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Research Article

Eigenvalue Problem of Nonlinear Semipositone Higher Order Fractional Differential Equations

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We study the eigenvalue interval for the existence of positive solutions to a semipositone higher order fractional differential equation $-\mathfrak{D}_t{}^{\mu}x(t) = \lambda f(t, x(t), \mathfrak{D}_t{}^{\mu_1}x(t), \mathfrak{D}_t{}^{\mu_2}x(t), \ldots, \mathfrak{D}_t{}^{\mu_{n-1}}x(t)) \ldots \mathfrak{D}_t{}^{\mu_i}x(0) = 0, 1 \le i \le n-1, \mathfrak{D}_t{}^{\mu_{n-1}+1}x(0) = 0, \mathfrak{D}_t{}^{\mu_{n-1}}x(1) = \sum_{j=1}^{m-2} a_j \mathfrak{D}_t{}^{\mu_{n-1}}x(\xi_j)$, where $n-1 < \mu \le n$, $n \ge 3$, $0 < \mu_1 < \mu_2 < \cdots < \mu_{n-2} < \mu_{n-1}$, $n-3 < \mu_{n-1} < \mu - 2$, $a_j \in \mathbb{R}, 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ satisfying $0 < \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1} < 1$, $\mathfrak{D}_t{}^{\mu}$ is the standard Riemann-Liouville derivative, $f \in C((0,1) \times \mathbb{R}^n, (-\infty, +\infty))$, and f is allowed to be changing-sign. By using reducing order method, the eigenvalue interval of existence for positive solutions is obtained.

1. Introduction

In this paper, we consider the eigenvalue interval for existence of positive solutions to the following semipositone higher order fractional differential equation:

$$\begin{gathered}
-\mathfrak{D}_t{}^{\mu}x(t) = \lambda f(t, x(t), \mathfrak{D}_t{}^{\mu_1}x(t), \mathfrak{D}_t{}^{\mu_2}x(t), \ldots, \mathfrak{D}_t{}^{\mu_{n-1}}x(t)), \\
\mathfrak{D}_t{}^{\mu_i}x(0) = 0, \quad 1 \le i \le n-1, \quad \mathfrak{D}_t{}^{\mu_{n-1}+1}x(0) = 0, \\
\mathfrak{D}_t{}^{\mu_{n-1}}x(1) = \sum_{j=1}^{m-2} a_j \mathfrak{D}_t{}^{\mu_{n-1}}x(\xi_j),
\end{gathered} \tag{1.1}$$

where $n-1<\mu\leq n,\ \ n\geq 3,\ n\in\mathbb{N}$, $0<\mu_1<\mu_2<\cdots<\mu_{n-2}<\mu_{n-1}$, and $n-3<\mu_{n-1}<\mu-2$, $a_j\in\mathbb{R}$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$ satisfying $0<\sum_{j=1}^{m-2}a_j\xi_j^{\mu-\mu_{n-1}-1}<1$, $\mathfrak{D}_t{}^{\mu}$ is the standard Riemann-Liouville derivative, $f:(0,1)\times\mathbb{R}^n\rightarrow(-\infty,+\infty)$ is continuous.

Recently, one has found that fractional models can sufficiently describe the operation of variety of computational, economic mathematics, physical, and biological processes and systems, see [1–9]. Accordingly, considerable attention has been paid to the solution of fractional differential equations, integral equations, and fractional partial differential equations of physical phenomena [10–24]. One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators. It possesses advantages of fast convergence, higher stability and higher accuracy to derive the solution of different types of fractional equations.

In this work, we will deal with the eigenvalue interval for existence of positive solutions to the higher order fractional differential equation when f may be negative. This type of differential equation is called semipositone problem which arises in many interesting applications as pointed out by Lions in [25]. For example, the semipositone differential equation which can be derived from chemical reactor theory, design of suspension bridges, combustion, and management of natural resources, see [26–28]. To our knowledge, few results were established, especially for higher order multipoint boundary value problems with the fractional derivatives.

