $\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$, as remarked in [15]; moreover, the density $u = u(y,t)$ of the latter is the solution to the space-time fractional equation:

$$\frac{\partial^\beta u}{\partial t^\beta} = \lambda \frac{\partial^\alpha u}{\partial |y|^\alpha}, \qquad u(y,0) = \delta(y), \quad y \in \mathbb{R}, \ t > 0, \tag{1.12}$$

where $\partial^\alpha / \partial |y|^\alpha$ denotes the Riesz-Feller derivative of order $\alpha \in (0,1]$ (see [16]). Thus, in the special case $\alpha = 1$, it reduces to the composition of a Cauchy process with $\mathcal{L}_\beta$.

Finally, we recall the following result proved in [17]: under the assumption of heavy tailed r.v.'s representing the jumps, that is,

$$\Pr\{|X_j| > x\} \sim x^{-\alpha}, \quad x \longrightarrow \infty, \tag{1.13}$$

the following convergence holds, as $c \to \infty$,

$$c^{-\beta/\alpha} Y_\beta(ct) \Longrightarrow \mathcal{O}(\mathcal{L}_\beta(t)). \tag{1.14}$$

In (1.14) $\mathcal{O}$ is a α-stable Lévy process with density $p_\alpha(x,t)$ and characteristic function

$$\widehat{p}_\alpha(\kappa, t) = e^{tb[q(-i\kappa)^\alpha + (1-q)(i\kappa)^\alpha]}, \quad \text{for } \begin{cases} b < 0, \ 0 < \alpha < 1 \\ \text{or } b > 0, \ 1 < \alpha < 2, \end{cases} \tag{1.15}$$

under the assumption that $\lim_{x\to\infty} \Pr\{X_j < -x\} / \Pr\{|X_j| > x\} = q \in 0,1]$. The density of the limiting process is proved to satisfy the following time and space fractional equation:

$$D^\beta_{0+,t} u = qbD^\alpha_{-,x} u + (1-q)bD^\alpha_{0+,x} u, \tag{1.16}$$

where the fractional derivatives are intended in the Riemann-Liouville sense (see [12], formulae (2.2.3) and (2.2.4), page 80).

We present, in this paper, different versions of the compound Poisson process (CPP), fractional (under different acceptions) with respect to time and space; we provide for them analytic expressions of the distributions and some composition relationships with stable and inverse-stable processes, holding not only in the scaling limit, but also in the finite domain.

Tables 1 and 2 provide a summary of these results in the finite and asymptotic domains, respectively.

We assume here exponential jumps (generalized later to Mittag-Leffler), since this allows to obtain explicit equations (fractional in most cases) driving these fractional CPP's for any finite value of the time and space arguments. This kind of explicit formulae, together with the knowledge of the related governing differential equations, is of great importance in many actuarial applications (see, for example, [18], Section 4.2). In risk theory it is related to the Tweedie's compound Poisson model (see [19]). The hypothesis of exponential jumps has been widely applied also in other fields: in natural sciences it leads to the so-called compound Poisson-Gamma model, which is used for rainfall prediction (see, for example, [20]).

2. Time-Fractional Compound Poisson Processes

We consider different forms of TFCPP, starting with the more familiar one given in (1.4) and then comparing the results with those obtained for an alternative definition of FPP.

2.1. The Standard Case

In order to get a form of the density of the TFCPP more explicit than (1.8), we assume that the X_j's are exponentially distributed: in this case it can be expressed in terms of the generalized Mittag-Leffler function:

$$E^{\gamma}_{\alpha,\delta}(x) = \sum_{j=0}^{\infty} \frac{(\gamma)^{(j)}}{j!} \frac{x^j}{\Gamma(\alpha j + \delta)}, \quad \alpha, \delta, \gamma \in \mathbb{C}, \ \mathcal{R}(\alpha), \mathcal{R}(\delta) > 0, \tag{2.1}$$

where $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ is the rising factorial (or Pochhammer symbol). Moreover, we can obtain the fractional partial-differential equation satisfied by the density of its absolutely continuous component.

Theorem 2.1. *The process*

$$Y_\beta(t) = \sum_{j=1}^{\mathcal{N}_\beta(t)} X_j, \quad t \geq 0, \tag{2.2}$$

with X_j, $j = 1, 2 \ldots$, independent and exponentially distributed with parameter ξ, has the following distribution:

$$\Pr\{Y_\beta(t) \leq y\} = E_{\beta,1}\left(-\lambda t^\beta\right) 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{Y_\beta}(z,t)\,dz, \quad t \geq 0, \ y \in \mathbb{R}, \tag{2.3}$$

where

$$f_{Y_\beta}(y,t) = \frac{e^{-\xi y}}{y} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)!} E^{n+1}_{\beta,\beta n+1}\left(-\lambda t^\beta\right) 1_{[0,+\infty)}(y), \quad t \geq 0. \tag{2.4}$$

The function $f_{Y_\beta}(y,t)$ given in (2.4) *satisfies the following partial differential equation:*

$$\xi \frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta} = -\left[\lambda + \frac{\partial^\beta}{\partial t^\beta}\right] \frac{\partial}{\partial y} f_{Y_\beta}, \quad t, y \geq 0, \tag{2.5}$$

where $\partial^\beta / \partial t^\beta$ denotes the Caputo fractional derivative with the conditions

$$\begin{gathered} f_{Y_\beta}(y,0) = 0, \\ \int_0^{+\infty} f_{Y_\beta}(y,t)\,dy = 1 - E_{\beta,1}\left(-\lambda t^\beta\right). \end{gathered} \tag{2.6}$$

Proof . Formula (1.8) can be rewritten by considering that $f_X^{*n}(y) = \xi^n y^{n-1} e^{-\xi y}/(n-1)!$ and using the expression of $\Pr\{\mathcal{N}_\beta(t) = n\}$ in terms of generalized Mittag-Leffler functions (see [21]), that is,

$$\Pr\left\{\mathcal{N}_\beta(t) = n\right\} = \lambda^n t^{n\beta} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right), \quad n \geq 0. \tag{2.7}$$

In order to derive (2.5), we evaluate the following partial derivatives of (2.4):

$$\begin{aligned}
\frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) &= \frac{e^{-\xi y}}{y t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)! n!} \sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)}, \\
\frac{\partial}{\partial y} f_{Y_\beta}(y,t) &= -\frac{\xi e^{-\xi y}}{y} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)! n!} \sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2} \sum_{n=2}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)! n!} \sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n + 1)}, \\
\frac{\partial}{\partial y} \frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) &= -\frac{\xi e^{-\xi y}}{y t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{n!(n-1)!} \sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2 t^\beta} \sum_{n=2}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)! n!} \sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)} \\
&= -\frac{\xi e^{-\xi y}}{y t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{((n-1)!)^2} \sum_{j=0}^\infty \frac{(n+j-1)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad - \frac{\xi e^{-\xi y}}{y t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{n!(n-1)!} \sum_{j=1}^\infty \frac{(n+j-1)!\left(-\lambda t^\beta\right)^j}{(j-1)! \Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2 t^\beta} \sum_{n=2}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)!(n-1)!} \sum_{j=0}^\infty \frac{(n+j-1)!\left(-\lambda t^\beta\right)^j}{j! \Gamma(\beta j + \beta n - \beta + 1)} + \\
&\quad + \frac{e^{-\xi y}}{y^2 t^\beta} \sum_{n=2}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-2)! n!} \sum_{j=0}^\infty \frac{(n+j-1)!\left(-\lambda t^\beta\right)^j}{(j-1)! \Gamma(\beta j + \beta n - \beta + 1)}.
\end{aligned} \tag{2.8}$$

By inserting (2.8) in (2.5), the equation is satisfied. Finally, it can be easily verified that the initial condition holds. In order to check the second condition in (2.6), we integrate f_{Y_β} with respect to y:

$$\int_0^\infty \frac{e^{-\xi y}}{y} \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta y\right)^n}{(n-1)!} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right) dy = \sum_{n=1}^\infty \frac{\left(\lambda \xi t^\beta\right)^n}{(n-1)!} \frac{(n-1)!}{\xi^n} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\lambda t^{\beta}\right)^{n} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^{\beta}\right) - E_{\beta,1}\left(-\lambda t^{\beta}\right) \\
&= 1 - E_{\beta,1}\left(-\lambda t^{\beta}\right),
\end{aligned}
\tag{2.9}
$$

where, in the last step, we have applied formula (2.30) of [21], for $u = 1$. □

2.1.1. *The Nonfractional Case* $\beta = 1$

From (2.4), we obtain the distribution of the standard CPP, defined as $Y(t) = \sum_{n=1}^{N(t)} X_j$, under the assumption of exponential jumps X_j, which reads

$$
\Pr\{Y(t) \le y\} = e^{-\lambda t} 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_Y(z,t)\,dz, \qquad t \ge 0,\ y \in \mathbb{R}, \tag{2.10}
$$

where

$$
\begin{aligned}
f_Y(y,t) &= \frac{e^{-\xi y-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{(\lambda \xi t y)^n}{n!(n-1)!} 1_{[0,+\infty)}(y) \\
&= \lambda \xi t e^{-\xi y-\lambda t} W_{1,2}(\lambda \xi t y) 1_{[0,+\infty)}(y), \quad t \ge 0,
\end{aligned}
\tag{2.11}
$$

$$
W_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(\alpha j+\beta)}, \quad \alpha > -1,\ \beta, z \in \mathbb{C}, \tag{2.12}
$$

is the Wright function. Equation (4.2.8) in [18] provides another expression of f_Y in terms of the modified Bessel function. The density (2.11) satisfies the following equation:

$$
\xi \frac{\partial}{\partial t} f_Y = -\left[\lambda + \frac{\partial}{\partial t}\right] \frac{\partial}{\partial y} f_Y, \tag{2.13}
$$

with conditions

$$
\begin{aligned}
&f_Y(y,0) = 0, \\
&\int_0^{+\infty} f_Y(y,t)\,dy = 1 - e^{-\lambda t},
\end{aligned}
\tag{2.14}
$$

as can be easily verified directly.