2. Preliminaries and Lemmas

We use the following assumptions in this paper:

(**B1**) $f:(0,1)\times\mathbb{R}^n\rightarrow(-\infty,+\infty)$ is continuous, and there exist functions $\alpha,\beta\in L^1[(0,1),(0,+\infty)]$ and continuous function $h:\mathbb{R}^n\rightarrow[0,+\infty)$ such that

$$-\alpha(t)\leq f(t,x_1,x_2,\ldots,x_n)\leq\beta(t)h(x_1,x_2,\ldots,x_n),(t,x_1,x_2,\ldots,x_n)\in(0,1)\times\mathbb{R}^n. \tag{2.1}$$

Now we begin this section with some preliminaries of fractional calculus. Let $\mu>0$ and $n=[\mu]+1=N+1$, where N is the smallest integer greater than or equal to μ. For a function $x:(0,1)\rightarrow\mathbb{R}$, we define the fractional integral of order μ of x as

$$I^{\mu}x(t)=\frac{1}{\Gamma(\mu)}\int_0^t(t-s)^{\mu-1}x(s)ds \tag{2.2}$$

provided the integral exists. The fractional derivative of order μ of a continuous function x is defined by

$$\mathfrak{D}_t{}^{\mu}x(t)=\frac{1}{\Gamma(n-\mu)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\mu-1}x(s)ds, \tag{2.3}$$

provided the right side is pointwise defined on $(0,+\infty)$. We recall the following properties [8, 9] which are useful for the sequel.

Lemma 2.1 (see [8, 9]).

(1) *If $x \in L^1(0,1)$, $\rho > \sigma > 0$, and $n \in \mathbb{N}$, then*

$$I^{\rho} I^{\sigma} x(t) = I^{\rho+\sigma} x(t), \qquad \mathfrak{D}_t{}^{\sigma} I^{\rho} x(t) = I^{\rho-\sigma} x(t), \tag{2.4}$$

$$\mathfrak{D}_t{}^{\sigma} I^{\sigma} x(t) = x(t), \qquad \frac{d}{dt^n}\left(\mathfrak{D}_t{}^{\sigma} x(t)\right) = \mathfrak{D}_t{}^{n+\sigma} x(t). \tag{2.5}$$

(2) *If $\nu > 0$, $\sigma > 0$, then*

$$\mathfrak{D}_t{}^{\nu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}. \tag{2.6}$$

Lemma 2.2 (see [8]). *Assume that $x \in L^1(0,1)$ and $\mu > 0$. Then*

$$I^{\mu} \mathfrak{D}_t{}^{\mu} x(t) = x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_n t^{\mu-n}, \tag{2.7}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, n is the smallest integer greater than or equal to μ.

Let $x(t) = I^{\mu_{n-1}} v(t)$, and consider the following modified integro-differential equation:

$$\begin{gathered} -\mathfrak{D}_t{}^{\mu-\mu_{n-1}} v(t) = \lambda f\left(t, I^{\mu_{n-1}} v(t), I^{\mu_{n-1}-\mu_1} v(t), \ldots, I^{\mu_{n-1}-\mu_{n-2}} v(t), v(t)\right), \\ v(0) = v'(0) = 0 \qquad v(1) = \sum_{j=1}^{m-2} a_j v(\xi_j). \end{gathered} \tag{2.8}$$

The following Lemmas 2.3–2.5 are obtained by Zhang et al. [10].

Lemma 2.3. *The higher order multipoint boundary value problem* (1.1) *has a positive solution if and only if nonlinear integro-differential equation* (2.8) *has a positive solution. Moreover, if v is a positive solution of* (2.8)*, then $x(t) = I^{\mu_{n-1}} v(t)$ is positive solution of the higher order multipoint boundary value problem* (1.1).