Now we recall the following subordination law presented in [6] in a more general setting:

$$
Y_\beta(t) \overset{d}{=} Y(\mathcal{L}_\beta(t)), \tag{2.15}
$$

where $\mathcal{L}_\beta(t),\ t \geq 0$ is the inverse stable subordinator defined by (1.2). We give an explicit proof of (2.15), which will be useful to prove analogous results in the next sections. We start with the evaluation of the Laplace transform (hereafter denoted by $\widetilde{\cdot}$) of $Y_\beta(t)$ with respect to y: by considering the probability generating function of $\mathcal{N}_\beta$, that is,

$$\mathbb{E}u^{\mathcal{N}_\beta(t)} = E_{\beta,1}\left(-\lambda t^\beta(1-u)\right), \quad |u| \leq 1, \tag{2.16}$$

we get

$$\widetilde{g}_{Y_\beta}(k,t) := \mathbb{E}e^{-kY_\beta(t)} = E_{\beta,1}\left(-\frac{\lambda k}{k+\xi}t^\beta\right). \tag{2.17}$$

Formula (2.17), Laplace transformed with respect to t, gives

$$\widetilde{\widetilde{g}}_{Y_\beta}(k,s) := \int_0^{+\infty} e^{-st}\widetilde{g}_{Y_\beta}(k,t)dt = \frac{s^{\beta-1}(k+\xi)}{s^\beta(k+\xi)+k\lambda}, \tag{2.18}$$

which can be rewritten as

$$\begin{aligned}\widetilde{\widetilde{g}}_{Y_\beta}(k,s) &= s^{\beta-1}\int_0^{+\infty} e^{-s^\beta t}\mathbb{E}e^{-kY(t)}dt \\ &= [\text{by (1.3)}] \\ &= \int_0^{+\infty} \mathbb{E}e^{-kY(z)}\widetilde{l}_\beta(z;s)dz,\end{aligned} \tag{2.19}$$

where $\widetilde{l}_\beta(z;s) := \int_0^{+\infty} e^{-st}l_\beta(z,t)dt$. Thus, by inverting the double Laplace transform, we get

$$\Pr\{Y_\beta(t) \in dy\} = \int_0^{+\infty} \Pr\{Y(z) \in dy\}l_\beta(z,t)dz. \tag{2.20}$$

Now it is also easy to derive (2.5), since we can write in particular from (2.20) that

$$f_{Y_\beta}(y,t) = \int_0^{+\infty} f_Y(y,z)l_\beta(z,t)dz \tag{2.21}$$

and thus we get

$$\begin{aligned}\frac{\partial^\beta}{\partial t^\beta}f_{Y_\beta}(y,t) &= \int_0^{+\infty} f_Y(y,z)\frac{\partial^\beta}{\partial t^\beta}l_\beta(z,t)dz \\ &= -\int_0^{+\infty} f_Y(y,z)\frac{\partial}{\partial z}l_\beta(z,t)dz.\end{aligned} \tag{2.22}$$

Indeed, it is well known that $\mathcal{L}_\beta(t)$ is governed by the following equation:

$$\frac{\partial^\beta}{\partial t^\beta} l_\beta(z,t) = -\frac{\partial}{\partial z} l_\beta(z,t), \qquad l_\beta(z,0) = \delta(z), \quad z,t \geq 0. \tag{2.23}$$

By integrating by parts and applying the initial condition, (2.22) becomes

$$\begin{aligned} \frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) &= \int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) l_\beta(z,t) dz \\ &= [\text{by (2.13)}] \\ &= -\frac{1}{\xi}\frac{\partial}{\partial y}\int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) l_\beta(z,t) dz - \frac{\lambda}{\xi}\frac{\partial}{\partial y}\int_0^{+\infty} f_Y(y,z) l_\beta(z,t) dz \\ &= -\frac{1}{\xi}\frac{\partial}{\partial y}\frac{\partial^\beta}{\partial t^\beta} f_{Y_\beta}(y,t) - \frac{\lambda}{\xi}\frac{\partial}{\partial y} f_{Y_\beta}(y,t). \end{aligned} \tag{2.24}$$

2.2. *An Alternative Case*

We consider now a different model of TFCPP, based on the alternative definition of FPP given in [4], that is,

$$\Pr\left\{\overline{N}_\beta(t) = k\right\} = \frac{\left(\lambda t^\beta\right)^k}{\Gamma(\beta k+1)} \frac{1}{E_{\beta,1}\left(\lambda t^\beta\right)}, \quad t,k \geq 0. \tag{2.25}$$

The process with the above state probabilities plays a crucial role in the evolution of some random motions (see [22]) and can be considered as a fractional version of the Poisson process because its probability generating function (displayed below) satisfies a fractional equation (see formula (4.5) of [4]). The distribution (2.25) can be interpreted as a weighted Poisson distribution (for the general concept of discrete weighted distribution see, e.g., [23], page 90, and the references cited therein) and, as explained in [8], the weights that do not depend on t; actually we have

$$\Pr\left\{\overline{N}_\beta(t) = k\right\} = \frac{w_k p_k\left(t^\beta\right)}{\sum_{j\geq 0} w_j p_j\left(t^\beta\right)}, \quad t,k \geq 0, \tag{2.26}$$

where $w_j = j!/\Gamma(\beta j+1)$, $j = 0,1,\ldots$ (for all t) and $p_j(t) = ((\lambda t)^j/j!)e^{-\lambda t}$, $j = 0,1,\ldots$ are the distribution of the standard Poisson process N with intensity λ. We also recall [24] where one can find a sample path version of the weighted Poisson process.

We remark that the corresponding process is not Markovian, as $\mathcal{N}_\beta$, and moreover is not a renewal. Nevertheless, it is, for some aspects, more similar to the standard Poisson process N than $\mathcal{N}_\beta$. For example, the rate of the asymptotic behavior of its moments is the same as for N.

The moment generating function is given by

$$\mathbb{E}e^{\theta\overline{N}_\beta(t)} = \frac{E_{\beta,1}(\lambda t^\beta e^\theta)}{E_{\beta,1}(\lambda t^\beta)}, \tag{2.27}$$

so that we get

$$\mathbb{E}\overline{N}_\beta(t) = \frac{\lambda t^\beta}{\beta}\frac{E_{\beta,\beta}(\lambda t^\beta)}{E_{\beta,1}(\lambda t^\beta)}. \tag{2.28}$$

By applying the following asymptotic formula of the Mittag-Leffler function

$$E_{\beta,\nu}(z) \simeq \frac{1}{\beta} z^{(1-\nu)/\beta} \exp\left\{z^{1/\beta}\right\}, \quad \text{as } z \longrightarrow \infty, \tag{2.29}$$

(see, for example, [25] or [26]) we get

$$\mathbb{E}\overline{N}_\beta(t) \simeq \frac{1}{\beta}\lambda^{1/\beta} t, \quad \text{as } t \longrightarrow \infty, \tag{2.30}$$

while for $\mathcal{N}_\beta$ the mean value behaves asymptotically as t^β.

We define the alternative TFCPP as

$$\overline{Y}_\beta(t) = \sum_{j=1}^{\overline{N}_\beta(t)} X_j, \quad t \geq 0,\ \beta \in (0,1], \tag{2.31}$$

where again X_j's are i.i.d. with exponential distribution, independent from $\overline{N}_\beta$. Under this assumption we obtain the following result on the distribution of $\overline{Y}_\beta$.

Theorem 2.2. *The process $\overline{Y}_\beta$ defined in* (2.31), *with X_j, $j = 1, 2, \ldots$, independent and exponentially distributed with parameter ξ, has the following distribution:*

$$\Pr\left\{\overline{Y}_\beta(t) \leq y\right\} = \frac{1}{E_{\beta,1}(\lambda t^\beta)} 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{\overline{Y}_\beta}(z,t)\,dz, \quad t \geq 0,\ y \in \mathbb{R}, \tag{2.32}$$

where

$$f_{\overline{Y}_\beta}(y,t) = \frac{\lambda\xi t^\beta e^{-\xi y}}{E_{\beta,1}(\lambda t^\beta)} W_{\beta,\beta+1}\left(\lambda\xi t^\beta y\right) 1_{[0,+\infty)}(y), \quad t \geq 0. \tag{2.33}$$

Proof. The density (2.33) can be obtained as follows:

$$\begin{aligned} f_{\overline{Y}_\beta}(y,t) &= \frac{e^{-\xi y}}{E_{\beta,1}(\lambda t^\beta)} \sum_{n=1}^{\infty} \frac{(\lambda t^\beta)^n}{\Gamma(\beta n+1)} \frac{\xi^n y^{n-1}}{(n-1)!} \\ &= \frac{\lambda \xi t^\beta e^{-\xi y}}{E_{\beta,1}(\lambda t^\beta)} \sum_{l=0}^{\infty} \frac{(\lambda \xi t^\beta)^l}{l!\Gamma(\beta l+\beta+1)}. \end{aligned} \tag{2.34}$$

Moreover, one can check that

$$\int_0^\infty f_{\overline{Y}_\beta}(y,t)dy = 1 - \frac{1}{E_{\beta,1}(\lambda t^\beta)}, \tag{2.35}$$

and this completes the proof. □

Remark 2.3. For $\beta = 1$, formula (2.33) reduces to (2.11). We note that, as happens for the standard case, the density in (2.33) is expressed in terms of a single Wright function instead of an infinite sum of generalized Mittag-Leffler functions (as for the process Y_β). Nevertheless, the presence of a Mittag-Leffler in the denominator does not allow to evaluate the equation satisfied by $f_{\overline{Y}_\beta}$.

2.2.1. *Asymptotic Results*

The analogy with the standard case is even more evident in the asymptotic behavior of the rescaled version of (2.31). Under the assumption (1.9) for the r.v.'s X_j^*, we can prove that, as $h, r \to 0$, s.t. $h^\alpha/r \to 1$ (not depending on β),

$$h\overline{Y}_\beta^*\left(\frac{t}{r}\right) = \sum_{j=1}^{\overline{N}_\beta(t/r)} hX_j^* \Longrightarrow \mathcal{S}_\alpha^\beta(t), \tag{2.36}$$

where $\mathcal{S}_\alpha^\beta$ is a symmetric α-stable Lévy process with $\mu = \theta = 0$ and $\sigma = ((1/\beta)\lambda^{1/\beta} t\cos(\pi\alpha/2))^{1/\alpha}$. Indeed, the characteristic function of (2.36) can be written as

$$\begin{aligned} \widehat{g}_{h\overline{Y}_\beta^*}\left(\kappa, \frac{t}{r}\right) &= \mathbb{E}e^{i\kappa h\overline{Y}_\beta^*(t/r)} = \frac{1}{E_{\beta,1}(\lambda(t^\beta/r^\beta))} \sum_{n=0}^{\infty} \frac{\left(\lambda(t^\beta/r^\beta)\widehat{f}_{hX}(\kappa)\right)^n}{\Gamma(\beta n+1)} \\ &= \frac{E_{\beta,1}\left(\lambda \widehat{f}_{hX}(\kappa)(t^\beta/r^\beta)\right)}{E_{\beta,1}(\lambda(t^\beta/r^\beta))} \\ &= [\text{by the assumption (1.9)}] \end{aligned}$$

$$\begin{aligned}&\simeq \frac{E_{\beta,1}\left(\lambda\left(t^{\beta}/r^{\beta}\right)-\lambda\left(t^{\beta}h^{\alpha}|\kappa|^{\alpha}/r^{\beta}\right)\right)}{E_{\beta,1}\left(\lambda\left(t^{\beta}/r^{\beta}\right)\right)}\\&=[\text{for (2.29)}]\\&\simeq \exp\left\{\frac{\lambda^{1/\beta}t}{r}\left[\left(1-h^{\alpha}|\kappa|^{\alpha}\right)^{1/\beta}-1\right]\right\}.\end{aligned} \tag{2.37}$$

By considering the generalized binomial theorem, we get from (2.37) that

$$\begin{aligned}\hat{g}_{h\overline{Y}^{*}_{\beta}}\left(\kappa,\frac{t}{r}\right)&\simeq \exp\left\{\frac{\lambda^{1/\beta}t}{r}\sum_{j=0}^{\infty}\binom{\frac{1}{\beta}}{j}\left(-h^{\alpha}|\kappa|^{\alpha}\right)^{j}\right\}\\&=\exp\left\{\frac{\lambda^{1/\beta}t}{r}\left[1-\frac{h^{\alpha}|\kappa|^{\alpha}}{\beta}+o(h^{\alpha})\right]\right\}.\end{aligned} \tag{2.38}$$

Therefore, the limiting process is represented by the α-stable process $\mathcal{S}^{\beta}_{\alpha}$ with characteristic function $e^{-(1/\beta)\lambda^{1/\beta}t|\kappa|^{\alpha}}$, instead of the subordinated process $\mathcal{S}_{\alpha}(\mathcal{L}_{\beta}(t))$ obtained in the limit when considering the FPP $\mathcal{N}_{\beta}$; note that $\mathcal{S}_{\alpha}(\mathcal{L}_{\beta}(t))$ coincides with $Z(t)$ in (1.10). It is clear that the dependence on β is limited to the scale parameter; the space-fractional equation satisfied by its density is therefore given by

$$\frac{\partial u}{\partial t}=\frac{\lambda^{1/\beta}}{\beta}\frac{\partial^{\alpha}u}{\partial|y|^{\alpha}},\qquad u(y,0)=\delta(y),\quad y\in\mathbb{R},\ t\geq 0, \tag{2.39}$$

instead of (1.12). For $\alpha=1$ the density of the limiting process reduces to a Cauchy with scale parameter $\lambda^{1/\beta}t/\beta$.

3. Space-Fractional Compound Poisson Process

We define now a space-fractional version of the compound Poisson process (which we will indicate hereafter by SFCPP): indeed, its distribution satisfies (2.5), but with integer time derivative and fractional space derivative. We consider the standard CPP

$$Y^{(\alpha)}(t)=\sum_{j=1}^{N(t)}X^{(\alpha)}_{j},\quad \alpha\in(0,1], \tag{3.1}$$

where, as usual, $N(t),\ t>0$ is a standard Poisson process with parameter λ and the random variables $X^{(\alpha)}_{j}$ have the following heavy tail distribution:

$$f_{X^{(\alpha)}}(x)=\xi x^{\alpha-1}E_{\alpha,\alpha}(-\xi x^{\alpha}),\quad x>0,\ \alpha\in(0,1] \tag{3.2}$$

for $\xi > 0$. The Laplace transform of (3.2) is $\widetilde{f}_{X^{(\alpha)}}(k) = \xi/(k^\alpha + \xi)$. The distribution of $X_j^{(\alpha)}$ given in (3.2) is usually called Mittag-Leffler and coincides with the geometric-stable law of index α (hereafter $\mathcal{GS}_\alpha$) with parameters $\mu = 0$, $\theta = 1$, and $\sigma = [\cos(\pi\alpha/2)/\xi]^{1/\alpha}$ (see [27]). The density of $\sum_{j=1}^n X_j^{(\alpha)}$ is given by

$$f^{*n}_{X^{(\alpha)}}(y) = \xi^n y^{\alpha n-1} E^n_{\alpha,\alpha n}(-\xi y^\alpha) \tag{3.3}$$

with Laplace transform

$$\widetilde{f}^{*n}_{X^{(\alpha)}}(k) = \frac{\xi^n}{(k^\alpha + \xi)^n}. \tag{3.4}$$

Note that (3.3) coincides with the density of the nth event waiting time for the fractional Poisson process $\mathcal{N}_\alpha$ (see [21]). It is easy to check that the variable $X_j^{(\alpha)}$ displays the asymptotic behavior (1.13).

Theorem 3.1. *The process $Y^{(\alpha)}$ defined in* (3.1)*, with $X_j^{(\alpha)}$, $j = 1, 2, \ldots$, independent and distributed according to* (3.2)*, has the following distribution:*

$$\Pr\left\{Y^{(\alpha)}(t) \leq y\right\} = e^{-\lambda t} 1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{Y^{(\alpha)}}(z,t)\,dz, \quad t \geq 0,\ y \in \mathbb{R}, \tag{3.5}$$

where

$$f_{Y^{(\alpha)}}(y,t) = \frac{e^{-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{(\xi \lambda t y^\alpha)^n}{n!} E^n_{\alpha,\alpha n}(-\xi y^\alpha) 1_{(0,+\infty)}(y), \quad t \geq 0. \tag{3.6}$$

The density (3.6) *satisfies the following equation:*

$$\xi \frac{\partial}{\partial t} f_{Y^{(\alpha)}} = -\left[\lambda + \frac{\partial}{\partial t}\right] \frac{\partial^\alpha}{\partial y^\alpha} f_{Y^{(\alpha)}}, \quad t \geq 0,\ y > 0, \tag{3.7}$$

with conditions

$$\begin{gathered} f_{Y^{(\alpha)}}(y,0) = 0, \\ \int_0^{+\infty} f_{Y^{(\alpha)}}(y,t)\,dy = 1 - e^{-\lambda t}. \end{gathered} \tag{3.8}$$

The following composition rule holds for the one-dimensional distribution of (3.1)*:*

$$Y^{(\alpha)}(t) \stackrel{d}{=} \mathcal{A}_\alpha(Y(t)), \tag{3.9}$$

where $\mathcal{A}_\alpha(t)$ is the stable subordinator defined in (1.1) *and Y is the standard CPP.*

Proof. We start by noting that the absolutely continuous part of the distribution is defined in $(0,\infty)$, with the exclusion of $y = 0$, where only the discrete component gives some contribution.

In order to check (3.7) we evaluate the following fractional derivatives, arguing as in the proof of Theorem 2.1:

$$\begin{aligned}
\frac{\partial}{\partial t} f_{Y^{(\alpha)}}(y,t) &= -\frac{\lambda e^{-\lambda t}}{y}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{(n-1)!n!}\sum_{j=0}^{\infty}\frac{(n+j-1)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n)}+ \\
&\quad + \frac{e^{-\lambda t}}{yt}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{((n-1)!)^2}\sum_{j=0}^{\infty}\frac{(n+j-1)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n)}, \\
\frac{\partial^{\alpha}}{\partial y^{\alpha}} f_{Y^{(\alpha)}}(y,t) &= \frac{e^{-\lambda t}}{y^{1+\alpha}}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{n!(n-1)!}\sum_{j=0}^{\infty}\frac{(n+j-1)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n-\alpha)}, \\
\frac{\partial^{\alpha}}{\partial y^{\alpha}}\frac{\partial}{\partial t} f_{Y^{(\alpha)}}(y,t) &= -\frac{e^{-\lambda t}}{y^{1+\alpha}}\sum_{n=2}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{n!(n-2)!}\sum_{j=0}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n-\alpha)}+ \\
&\quad + \frac{e^{-\lambda t}}{y^{1+\alpha}t}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{((n-1)!)^2}\sum_{j=1}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{(j-1)!\Gamma(\alpha j+\alpha n-\alpha)}+ \\
&\quad + \frac{e^{-\lambda t}}{y^{1+\alpha}t}\sum_{n=2}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{(n-1)!(n-2)!}\sum_{j=0}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{j!\Gamma(\alpha j+\alpha n-\alpha)}+ \\
&\quad + \frac{e^{-\lambda t}}{y^{1+\alpha}}\sum_{n=1}^{\infty}\frac{(\lambda\xi t y^{\alpha})^n}{n!(n-1)!}\sum_{j=1}^{\infty}\frac{(n+j-2)!(-\xi y^{\alpha})^j}{(j-1)!\Gamma(\alpha j+\alpha n-\alpha)}.
\end{aligned} \tag{3.10}$$