Lemma 2.4. *If $2 < \mu - \mu_{n-1} < 3$ and $\alpha \in L^1[0,1]$, then the boundary value problem*

$$\begin{gathered} \mathfrak{D}_t{}^{\mu-\mu_{n-1}} w(t) + \lambda \alpha(t) = 0, \\ w(0) = w'(0) = 0, \qquad w(1) = \sum_{j=1}^{m-2} a_j w(\xi_j) \end{gathered} \tag{2.9}$$

has the unique solution

$$w(t) = \lambda \int_0^1 K(t,s) \alpha(s) ds, \tag{2.10}$$

where

$$K(t,s) = k(t,s) + \frac{t^{\mu-\mu_{n-1}-1}}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1}} \sum_{j=1}^{m-2} a_j k(\xi_j, s), \tag{2.11}$$

is the Green function of the boundary value problem (2.9), *and*

$$k(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\mu-\mu_{n-1}-1} - (t-s)^{\mu-\mu_{n-1}-1}}{\Gamma(\mu-\mu_{n-1})}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\mu-\mu_{n-1}-1}}{\Gamma(\mu-\mu_{n-1})}, & 0 \le t \le s \le 1. \end{cases} \tag{2.12}$$

Lemma 2.5. *The Green function of the boundary value problem* (2.9) *satisfies*

$$K(t,s) \le \tau, \tag{2.13}$$

where

$$\tau = \frac{1+\sum_{j=1}^{m-2} a_j \left(1-\xi_j^{\mu-\mu_{n-1}-1}\right)}{\Gamma(\mu-\mu_{n-1})\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1}\right)}. \tag{2.14}$$

Define a modified function $[\cdot]^*$ *for any* $z \in C[0,1]$ *by*

$$[z(t)]^* = \begin{cases} z(t), & z(t) \ge 0, \\ 0, & z(t) < 0, \end{cases} \tag{2.15}$$

and consider the following boundary value problem

$$\begin{gathered} -\mathfrak{D}_t^{\mu-\mu_{n-1}} u(t) = \lambda\Big[f\big(t, I^{\mu_{n-1}}[u(t)-w(t)]^*, I^{\mu_{n-1}-\mu_1}[u(t)-w(t)]^* \cdots, \\ I^{\mu_{n-1}-\mu_{n-2}}[u(t)-w(t)]^*, [u(t)-w(t)]^*\big) + \alpha(t)\Big], \\ u(0) = u'(0) = 0, \qquad u(1) = \sum_{j=1}^{m-2} a_j u(\xi_j). \end{gathered} \tag{2.16}$$

Lemma 2.6. *Suppose* $u(t) \ge w(t)$, $t \in [0,1]$ *is a solution of the problem* (2.16), *then* $u-w$ *is a positive solution of the problem* (2.8), *consequently,* $I^{\mu_{n-1}}[u(t)-w(t)]$ *is also a positive solution of the semipositone higher differential equation* (1.1).

Proof. Since u is a solution of the BVP (2.16) and $u(t) \geq w(t)$ for any $t \in [0,1]$, then we have

$$\begin{gathered} -\mathfrak{D}_t{}^{\mu-\mu_{n-1}} u(t) = \lambda\big[f\big(t, I^{\mu_{n-1}}[u(t) - w(t)], I^{\mu_{n-1}-\mu_1}[u(t) - w(t)], \dots, \\ I^{\mu_{n-1}-\mu_{n-2}}[u(t) - w(t)], [u(t) - w(t)]\big) + \alpha(t)\big], \\ u(0) = u'(0) = 0, \qquad u(1) = \sum_{j=1}^{m-2} a_j u(\xi_j). \end{gathered} \tag{2.17}$$

Let $v = u - w$, then we have

$$w(0) = w'(0) = 0, \qquad w(1) = \sum_{j=1}^{m-2} a_j w(\xi_j), \tag{2.18}$$

and $\mathfrak{D}_t{}^{\mu-\mu_{n-1}} v(t) = \mathfrak{D}_t{}^{\mu-\mu_{n-1}} u(t) - \mathfrak{D}_t{}^{\mu-\mu_{n-1}} w(t)$, which implies that

$$-\mathfrak{D}_t{}^{\mu-\mu_{n-1}} u(t) = -\mathfrak{D}_t{}^{\mu-\mu_{n-1}} v(t) + \lambda\alpha(t). \tag{2.19}$$