The initial condition is immediately satisfied by (3.6), while the second condition in (3.8) can be verified as follows:

$$\begin{aligned}
\int_0^{\infty} e^{-ky} f_{Y^{(\alpha)}}(y,t)\,dy &= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\xi\lambda t)^n}{n!k^{\alpha n}}\sum_{j=0}^{\infty}\frac{(n+j-1)!}{j!k^{\alpha j}} \\
&= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\xi\lambda t}{k^{\alpha}+\xi}\right)^n \\
&= e^{-\lambda t}\left(e^{\lambda t\xi/(k^{\alpha}+\xi)}-1\right),
\end{aligned} \tag{3.11}$$

which, for $k = 0$, becomes $1 - e^{-\lambda t}$. The composition rule given in (3.9) can be verified by taking the Laplace transform of $Y^{(\alpha)}$,

$$\widetilde{g}_{Y^{(\alpha)}}(k,t) := \mathbb{E}e^{-kY^{(\alpha)}(t)} = e^{-(\lambda k^{\alpha}/(k^{\alpha}+\xi))t}, \tag{3.12}$$

which Laplace transformed with respect to t gets

$$\tilde{\tilde{g}}_{Y^{(\alpha)}}(k,s) := \int_0^\infty e^{-st}\tilde{g}_{Y^{(\alpha)}}(k,t)dt = \frac{k^\alpha+\xi}{k^\alpha(\lambda+s)+s\xi} = \int_0^{+\infty} \mathbb{E}e^{-k^\alpha Y(z)}e^{-sz}dz. \tag{3.13}$$

Thus,

$$\tilde{g}_{Y^{(\alpha)}}(k,t) = \mathbb{E}e^{-k^\alpha Y(t)} = \int_0^{+\infty} e^{-k^\alpha v}\Pr\{Y(t)\in dv\}, \tag{3.14}$$

so that, by (1.1), we get

$$\Pr\left\{Y^{(\alpha)}(t)\in dy\right\} = \int_0^{+\infty} h_\alpha(y,v)\Pr\{Y(t)\in dv\}dy, \tag{3.15}$$

and formula (3.9) follows. □

Remark 3.2. Equation (3.15) yields an alternative proof of (3.7) noting that the density of $\mathcal{A}_\alpha^{\lambda,\xi}$ satisfies the following equation (where the space-fractional derivative is defined now in the Caputo sense):

$$\frac{\partial u}{\partial t} = -\frac{\partial^\alpha u}{\partial y^\alpha}, \qquad u(y,0)=\delta(y), \quad y,t\geq 0. \tag{3.16}$$

Indeed, we get

$$\begin{aligned}
\frac{\partial^\alpha}{\partial y^\alpha}f_{Y^{(\alpha)}}(y,t) &= \int_0^{+\infty}\frac{\partial^\alpha}{\partial y^\alpha}h_\alpha(y,v)f_Y(v,t)dv \\
&= -\int_0^{+\infty}\frac{\partial}{\partial v}h_\alpha(y,v)f_Y(v,t)dv \\
&= \int_0^{+\infty}h_\alpha(y,v)\frac{\partial}{\partial v}f_Y(v,t)dv \\
&= [\text{by (2.5)}] \\
&= -\frac{\xi}{\lambda}\frac{\partial}{\partial t}\int_0^{+\infty}h_\alpha(y,v)f_Y(v,t)dv - \frac{1}{\lambda}\frac{\partial}{\partial t}\int_0^{+\infty}h_\alpha(y,v)\frac{\partial}{\partial v}f_Y(v,t)dv \\
&= -\frac{\xi}{\lambda}\frac{\partial}{\partial t}f_{Y^{(\alpha)}}(y,t) - \frac{1}{\lambda}\frac{\partial}{\partial t}\frac{\partial^\alpha}{\partial y^\alpha}f_{Y^{(\alpha)}}(y,t).
\end{aligned} \tag{3.17}$$

By considering (3.9) together with (1.2), we can write the following relationship:

$$F_{Y^{(\alpha)}(t)}(z) = \Pr\left\{Y^{(\alpha)}(t)\leq z\right\} = \Pr\{\mathcal{A}_\alpha(Y(t))\leq z\} = \Pr\{Y(t)\geq \mathcal{L}_\alpha(z)\}, \tag{3.18}$$

while for the first version of TFCPP we had, from (2.15), that $F_{Y_\beta(t)}(z) = \Pr\{Y(\mathcal{L}_\beta(t))\leq z\}$.

We finally note that the process $Y^{(\alpha)}$ is still a Markovian and Lévy process, since it is substantially a special case of CPP.

3.1. Special Cases

For $\alpha = 1$, since the X_j's reduce to exponential r.v.'s, from (3.6) and (3.7) we retrieve the results (2.11) and (2.13) valid for the standard CPP, under the exponential assumption for X_j's. As a direct check of (3.9), we can consider the special case $\alpha = 1/2$, so that the law $h_{1/2}(\cdot, z)$ can be written explicitly as the density of the first passage time of a standard Brownian motion through the level $z > 0$. Then by considering (3.15) we can write

$$\begin{aligned}
\Pr\{Y_{1/2}(t) \in dy\} &= \int_0^{+\infty} h_{1/2}(y,v) f_Y(v,t) dv\, dy \\
&= \int_0^{+\infty} \frac{z e^{-z^2/2y}}{\sqrt{2\pi y^3}} \frac{e^{-\xi z-\lambda t}}{z} \sum_{n=1}^{\infty} \frac{(\lambda \xi t z)^n}{n!(n-1)!} dz\, dy \\
&= \frac{e^{-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{(\lambda\xi t)^n}{n!(n-1)!} (-1)^n \frac{d^n}{d\xi^n} \int_0^{+\infty} \frac{e^{-z^2/2y}}{\sqrt{2\pi y}} e^{-\xi z} dz\, dy \\
&= \frac{e^{-\lambda t}}{2y} \sum_{n=1}^{\infty} \frac{(\lambda\xi t)^n}{n!(n-1)!} (-1)^n \frac{d^n}{d\xi^n} E_{1/2,1}\left(-\xi y^{1/2}\right) dy,
\end{aligned} \tag{3.19}$$

where the last equality holds by (2.11)-(2.12) in [28]; then, by (1.10.3) in [12], we get

$$\begin{aligned}
\Pr\{Y_{1/2}(t) \in dy\} &= \frac{e^{-\lambda t}}{2y} \sum_{n=1}^{\infty} \frac{(\lambda\xi t)^n}{(n-1)!} \frac{y^{n/2}}{n!} \sum_{j=0}^{\infty} \frac{(n+j)!\left(-\xi y^{1/2}\right)^j}{j!\Gamma(j/2+n/2+1)} dy \\
&= \frac{e^{-\lambda t}}{y} \sum_{n=1}^{\infty} \frac{\left(\lambda \xi t y^{1/2}\right)^n}{n!} E^n_{1/2,n/2}\left(-\xi y^{1/2}\right) dy.
\end{aligned} \tag{3.20}$$

3.2. Asymptotic Results

We study now the asymptotic behavior of the rescaled version of $Y^{(\alpha)}$ defined as

$$hY^{(\alpha)}\left(\frac{t}{r}\right) = \sum_{j=1}^{N(t/r)} hX_j^{(\alpha)}, \tag{3.21}$$

for $h, r \to 0$. The Fourier transform of the r.v.'s, $X_j^{(\alpha)}$, for any $\alpha \in (0,1)$, is given by

$$\widehat{f}_{X^{(\alpha)}}(\kappa) = \frac{1}{1 + (1/\xi)\cos(\pi\alpha/2)|\kappa|^{\alpha}(1 - i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))} \tag{3.22}$$

(see [27], formula (2.4.1)), which, in the limit, behaves as

$$\widehat{f}_{X^{(\alpha)}}(h\kappa) \simeq 1 - Ah^{\alpha}|\kappa|^{\alpha}, \quad h \longrightarrow 0, \tag{3.23}$$

where $A = (1/\xi)\cos(\pi\alpha/2)(1-i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))$. Thus, the characteristic function of (3.21) can be written as

$$\begin{aligned}\widehat{g}_{hY^{(\alpha)}}\left(\kappa, \frac{t}{r}\right) &= e^{\lambda(t/r)[\widehat{f}_{X^{(\alpha)}}(h\kappa)-1]} \\ &\simeq e^{-(\lambda t/\xi)\cos(\pi\alpha/2)|\kappa|^{\alpha}(1-i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))}, \quad \alpha \in (0,1),\end{aligned} \tag{3.24}$$

for $h, r \to 0$, s.t. $h^{\alpha}/r \to 1$. We can conclude that

$$hY^{(\alpha)}\left(\frac{t}{r}\right) \Longrightarrow \mathcal{A}_{\alpha}^{\lambda,\xi}(t), \tag{3.25}$$

where the limiting process is represented, in this case, by an α-stable subordinator $\mathcal{A}_{\alpha}^{\lambda,\xi}(t)$ with parameters $\mu = 0$, $\theta = 1$, $\sigma = ((\lambda t/\xi)\cos\pi\alpha/2)^{1/\alpha}$, whose density satisfies

$$\frac{\partial u}{\partial t} = -\frac{\lambda}{\xi}\frac{\partial^{\alpha} u}{\partial y^{\alpha}}, \qquad u(y,0) = \delta(y), \quad y > 0, \; t > 0. \tag{3.26}$$

4. Compound Poisson Processes Fractional in Time and Space

We consider now together the results obtained in the previous sections, by defining a CPP fractional both in space and time (STFCPP), that is,

$$Y_{\beta}^{(\alpha)}(t) = \sum_{j=1}^{\mathcal{N}_{\beta}(t)} X_j^{(\alpha)}, \quad t > 0, \tag{4.1}$$

where $X_j^{(\alpha)}$'s are i.i.d. with density (3.2) and $\mathcal{N}_{\beta}(t)$, $t > 0$ is again the FPP.

Theorem 4.1. *The process $Y_{\beta}^{(\alpha)}(t), t > 0$, defined in* (4.1) *has the following distribution:*

$$\Pr\left\{Y_{\beta}^{(\alpha)}(t) \le y\right\} = E_{\beta,1}\left(-\lambda t^{\beta}\right)1_{[0,+\infty)}(y) + \int_{-\infty}^{y} f_{Y_{\beta}^{(\alpha)}}(z,t)dz, \quad t \ge 0, \; y \in \mathbb{R}, \tag{4.2}$$

where

$$f_{Y_{\beta}^{(\alpha)}}(y,t) = \frac{1}{y}\sum_{n=1}^{\infty}\left(\lambda\xi t^{\beta} y^{\alpha}\right)^{n} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^{\beta}\right)E_{\alpha,\alpha n}^{n}(-\xi y^{\alpha})1_{(0,+\infty)}(y), \quad t \ge 0. \tag{4.3}$$

The density $f_{Y_\beta^{(\alpha)}}$ solves the following equation:

$$\xi \frac{\partial^\beta}{\partial t^\beta} f_{Y_{\alpha\beta}} = -\left[\lambda + \frac{\partial^\beta}{\partial t^\beta}\right] \frac{\partial^\alpha}{\partial y^\alpha} f_{Y_\beta^{(\alpha)}}, \quad t \geq 0,\ y > 0, \tag{4.4}$$

with conditions

$$\begin{gathered} f_{Y_\beta^{(\alpha)}}(y,0) = 0, \\ \int_0^{+\infty} f_{Y_\beta^{(\alpha)}}(y,t)\,dy = 1 - E_{\beta,1}\left(-\lambda t^\beta\right). \end{gathered} \tag{4.5}$$

The following equality of the one-dimensional distributions holds:

$$Y_\beta^{(\alpha)}(t) \overset{d}{=} \mathcal{S}_\alpha(Y_\beta(t)). \tag{4.6}$$

Proof. In order to check (4.4) we evaluate the following fractional derivatives:

$$\begin{aligned} \frac{\partial}{\partial t^\beta} f_{Y_\beta^{(\alpha)}}(y,t) &= \frac{1}{y t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda\xi t^\beta y^\alpha\right)^n}{(n-1)!n!} \left(\sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)} \right) \left(\sum_{r=0}^\infty \frac{(n+r-1)!\left(-\xi y^\alpha\right)^r}{r!\Gamma(\alpha r + \alpha n)} \right), \\ \frac{\partial^\alpha}{\partial y^\alpha} f_{Y_\beta^{(\alpha)}}(y,t) &= \frac{1}{y^{1+\alpha}} \sum_{n=1}^\infty \frac{\left(\lambda\xi t^\beta y^\alpha\right)^n}{n!(n-1)!} \left(\sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j!\Gamma(\beta j + \beta n + 1)} \right) \left(\sum_{r=0}^\infty \frac{(n+r-1)!\left(-\xi y^\alpha\right)^r}{r!\Gamma(\alpha r + \alpha n - \alpha)} \right), \\ \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial}{\partial t^\beta} f_{Y_\beta^{(\alpha)}}(y,t) &= \frac{1}{y^{1+\alpha} t^\beta} \sum_{n=1}^\infty \frac{\left(\lambda\xi t^\beta y^\alpha\right)^n}{n!(n-1)!} \left(\sum_{j=0}^\infty \frac{(n+j)!\left(-\lambda t^\beta\right)^j}{j!\Gamma(\beta j + \beta n - \beta + 1)} \right) \\ &\quad \times \left(\sum_{r=0}^\infty \frac{(n+r-1)!\left(-\xi y^\alpha\right)^r}{r!\Gamma(\alpha r + \alpha n - \alpha)} \right). \end{aligned} \tag{4.7}$$

By some algebraic manipulations we finally get (4.4). While the initial condition is trivially satisfied, the second condition in (4.5) can be checked as follows:

$$\begin{aligned} \int_0^\infty e^{-ky} f_{Y_\beta^{(\alpha)}}(y,t)\,dy &= \sum_{n=1}^\infty \frac{\left(\lambda\xi t^\beta\right)^n}{k^{\alpha n}} E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right) \sum_{r=0}^\infty \binom{n+r-1}{r} \left(-\frac{\xi}{k^\alpha}\right)^r \\ &= \sum_{n=1}^\infty \left(\frac{\lambda\xi t^\beta}{k^\alpha + \xi}\right)^n E_{\beta,\beta n+1}^{n+1}\left(-\lambda t^\beta\right) \\ &= \left[\text{by (2.30) of [21]}\right] \\ &= E_{\beta,1}\left(-\frac{\lambda\xi t^\beta k^\alpha}{k^\alpha + \xi}\right) - E_{\beta,1}\left(-\lambda t^\beta\right), \end{aligned} \tag{4.8}$$

which, for $k = 0$, becomes $1 - E_{\beta,1}(-\lambda t^\beta)$.

The relationship (4.6) can be checked by evaluating the double Laplace transform of $\mathcal{Y}_\beta^{(\alpha)}$ as follows:

$$\tilde{\tilde{g}}_{\mathcal{Y}_\beta^{(\alpha)}}(k,s) = \int_0^{+\infty} \mathbb{E} e^{-k\mathcal{Y}_\beta^{(\alpha)}(t)} e^{-st} dt = \frac{s^{\beta-1}(k^\alpha + \xi)}{s^\beta(k^\alpha + \xi) + \lambda k^\alpha}. \tag{4.9}$$

We then rewrite formula (4.9) as

$$\tilde{\tilde{g}}_{\mathcal{Y}_\beta^{(\alpha)}}(k,s) = s^{\beta-1} \int_0^{+\infty} e^{-s^\beta z} \mathbb{E} e^{-k^\alpha \mathcal{Y}(z)} dz \tag{4.10}$$

and we follow the same lines which lead to (2.15) to get the conclusion. □

Remark 4.2. For $\alpha = 1$ formulae (4.3) and (4.4) coincide with (2.4) and (2.5), while for $\alpha = \beta = 1$ we get (3.6) and (3.7).

From (4.6), by considering (1.2), we get the following relation:

$$F_{\mathcal{Y}_\beta^{(\alpha)}(t)}(z) = \Pr\left\{\mathcal{Y}_\beta^{(\alpha)}(t) \le z\right\} = \Pr\{\mathcal{S}_\alpha(\mathcal{Y}_\beta(t)) \le z\} = \Pr\{\mathcal{Y}_\beta(t) \ge \mathcal{L}_\alpha(z)\}, \tag{4.11}$$

where $\mathcal{L}_\alpha$ is the inverse stable subordinator.

4.1. Asymptotic Results

For the rescaled version of $\mathcal{Y}_\beta^{(\alpha)}$ we obtain the following asymptotic result, which agrees with (1.14) and (1.15) proved in [17]: the characteristic function of the process

$$h\mathcal{Y}_\beta^{(\alpha)}\left(\frac{t}{r}\right) = \sum_{j=1}^{\mathcal{N}_\beta(t/r)} hX_j^{(\alpha)} \tag{4.12}$$

can be written as

$$\widehat{g}_{h\mathcal{Y}_\beta^{(\alpha)}}\left(\kappa, \frac{t}{r}\right) = E_{\beta,1}\left(\lambda \frac{t^\beta}{r^\beta}\left[\widehat{f}_{hX^{(\alpha)}}(\kappa) - 1\right]\right). \tag{4.13}$$

By applying formula (3.23) we conclude that (4.13) converges, for $h, r \to 0$ s.t. $h^\alpha / r^\beta \to 1$, to

$$E_{\beta,1}\left(-\frac{\lambda t^\beta}{\xi}|\kappa|^\alpha \cos\frac{\pi\alpha}{2}\left(1 - i\,\mathrm{sgn}(\kappa)\tan\frac{\pi\alpha}{2}\right)\right) \tag{4.14}$$

so that the process $hY_\beta^{(\alpha)}(t/r)$ converges weakly to the α-stable subordinator $\mathcal{A}_\alpha^{\lambda,\xi}(t)$, composed with the inverse β-stable subordinator $\mathcal{L}_\beta$. Indeed, the characteristic function of $\mathcal{A}_\alpha^{\lambda,\xi}(\mathcal{L}_\beta(t))$ can be evaluated as follows:

$$\begin{aligned}\int_0^{+\infty} e^{-st}\widehat{f}_{\mathcal{A}_\alpha^{\lambda,\xi}(\mathcal{L}_\beta)}(\kappa,t)dt &= \int_0^{+\infty} e^{-st}dt\int_0^{+\infty} e^{i\kappa y}dy\int_0^{+\infty} p_\alpha^{\lambda,\xi}(y;z)l_\beta(z,t)dz \\ &= s^{\beta-1}\int_0^{+\infty} e^{-z\lambda|\kappa|^\alpha A}e^{-zs^\beta}dz \\ &= \frac{s^{\beta-1}}{\lambda|\kappa|^\alpha A+s^\beta},\end{aligned} \tag{4.15}$$

where $h_\alpha^{\lambda,\xi}(y,z)$ is the law of $\mathcal{A}_\alpha^{\lambda,\xi}(z)$ and $A=(1/\xi)\cos(\pi\alpha/2)(1-i\,\mathrm{sgn}(\kappa)\tan(\pi\alpha/2))$. By inverting the Laplace transform in (4.15) we get (4.14). The density of $\mathcal{A}_\alpha^{\lambda,\xi}(\mathcal{L}_\beta(t))$ satisfies the following equation:

$$\frac{\partial^\beta u}{\partial t^\beta}=-\frac{\lambda}{\xi}\frac{\partial^\alpha u}{\partial y^\alpha},\quad y,t>0, \tag{4.16}$$

as can be easily seen from (4.15) (see also [29]). A relevant special case of this result can be obtained by taking $\alpha=\beta=\nu$, so that the composition $\mathcal{A}_\nu^{\lambda,\xi}(\mathcal{L}_\nu(t))$ is proved to display a Lamperti-type law (see on this topic [30, 31]); therefore, the latter can be seen as the weak limit of the STFCPP.

We note that in the particular case $\beta=1$, the Fourier transform (4.14) reduces to (3.24) and correspondingly (4.16) coincides with (3.26).