Substituting the above into (2.17), then $v = u - w$ solves the (2.8), that is, $u - w$ is a positive solution of the semipositone differential equation (2.8). By Lemma 2.3, $I^{\mu_{n-1}}[u(t) - w(t)]$ is a positive solution of the singular semipositone differential equation (1.1). This completes the proof of Lemma 2.5. □

Let

$$\gamma(t) = \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1}\right) t^{\mu-\mu_{n-1}-1}(1-t) + \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1}(1-\xi_j) t^{\mu-\mu_{n-1}-1}. \tag{2.20}$$

Lemma 2.7 (see [10]). *The solution $w(t)$ of (2.9) satisfies*

$$w(t) \leq \lambda\eta\gamma(t), \quad t \in [0,1], \tag{2.21}$$

where

$$\eta = \frac{\int_0^1 \alpha(s)ds}{\Gamma(\mu - \mu_{n-1})\left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1}\right)}. \tag{2.22}$$

It is well known that the BVP (2.17) is equivalent to the fixed points for the mapping T by

$$\begin{gathered} (Tu)(t) = \lambda \int_0^1 K(t,s)\big[f\big(s, I^{\mu_{n-1}}[u(s) - w(s)], I^{\mu_{n-1}-\mu_1}[u(s) - w(s)], \dots, \\ I^{\mu_{n-1}-\mu_{n-2}}[u(s) - w(s)], [u(s) - w(s)]\big) + \alpha(s)\big] ds. \end{gathered} \tag{2.23}$$

The basic space used in this paper is $E = C([0,1];\mathbb{R})$, where $\mathbb{R}$ is the set of real numbers. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|u\| = \max_{t\in[0,1]} |u(t)|, \tag{2.24}$$

for any $u \in E$. Let

$$P = \left\{ u \in E : u(t) \geq \frac{1}{2}\gamma(t)\|u\| \right\}, \tag{2.25}$$

then P is a cone of E.

Lemma 2.8. *Assume that* (**B1**) *holds. Then $T : P \to P$ is a completely continuous operator.*

Proof. By using similar method to [10] and standard arguments, according to the Ascoli-Arzela Theorem, one can show that $T : P \to P$ is a completely continuous operator. □

Lemma 2.9 (see [29]). *Let E be a real Banach space, $P \subset E$ be a cone. Assume Ω_1, Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_2$, *or*

(2) $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

Theorem 3.1. *Suppose that* (**B1**) *holds, and*

$$\lim_{\sum_{i=1}^{n}|x_i|\to\infty} \min_{t\in[1/4,3/4]} \frac{f(t, x_1, x_2, \ldots, x_n)}{\sum_{i=1}^{n}|x_i|} = +\infty. \tag{3.1}$$

Then there exists some constant $\lambda^* > 0$ such that the higher order multipoint boundary value problem (1.1) *has at least one positive solution for any $\lambda \in (0, \lambda^*)$.*

Proof. By Lemma 2.8, we know T is a completely continuous operator. Take

$$r = \tau \int_0^1 \left[\beta(s) + \alpha(s)\right] ds, \tag{3.2}$$

where τ is defined by Lemma 2.5. Let $\Omega_1 = \{u \in P : \|u\| < r\}$. Then, for any $u \in \partial\Omega_1$, $s \in [0,1]$, we have

$$\begin{aligned}
&0 \le [u(s) - w(s)]^* \le u(s) \le \|u\| \le r, \\
&0 \le I^{\mu_{n-1}}[u(s) - w(s)]^* \le \frac{r}{\Gamma(\mu_{n-1})}, \\
&0 \le I^{\mu_{n-1}-\mu_i}[u(s) - w(s)]^* \le \frac{r}{\Gamma(\mu_{n-1} - \mu_i)}, \quad i = 1, \dots, n-2.
\end{aligned} \tag{3.3}$$