Finally, we consider the case where we have $\overline{N}_\beta(t)$ in place of $N_\beta(t)$. If the jumps are Mittag-Leffler distributed, we get the following space-time fractional CPP:

$$\overline{Y}_{\alpha,\beta}(t)=\sum_{j=1}^{\overline{N}_\beta(t)}X_j^{(\alpha)}, \tag{4.17}$$

whose distribution is given by

$$\Pr\left\{\overline{Y}_{\alpha,\beta}\le y\right\}=\frac{1}{E_{\beta,1}(-\lambda t^\beta)}1_{[0,+\infty)}(y)+\int_{-\infty}^{y} f_{\overline{Y}_{\alpha,\beta}}(z,t)dz,\quad t\ge 0,\ y\in\mathbb{R}, \tag{4.18}$$

where

$$f_{\overline{Y}_{\alpha,\beta}}(y,t)=\frac{1}{E_{\beta,1}(\lambda t^\beta)}\sum_{n=1}^{\infty}\frac{(\lambda\xi t^\beta y^\alpha)^n}{\Gamma(\beta n+1)}E_{\alpha,\alpha n}^n(-\xi y^\alpha)1_{[0,+\infty)}(y),\quad t\ge 0. \tag{4.19}$$

The rescaled version of (4.17) is defined as

$$h\overline{Y}_{\alpha,\beta}\left(\frac{t}{r}\right) = \sum_{j=1}^{\overline{N}_\beta(t/r)} hX_j^{(\alpha)} \Longrightarrow \mathcal{A}_\alpha^{\lambda,\xi}(t) \tag{4.20}$$

for $h, r \to 0$, s.t. $h^\alpha/r \to 1$, where again $\mathcal{A}_\alpha^{\lambda,\xi}$ denotes the α-stable subordinator with characteristic function given in (3.24) (last line). Thus, in the limit, the fractional nature of the counting process $\overline{N}_\beta$ does not exert any influence, in analogy with the result given in (3.25).

5. Fractional-Difference Compound Poisson Process

We present now a final version of the fractional CPP, where the fractionality of the counting process is referred to the difference operator involved in the recursive equation governing its distribution. Let B denote the standard backward shift operator, $\Delta = 1 - B$, and let γ be a fractional parameter in $(0,1]$, then the fractional recursive differential equation

$$\frac{d}{dt}p_k^\Delta(t) = -\lambda^\gamma \Delta^\gamma p_k^\Delta(t), \qquad p_k^\Delta(0) = 1_{[k=0]}, \tag{5.1}$$

has been introduced in [9]. In (5.1) the following definition of the fractional difference operator Δ^γ of a function $f(n)$ has been used (see [12], formula (2.8.2), page121):

$$\Delta^\gamma f(n) = \sum_{j=0}^{\infty} \frac{(-1)^j (\gamma)_j}{j!} f(n-j), \tag{5.2}$$

where $(x)_n = x(x-1)\cdots(x-(n-1))$ is the falling factorial. We use the notation $p_k^\Delta(t) := \Pr\{N_\Delta(t) = k\},\ k \geq 0,\ t > 0$, and we have

$$p_k^\Delta(t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\gamma t)^r}{r!} (\gamma r)_k, \quad \gamma \in (0,1]. \tag{5.3}$$

It can be proved that N_Δ is not a renewal process, by verifying that the density of the kth event waiting time cannot be expressed as kth convolution of i.i.d. random variables. Nevertheless, $N_\Delta(t)$ is a Lévy process, with infinite expected value for any t. Moreover, by (5.3), one can check that (as $h \to 0$)

$$\Pr\{N_\Delta(h) = k\} = (-1)^{k+1} \frac{\lambda^\gamma (\gamma)_k}{k!} h + o(h), \quad \forall k \geq 1 \tag{5.4}$$

instead of $o(h)$ for $k \geq 2$, as for the standard or the time-fractional Poisson process. We can obtain (5.1) from (5.4) by taking into account that the increments are independent and stationary.

Let us define the corresponding fractional-difference compound Poisson process (hereafter ΔFCPP) as

$$Y_\Delta(t) = \sum_{j=1}^{N_\Delta(t)} X_j, \quad t \geq 0, \ \gamma \in (0,1] \tag{5.5}$$

so that we can obtain, under the assumption of i.i.d. exponential X_j's, the distribution of Y_Δ together with the differential equation which is satisfied by its absolutely continuous component.

Theorem 5.1. *For $\gamma \in (0,1]$, the distribution of the process Y_Δ defined in* (5.5)*, with X_j, $j = 1,2,\ldots$, independent and exponentially distributed with parameter ξ, is given by*

$$\Pr\{Y_\Delta(t) < y\} = e^{-\lambda^\gamma t} 1_{[0,+\infty)}(y) + \int_0^y f_{Y_\Delta}(z,t)dz, \quad t, y \geq 0, \tag{5.6}$$

where

$$f_{Y_\Delta}(y,t) = \frac{e^{-\xi y}}{y} \sum_{n=1}^{\infty} \frac{(-\xi y)^n}{n!(n-1)!} \sum_{r=0}^{\infty} \frac{(-\lambda^\gamma t)^r}{r!} (\gamma r)_n 1_{[0,+\infty)}(y), \quad t \geq 0. \tag{5.7}$$

The density f_{Y_Δ} solves the differential equation:

$$\xi D_{-,t}^{1/\gamma} f_{Y_\Delta} = \left[\lambda - D_{-,t}^{1/\gamma}\right] \frac{\partial}{\partial y} f_{Y_\Delta}, \tag{5.8}$$

where $D_{0-;\,t}^{1/\gamma}$ is the right-sided fractional Riemann-Liouville derivative on the half-axis $\mathbb{R}^+$, with conditions

$$\begin{aligned} f_{Y_\Delta}(y,0) &= 0, \\ D_{-,t}^{r} f_{Y_\Delta}(y,t)\Big|_{t=0} &= \Phi_r(y), \\ \int_0^{+\infty} f_{Y_\Delta}(y,t)dy &= 1 - e^{-\lambda^\gamma t}, \end{aligned} \tag{5.9}$$

where $\Phi_r(y) = (\lambda^{\gamma r} e^{-\xi y}/y) \sum_{n=1}^{\infty} ((-\xi y)^n (\gamma r)_n / n!(n-1)!)$.

The following subordinating relationship holds for (5.5)*:*

$$Y_\Delta(t) \stackrel{d}{=} Y(\mathcal{A}_\gamma(t)), \tag{5.10}$$

where, as usual, $\mathcal{A}_\gamma$ denotes the γ-stable subordinator.

Proof. Formula (5.7) can be easily derived by (5.3) and can be checked by verifying that, for $\gamma = 1$, it reduces to (2.11):

$$\begin{aligned} f_{Y_\Delta}(y,t)\big|_{\gamma=1} &= \frac{e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(-\xi y)^n}{n!(n-1)!}\sum_{r=n}^{\infty}\frac{(-\lambda t)^r}{r!}(r)_n \\ &= \frac{e^{-\xi y}}{y}\sum_{n=1}^{\infty}\frac{(-\xi y)^n}{n!(n-1)!}(-\lambda t)^n e^{-\lambda t}. \end{aligned} \tag{5.11}$$

We now prove the relationship (5.10) as follows. The Laplace transform of $Y_\Delta(t)$ is given by

$$\begin{aligned} \widetilde{g}_{Y_\Delta}(k,t) = \mathbb{E}e^{-kY_\Delta(t)} &= \sum_{n=0}^{\infty}\frac{(-\xi)^n}{n!}\frac{1}{(k+\xi)^n}\sum_{r=0}^{\infty}\frac{(-\lambda^\gamma t)^r}{r!}(\gamma r)_n \\ &= \sum_{r=0}^{\infty}\frac{(-\lambda^\gamma t)^r}{r!}\sum_{n=0}^{\infty}\frac{(\gamma r)_n}{n!}\left(-\frac{\xi}{k+\xi}\right)^n \\ &= \sum_{r=0}^{\infty}\frac{(-\lambda^\gamma t)^r}{r!}\left(1-\frac{\xi}{k+\xi}\right)^{\gamma r} = e^{-\lambda^\gamma k^\gamma t/(k+\xi)^\gamma}; \end{aligned} \tag{5.12}$$

moreover, (5.12) can be rewritten as

$$\begin{aligned} \widetilde{g}_{Y_\Delta}(k,t) &= \int_0^{+\infty} e^{-\lambda z}e^{\lambda\xi z/(k+\xi)}h_\gamma(z,t)dz \\ &= \sum_{n=0}^{\infty}\frac{(\lambda\xi)^n}{n!(\xi+k)^n}\int_0^{+\infty} z^n e^{-\lambda z}h_\gamma(z,t)dz \\ &= \int_0^{+\infty}\mathbb{E}e^{-kY(z)}h_\gamma(z,t)dz, \end{aligned} \tag{5.13}$$

which gives (5.10). Thus, we also have that

$$\mathrm{f}_{Y_\Delta}(y,t) = \int_0^{+\infty} f_Y(y,z)h_\gamma(z,t)dz. \tag{5.14}$$

In order to prove that (5.7) satisfies (5.8), we recall the following result proved in [32]: the density $h_{\nu/n}(y,t)$ of the stable subordinator $\mathcal{A}_{\nu/n}$ is governed by the following equation (as well as by (3.16) for $\alpha = \nu/n$):

$$D^n_{-,t}h_{\nu/n} = \frac{\partial^\nu}{\partial y^\nu}h_{\nu/n}, \quad y,t\geq 0,\ \nu\in(0,1], \tag{5.15}$$

for $n \in \mathbb{N}$, with conditions

$$\begin{aligned} h_{\nu/n}(0,t) &= 0, \\ h_{\nu/n}(y,0) &= \delta(y), \\ D^{r}_{-,t} h_{\nu/n}(y,t)\Big|_{t=0} &= \frac{y^{-(\nu r/n)-1}}{\Gamma(-r\nu/n)}, \quad r = 1,\dots,n-1. \end{aligned} \tag{5.16}$$

We can prove that the slightly different result holds:

$$D^{1/\gamma}_{-,t} h_\gamma = \frac{\partial}{\partial y} h_\gamma, \quad y,t \geq 0,\ \gamma \in (0,1],\ n = \left\lfloor \frac{1}{\gamma} \right\rfloor + 1 \tag{5.17}$$

with the following conditions

$$\begin{aligned} h_{\gamma}(0,t) &= 0, \\ h_{\gamma}(y,0) &= \delta(y), \\ D^{r}_{-,t} h_{\gamma}(y,t)\Big|_{t=0} &= \frac{y^{-\gamma r-1}}{\Gamma(-\gamma r)}, \quad r = 1,\dots,n-1. \end{aligned} \tag{5.18}$$

Equation (5.17) can be checked by resorting to the Laplace transform with respect to y as follows:

$$\begin{aligned} D^{1/\gamma}_{-,t} \widetilde{h}_\gamma(k,t) &= D^{1/\gamma}_{-,t} e^{-k^\gamma t} \\ &= \left[\text{by (2.2.15) of [12]}\right] \\ &= k e^{-k^\gamma t} \\ &= \int_0^{+\infty} e^{-ky} \frac{\partial}{\partial y} h_\gamma(y,t) dy. \end{aligned} \tag{5.19}$$

Analogously, we can check (5.18): in particular we get

$$\begin{aligned} D^{r}_{-,t} \widetilde{h}_\gamma(k,t)\Big|_{t=0} &= (-1)^r \frac{\partial^r}{\partial t^r} \widetilde{h}_\gamma(k,t)\Big|_{t=0} \\ &= k^{\gamma r} \\ &= \int_0^{+\infty} e^{-ky} \frac{y^{-\gamma r-1}}{\Gamma(-\gamma r)} dy. \end{aligned} \tag{5.20}$$

We now take the derivative of (5.14) of order $1/\gamma$ with respect to t:

$$\begin{aligned} D^{1/\gamma}_{-,t} f_{Y_\Delta}(y,t) &= \int_0^{+\infty} f_Y(y,z) D^{1/\gamma}_{-,t} h_\gamma(z,t)dz \\ &= \int_0^{+\infty} f_Y(y,z) \frac{\partial}{\partial z} h_\gamma(z,t)dz \\ &= [\text{by considering } (2.11)] \\ &= -\int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) h_\gamma(z,t)dz \\ &= \frac{1}{\xi}\frac{\partial}{\partial y}\int_0^{+\infty} \frac{\partial}{\partial z} f_Y(y,z) h_\gamma(z,t)dz + \frac{\lambda}{\xi}\frac{\partial}{\partial y} f_{Y_\Delta}(y,t) \\ &= -\frac{1}{\xi}\frac{\partial}{\partial y} D^{1/\gamma}_{-,t} f_{Y_\Delta}(y,t) + \frac{\lambda}{\xi}\frac{\partial}{\partial y} f_{Y_\Delta}(y,t). \end{aligned} \tag{5.21}$$

We remark that, for $\gamma = 1$, $D^{1/\gamma}_{-,t} f_{Y_\Delta} = -\partial f_{Y_\Delta}/\partial t$, and therefore the previous equation reduces to (2.13). Finally, we have to check (5.9): the first initial condition is trivially satisfied, while the second condition can be checked either directly by taking the derivatives of (5.7) or by noting that

$$\begin{aligned} D^{r}_{-,t} f_{Y_\Delta}(y,t) &= \int_0^{+\infty} f_Y(y,z) D^{r}_{-,t} h_\gamma(z,t)dz \\ &= [\text{by } (5.18)] \\ &= \frac{e^{-\xi y}}{y}\sum_{n=1}^{\infty} \frac{(\lambda\xi y)^n}{n!(n-1)!}\int_0^{+\infty} e^{-\lambda z}\frac{z^{n-r\gamma-1}}{\Gamma(-r\gamma)}dz \\ &= \frac{\lambda^{\gamma n} e^{-\xi y}}{y}\sum_{n=1}^{\infty} \frac{(\xi y)^n}{n!(n-1)!}\frac{\Gamma(n-r\gamma)}{\Gamma(-r\gamma)} \\ &= \frac{\lambda^{\gamma n} e^{-\xi y}}{y}\sum_{n=1}^{\infty} \frac{(\xi y)^n}{n!(n-1)!}(-\gamma r)^{(n)} \\ &= \frac{\lambda^{\gamma n} e^{-\xi y}}{y}\sum_{n=1}^{\infty} \frac{(-\xi y)^n}{n!(n-1)!}(\gamma r)_n, \end{aligned} \tag{5.22}$$

where, in the last step, we have applied the following relationship between falling and rising factorial $(x)_n = (-1)^n (x)^{(n)}$. The last condition in (5.9) holds, since $\widetilde{g}_{Y_\Delta}(0,t) = 1$ by (5.12), so that $\widetilde{f}_{Y_\Delta}(0,t) = \widetilde{g}_{Y_\Delta}(0,t) - e^{-\lambda^\gamma t} = 1 - e^{-\lambda^\gamma t}$. □

Remark 5.2. We show that the distribution of the ΔFCPP satisfies a fractional master equation of order $1/\gamma$ greater than one, when the jumps have an arbitrary continuous density f_X. If we consider the (generalized) density function of $Y_\Delta(t)$,

$$g_{Y_\Delta}(y,t) := e^{-\lambda^\gamma t}\delta(y) + f_{Y_\Delta}(y,t), \quad y,t \geq 0, \tag{5.23}$$

then we get

$$D_{-,t}^{1/\gamma} g_{Y_\Delta}(y,t) = \lambda g_{Y_\Delta}(y,t) - \lambda \int_{-\infty}^{+\infty} g_{Y_\Delta}(y-x,t) f_X(x) dx, \tag{5.24}$$

which is analogue to (1.6) for the TFCPP Y_β. Indeed, by (5.10), we can write (5.23) as

$$g_{Y_\Delta}(y,t) = \int_0^{+\infty} g_Y(y,z) h_\gamma(z,t) dz, \tag{5.25}$$

where $g_Y(y,t) = e^{-\lambda t}\delta(y) + f_Y(y,t)$ and $f_Y(y,t)$ are the density of the standard CPP. By taking the fractional time-derivative of (5.25) we get

$$\begin{aligned}
D_{-,t}^{1/\gamma} f_{Y_\Delta}(y,t) &= \int_0^{+\infty} g_Y(y,z) D_{-,t}^{1/\gamma} h_\gamma(z,t) dz \\
&= [\text{by } (5.17)] \\
&= \int_0^{+\infty} g_Y(y,z) \frac{\partial}{\partial z} h_\gamma(z,t) dz \\
&= [\text{by } (5.18)] \\
&= -\int_0^{+\infty} \frac{\partial}{\partial z} g_Y(y,z) h_\gamma(z,t) dz \\
&= [\text{by the Kolmogorov master equation}] \\
&= \lambda \int_0^{+\infty} g_Y(y,z) h_\gamma(z,t) dz - \lambda \int_{-\infty}^{+\infty} f_X(x) \int_0^{+\infty} g_Y(y-x,z) h_\gamma(z,t) dz\, dx,
\end{aligned} \tag{5.26}$$

which coincides with (5.24). For $\gamma = 1$ (5.24) reduces to the well-known master equation of the standard CPP, by considering again that $D_{-,t}^{1/\gamma} f_{Y_\Delta} = -\partial f_{Y_\Delta}/\partial t$.

5.1. Asymptotic Results

We study the asymptotic behavior of the rescaled version of (5.5) under the two alternative assumptions on the r.v.'s representing the jumps: for X_j^* distributed according to (1.9) and for $X_j^{(\alpha)}$ with density (3.2). In the first case, we have that

$$hY_\Delta\left(\frac{t}{r}\right) = \sum_{j=1}^{N_\Delta(t/r)} hX_j^* \Longrightarrow \mathcal{S}_{\alpha\gamma}(t) \tag{5.27}$$

for $h, r \to 0$, s.t. $h^{\alpha\gamma}/r \to 1$, where $\mathcal{S}_{\alpha\gamma}(t)$ is a symmetric stable process of index $\alpha\gamma$ (which is strictly less than one) and parameters $\mu = 0$, $\theta = 0$, and $\sigma = (t\cos\pi\alpha\gamma/2)^{1/\alpha\gamma}$. Indeed, the characteristic function of (5.27) can be evaluated, by considering that the probability generating function of N_Δ is $G(u,t) = e^{-\lambda^\gamma t(1-u)^\gamma}$ (see [9]) as follows:

$$\begin{aligned}\widehat{f}_{hY_\Delta}\left(\kappa, \frac{t}{r}\right) &= e^{-(\lambda^\gamma t/r)(1-\widehat{f}_{hX^*}(\kappa))^\gamma} \\ &\simeq e^{-\lambda^\gamma t|\kappa|^{\alpha\gamma}}.\end{aligned} \tag{5.28}$$

Under the assumption of Mittag-Leffler distributed $X_j^{(\alpha)}$'s, we get instead the following result: the rescaled process

$$hY_\Delta^{(\alpha)}\left(\frac{t}{r}\right) = \sum_{j=1}^{N_\Delta(t/r)} hX_j^{(\alpha)} \Longrightarrow \mathcal{A}_{\alpha\gamma}^{\lambda,\xi}(t) \tag{5.29}$$

can be written as

$$\begin{aligned}\widehat{f}_{hY_\Delta^{(\alpha)}}\left(\kappa, \frac{t}{r}\right) &\simeq \exp\left\{-\frac{\lambda^\gamma t}{\xi^\gamma r}\left(\cos\frac{\pi\alpha}{2}\right)^\gamma h^{\alpha\gamma}|\kappa|^{\alpha\gamma}\left(1 - i\,\mathrm{sgn}(\kappa)\tan\frac{\pi\alpha}{2}\right)^\gamma\right\} \\ &\simeq \exp\left\{-\frac{\lambda^\gamma t}{\xi^\gamma}|\kappa|^{\alpha\gamma}\exp\left\{-i\,\mathrm{sgn}(\kappa)\frac{\pi\alpha\gamma}{2}\right\}\right\},\end{aligned} \tag{5.30}$$

for $h, r \to 0$, s.t. $h^{\alpha\gamma}/r \to 1$. The last line of (5.30) corresponds to the Fourier transform of a stable subordinator $\mathcal{A}_{\alpha\gamma}^{\lambda,\xi}$ of index $\alpha\gamma$ and with parameters $\mu = 0$, $\theta = 1$, and $\sigma = ((\lambda^\gamma t/\xi^\gamma)\cos\pi\alpha\gamma/2)^{1/\alpha\gamma}$. Therefore, in both cases, the limiting processes are simply the stable symmetric process and the stable subordinator of index $\alpha\gamma$, respectively, instead of their compositions with the inverse stable subordinator as happened when $\mathcal{N}_\beta$ was used as counting process.