Choose

$$\lambda^* = \min\left\{\frac{r}{4\eta}, \frac{1}{N+1}\right\}, \tag{3.4}$$

where

$$N = \max_{(u_1,u_2,\dots,u_n)\in[0,1]\times[0,r/\Gamma(\mu_{n-1})]\times\cdots\times[0,r/\Gamma(\mu_{n-1}-\mu_{n-2})]\times[0,r]} h(u_1, u_2, \dots, u_n), \tag{3.5}$$

and η is defined by Lemma 2.7. Thus, for any $u \in \partial\Omega_1$, $s \in [0,1]$, and $\lambda \in (0, \lambda^*)$, by (3.3), we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} (Tu)(t) \\
&= \lambda \max_{t\in[0,1]} \int_0^1 K(t,s) \big[f\big(s, I^{\mu_{n-1}}[u(s) - w(s)]^*, I^{\mu_{n-1}-\mu_1}[u(s) - w(s)]^*, \dots, \\
&\qquad I^{\mu_{n-1}-\mu_{n-2}}[u(s) - w(s)]^*, [u(s) - w(s)]^*\big) + \alpha(s)\big] ds \\
&\le \lambda\tau \int_0^1 \big[\beta(s) h\big(I^{\mu_{n-1}}[u(s) - w(s)]^*, I^{\mu_{n-1}-\mu_1}[u(s) - w(s)]^*, \dots, \\
&\qquad I^{\mu_{n-1}-\mu_{n-2}}[u(s) - w(s)]^*, [u(s) - w(s)]^*\big) + \alpha(s)\big] ds \\
&\le \lambda\tau(N+1) \int_0^1 \big[\beta(s) + \alpha(s)\big] ds \le r = \|u\|.
\end{aligned} \tag{3.6}$$

Therefore,

$$\|Tu\| \le \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.7}$$

On the other hand, choose a real number $L > 0$ such that

$$\lambda L \int_{1/4}^{3/4} K\left(\frac{1}{2}, s\right) ds \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}}\right) \left(\frac{1}{4}\right)^{\mu-\mu_{n-1}+1} \ge 1. \tag{3.8}$$

By (3.1), for any $t \in [1/4, 3/4]$, there exists a constant $B > 0$ such that

$$\frac{f(t, x_1, x_2, \dots, x_n)}{\sum_{i=1}^{n} |x_i|} > L, \quad \text{for } \sum_{i=1}^{n} |x_i| \geq B. \tag{3.9}$$

Take

$$R = \max\left\{ 2r, 4\lambda\eta, \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}}\right)^{-1} 4^{\mu-\mu_{n-1}+1} B \right\}, \tag{3.10}$$

let $\Omega_2 = \{u \in P : \|u\| < R\}$ and $\partial\Omega_2 = \{u \in P : \|u\| = R\}$. Then for any $u \in \partial\Omega_2$, $s \in [0,1]$, by Lemma 2.7, we have

$$u(s) - w(s) \geq u(s) - \lambda\eta\gamma(s) \geq \frac{1}{2} R\gamma(s) - \lambda\eta\gamma(s) \geq \frac{1}{4} R\gamma(s) \geq 0. \tag{3.11}$$

And then, for any $u \in \partial\Omega_2$, $s \in [1/4, 3/4]$, one gets

$$\begin{aligned} &\left(\sum_{i=1}^{n-2} \left| I^{\mu_{n-1}-\mu_i} [u(s) - w(s)]^* \right| \right) + \left| I^{\mu_{n-1}} [u(s) - w(s)]^* \right| + \left| [u(s) - w(s)]^* \right| \\ &\quad \geq u(s) - w(s) \geq \frac{1}{4} R\gamma(t) \geq \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1}\right) \left(\frac{1}{4}\right)^{\mu-\mu_{n-1}+1} R \\ &\qquad + \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1} (1 - \xi_j) \left(\frac{1}{4}\right)^{\mu-\mu_{n-1}} R \geq \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}}\right) \left(\frac{1}{4}\right)^{\mu-\mu_{n-1}+1} R \geq B. \end{aligned} \tag{3.12}$$

It follows from (3.12) that, for any $u \in \partial\Omega_2$,

$$\begin{aligned} \|Tu\| &\geq \lambda \int_0^1 K\left(\frac{1}{2}, s\right) \Big[f\big(s, I^{\mu_{n-1}} [u(s) - w(s)]^*, I^{\mu_{n-1}-\mu_1} [u(s) - w(s)]^*, \dots, \\ &\qquad I^{\mu_{n-1}-\mu_{n-2}} [u(s) - w(s)]^*, [u(s) - w(s)]^* \big) + \alpha(s) \Big] ds \\ &\geq \lambda \int_{1/4}^{3/4} K\left(\frac{1}{2}, s\right) f\big(s, I^{\mu_{n-1}} [u(s) - w(s)], I^{\mu_{n-1}-\mu_1} [u(s) - w(s)], \dots, \\ &\qquad I^{\mu_{n-1}-\mu_{n-2}} [u(s) - w(s)], [u(s) - w(s)] \big) ds \end{aligned}$$

$$\begin{aligned}
&\geq \lambda \int_{1/4}^{3/4} K\left(\frac{1}{2}, s\right) L\left(\left|I^{\mu_{n-1}}[u(s)-w(s)]\right| + \left|I^{\mu_{n-1}-\mu_1}[u(s)-w(s)]\right|\right. \\
&\qquad\qquad \left. + \cdots + \left|I^{\mu_{n-1}-\mu_{n-2}}[u(s)-w(s)]\right| + |[u(s)-w(s)]|\right) ds \\
&\geq \lambda L \int_{1/4}^{3/4} K\left(\frac{1}{2}, s\right) |[u(s)-w(s)]| ds \\
&\geq \lambda L \int_{1/4}^{3/4} K\left(\frac{1}{2}, s\right) ds \left(1 - \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}}\right) \left(\frac{1}{4}\right)^{\mu-\mu_{n-1}+1} R = R = \|u\|.
\end{aligned} \tag{3.13}$$

So, we have

$$\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.14}$$

By Lemma 2.9, T has at least a fixed point $u \in (P \cap \overline{\Omega}_2) \setminus \Omega_1$ such that $r \leq \|u\| \leq R$.

It follows from $\lambda^* \leq r/4\eta$ and $\lambda \in (0, \lambda^*)$ that

$$u(t) \ - w(t) \geq u(t) - \lambda\eta\gamma(t) \geq \frac{1}{2} r\gamma(t) - \lambda\eta\gamma(t) \geq \frac{r}{4}\gamma(t) > 0, \quad t \in (0,1). \tag{3.15}$$

Let $x(t) = I^{\mu_{n-1}}[u(t) - w(t)]$, then

$$x(t) > 0, \quad t \in (0,1). \tag{3.16}$$

By Lemma 2.6, we know that the differential equation (1.1) has at least a positive solutions x. □

Theorem 3.2. *Suppose* (**B1**) *holds and*

$$\liminf_{\sum_{i=1}^n |u_i| \to \infty} \min_{t \in [1/4, 3/4]} f(t, u_1, u_2, \ldots, u_n) = +\infty, \qquad \lim_{\sum_{i=1}^n |u_i| \to \infty} \frac{h(u_1, u_2, \ldots, u_n)}{\sum_{i=1}^n |u_i|} = 0. \tag{3.17}$$

Then there exists $\lambda^* > 0$ such that the higher order multipoint boundary value problem (1.1) *has at least one positive solution for any $\lambda \in (\lambda^*, +\infty)$.*

Proof. By (3.17), there exists $M > 0$ such that for any $t \in [1/4, 3/4]$ we have

$$f(t, u_1, u_2, \ldots, u_n) \geq \frac{4\eta}{\int_{1/4}^{3/4} K(1/2, s) ds}, \quad \text{if } \sum_{i=1}^n |u_i| \geq M. \tag{3.18}$$

Let

$$\lambda^* = \frac{M}{\eta\left(1-\sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}}\right)(1/4)^{\mu-\mu_{n-1}+1}}. \tag{3.19}$$

In the following of the proof, we suppose $\lambda > \lambda^*$. Take

$$r = 4\lambda\eta, \tag{3.20}$$

and let $\Omega_1 = \{u \in P : \|u\| < r\}$ and $\partial\Omega_1 = \{u \in P : \|u\| = r\}$. Then for any $u \in \partial\Omega_1$, $s \in [0,1]$, by Lemma 2.7, we have

$$u(s) - w(s) \geq u(s) - \lambda\eta\gamma(s) \geq \frac{1}{2}r\gamma(s) - \lambda\eta\gamma(s) = \lambda\eta\gamma(s) \geq 0. \tag{3.21}$$