Table 1: Main results in finite domain.

Process		Equation
CPP	$Y(t)$	$\xi \frac{\partial}{\partial t} = -\left[\lambda + \frac{\partial}{\partial t}\right] \frac{\partial}{\partial y}$
TFCPP	$Y_\beta(t) \overset{d}{=} Y(\mathcal{L}_\beta(t))$	$\xi \frac{\partial^\beta}{\partial t^\beta} = -\left[\lambda + \frac{\partial^\beta}{\partial t^\beta}\right] \frac{\partial}{\partial y}$
SFCPP	$Y^{(\alpha)}(t) \overset{d}{=} \mathcal{S}_\alpha(Y(t))$	$\xi \frac{\partial}{\partial t} = -\left[\lambda + \frac{\partial}{\partial t}\right] \frac{\partial^\alpha}{\partial y^\alpha}$
STFCPP	$Y_\beta^{(\alpha)}(t) \overset{d}{=} \mathcal{S}_\alpha(Y_\beta(t))$	$\xi \frac{\partial^\beta}{\partial t^\beta} = -\left[\lambda + \frac{\partial^\beta}{\partial t^\beta}\right] \frac{\partial^\alpha}{\partial y^\alpha}$
ΔFCPP	$Y_\Delta(t) \overset{d}{=} Y(\mathcal{A}_\gamma(t))$	$\xi D_{-,t}^{1/\gamma} = \left[\lambda - D_{-,t}^{1/\gamma}\right] \frac{\partial}{\partial y}$

Table 2: Main results in asymptotic domain.

Process	Hypothesis on jumps	Limiting process	Limiting equation
TFCPP Y_β	X_j^*	$\mathcal{S}_\alpha(\mathcal{L}_\beta(t))$	$\frac{\partial^\beta u}{\partial t^\beta} = \lambda \frac{\partial^\alpha u}{\partial \lvert y \rvert^\alpha}$
″	$X_j^{(\alpha)}$	$\mathcal{A}_\alpha(\mathcal{L}_\beta(t))$	$\frac{\partial^\beta u}{\partial t^\beta} = -\lambda \frac{\partial^\alpha u}{\partial y^\alpha}$
Altern. TFCPP $\overline{Y}_\beta$	X_j^*	$\mathcal{S}_\alpha^\beta(t)$	$\frac{\partial u}{\partial t} = \frac{\lambda^{1/\beta}}{\beta} \frac{\partial^\alpha u}{\partial \lvert y \rvert^\alpha}$
″	$X_j^{(\alpha)}$	$\mathcal{A}_\alpha^{\lambda,\xi}(t)$	$\frac{\partial u}{\partial t} = -\frac{\lambda}{\xi} \frac{\partial^\alpha u}{\partial y^\alpha}$
ΔFCPP Y_Δ	X_j^*	$\mathcal{S}_{\alpha\gamma}(t)$	$\frac{\partial u}{\partial t} = \lambda^\gamma \frac{\partial^{\alpha\gamma} u}{\partial \lvert y \rvert^{\alpha\gamma}}$
″	$X_j^{(\alpha)}$	$\mathcal{A}_{\alpha\gamma}^{\lambda,\xi}(t)$	$\frac{\partial u}{\partial t} = -\frac{\lambda^\gamma}{\xi} \frac{\partial^{\alpha\gamma} u}{\partial y^{\alpha\gamma}}$

Acronym

CPP: Compound Poisson process
TFCPP: Time-fractional compound Poisson process
SFCPP: Space-fractional compound Poisson process
STFCPP: Space-time fractional compound Poisson process
ΔFCPP: Fractional-difference compound Poisson process.

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References

[1] O. N. Repin and A. I. Saichev, "Fractional Poisson law," *Radiophysics and Quantum Electronics*, vol. 43, no. 9, pp. 738–741, 2000.

[2] F. Mainardi, R. Gorenflo, and E. Scalas, "A fractional generalization of the Poisson processes," *Vietnam Journal of Mathematics*, vol. 32, pp. 53–64, 2004.

[3] F. Mainardi, R. Gorenflo, and A. Vivoli, "Beyond the Poisson renewal process: a tutorial survey," *Journal of Computational and Applied Mathematics*, vol. 205, no. 2, pp. 725–735, 2007.

[4] L. Beghin and E. Orsingher, "Fractional Poisson processes and related planar random motions," *Electronic Journal of Probability*, vol. 14, no. 61, pp. 1790–1827, 2009.

[5] M. Politi, T. Kaizoji, and E. Scalas, "Full characterization of the fractional Poisson process," *Europhysics Letters*, vol. 96, no. 2, Article ID 20004, 6 pages, 2011.

[6] M. M. Meerschaert, E. Nane, and P. Vellaisamy, "The fractional Poisson process and the inverse stable subordinator," *Electronic Journal of Probability*, vol. 16, no. 59, pp. 1600–1620, 2011.

[7] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, NY, USA, 1994.

[8] L. Beghin and C. Macci, "Large deviations for fractional Poisson processes," submitted, http://128.84.158.119/abs/1204.1446.

[9] E. Orsingher and F. Polito, "The space-fractional Poisson process," *Statistics & Probability Letters*, vol. 82, no. 4, pp. 852–858, 2012.

[10] M. M. Meerschaert and H.-P. Scheffler, "Limit theorems for continuous-time random walks with infinite mean waiting times," *Journal of Applied Probability*, vol. 41, no. 3, pp. 623–638, 2004.

[11] R. Hilfer and L. Anton, "Fractional master equations and fractal random walks," *Physical Review E*, vol. 51, no. 2, pp. R848–R851, 1995.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[13] M. M. Meerschaert, D. A. Benson, H.-P. Scheffler, and B. Baeumer, "Stochastic solution of space-time fractional diffusion equations," *Physical Review E*, vol. 65, no. 4, Article ID 041103, 2002.

[14] E. Scalas, "The application of continuous-time random walks in finance and economics," *Physica A*, vol. 362, no. 2, pp. 225–239, 2006.

[15] E. Scalas, "A class of CTRWs: compound fractional Poisson processes," in *Fractional Dynamics*, pp. 353–374, World Scientific Publishers, Hackensack, NJ, USA, 2012.

[16] F. Mainardi, Y. Luchko, and G. Pagnini, "The fundamental solution of the space-time fractional diffusion equation," *Fractional Calculus & Applied Analysis*, vol. 4, no. 2, pp. 153–192, 2001.

[17] B. Baeumer, M. M. Meerschaert, and E. Nane, "Space-time duality for fractional diffusion," *Journal of Applied Probability*, vol. 46, no. 4, pp. 1100–1115, 2009.

[18] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels, *Stochastic Processes for Insurance and Finance*, Wiley Series in Probability and Statistics, John Wiley & Sons, Chichester, UK, 1999.

[19] G. K. Smyth and B. Jørgensen, "Fitting Tweedie's compound Poisson model to insurance claims data: dispersion modelling," *Astin Bulletin*, vol. 32, no. 1, pp. 143–157, 2002.

[20] C. S. Withers and S. Nadarajah, "On the compound Poisson-gamma distribution," *Kybernetika*, vol. 47, no. 1, pp. 15–37, 2011.

[21] L. Beghin and E. Orsingher, "Poisson-type processes governed by fractional and higher-order recursive differential equations," *Electronic Journal of Probability*, vol. 15, no. 22, pp. 684–709, 2010.

[22] A. De Gregorio and E. Orsingher, "Flying randomly in $\mathbb{R}^d$ with Dirichlet displacements," *Stochastic Processes and their Applications*, vol. 122, no. 2, pp. 676–713, 2012.

[23] N. L. Johnson, S. Kotz, and A. W. Kemp, *Univariate Discrete Distributions*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, NY, USA, Second edition, 1992.

[24] N. Balakrishnan and T. J. Kozubowski, "A class of weighted Poisson processes," *Statistics & Probability Letters*, vol. 78, no. 15, pp. 2346–2352, 2008.

[25] S. Gerhold, "Asymptotics for a variant of the Mittag-Leffler function," *Integral Transforms and Special Functions*, vol. 23, no. 6, pp. 397–403, 2012.

[26] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.

[27] K. Jayakumar and R. P. Suresh, "Mittag-Leffler distributions," *Journal of Indian Society of Probability and Statistics*, vol. 7, pp. 51–71, 2003.

[28] L. Beghin, “Fractional relaxation equations and Brownian crossing probabilities of a random boundary,” *Advances in Applied Probability*, vol. 44, no. 2, pp. 479–505, 2012.

[29] M. D’Ovidio, “From Sturm-Liouville problems to fractional and anomalous diffusions,” *Stochastic Processes and their Applications*, vol. 122, no. 10, pp. 3513–3544, 2012.

[30] M. D’Ovidio, “Explicit solutions to fractional diffusion equations via generalized gamma convolution,” *Electronic Communications in Probability*, vol. 15, pp. 457–474, 2010.

[31] L. F. James, “Lamperti-type laws,” *The Annals of Applied Probability*, vol. 20, no. 4, pp. 1303–1340, 2010.

[32] M. D’Ovidio, “On the fractional counterpart of the higher-order equations,” *Statistics & Probability Letters*, vol. 81, no. 12, pp. 1929–1939, 2011.