So for any $u \in \partial\Omega_1$, $s \in [1/4, 3/4]$, one gets

$$\begin{aligned}
&\left(\sum_{i=1}^{n-2}\left|I^{\mu_{n-1}-\mu_i}[u(s)-w(s)]^*\right|\right) + \left|I^{\mu_{n-1}}[u(s)-w(s)]^*\right| + \left|[u(s)-w(s)]^*\right| \\
&\quad \geq u(s)-w(s) \geq \lambda\eta\gamma(s) \geq \lambda\eta\left(1-\sum_{j=1}^{m-2} a_j\xi_j^{\mu-\mu_{n-1}}\right)\left(\frac{1}{4}\right)^{\mu-\mu_{n-1}+1} \\
&\quad \geq \lambda^*\eta\left(1-\sum_{j=1}^{m-2} a_j\xi_j^{\mu-\mu_{n-1}}\right)\left(\frac{1}{4}\right)^{\mu-\mu_{n-1}+1} \geq M.
\end{aligned} \tag{3.22}$$

Thus, by (3.22), for any $u \in \partial\Omega_1$, we have

$$\begin{aligned}
\|Tu\| &\geq \lambda\int_0^1 K\left(\frac{1}{2},s\right)\Big[f\big(s, I^{\mu_{n-1}}[u(s)-w(s)]^*, I^{\mu_{n-1}-\mu_1}[u(s)-w(s)]^*, \dots, \\
&\qquad I^{\mu_{n-1}-\mu_{n-2}}[u(s)-w(s)]^*, [u(s)-w(s)]^*\big) + \alpha(s)\Big]ds \\
&\geq \lambda\int_{1/4}^{3/4} K\left(\frac{1}{2},s\right)f\big(s, I^{\mu_{n-1}}[u(s)-w(s)]^*, I^{\mu_{n-1}-\mu_1}[u(s)-w(s)]^*, \dots, \\
&\qquad I^{\mu_{n-1}-\mu_{n-2}}[u(s)-w(s)]^*, [u(s)-w(s)]^*\big)ds \\
&\geq \lambda\int_{1/4}^{3/4} K\left(\frac{1}{2},s\right)ds \times \frac{4\eta}{\int_{1/4}^{3/4} K(1/2,s)ds} = r = \|u\|.
\end{aligned} \tag{3.23}$$

So, we have

$$\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.24}$$

Next, take

$$\sigma = \sum_{i=1}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)} + \frac{1}{\Gamma(\mu_{n-1})} + 1. \tag{3.25}$$

Let us choose $\varepsilon > 0$ such that

$$\lambda\tau\sigma\varepsilon \int_0^1 \beta(s)ds < 1. \tag{3.26}$$

Then for the above ε, by (3.17), there exists $N > r > 0$ such that, for any $t \in [0, 1]$,

$$h(u_1, u_2, \dots, u_n) \le \varepsilon \sum_{i=1}^{n} |u_i|, \quad \text{if } \sum_{i=1}^{n} |u_i| > N. \tag{3.27}$$

Thus, by (3.3) and (3.27), if

$$\left(\sum_{i=1}^{n-2} \left| I^{\mu_{n-1}-\mu_i} [u(s) - w(s)]^* \right| \right) + \left| I^{\mu_{n-1}} [u(s) - w(s)]^* \right| + \left| [u(s) - w(s)]^* \right| > N, \tag{3.28}$$

we have

$$\begin{aligned}
&h\big(I^{\mu_{n-1}} [u(s) - w(s)]^*, I^{\mu_{n-1}-\mu_1} [u(s) - w(s)]^*, \dots, \\
&I^{\mu_{n-1}-\mu_{n-2}} [u(s) - w(s)]^*, [u(s) - w(s)]^* \big) ds \\
&\quad \le \left| I^{\mu_{n-1}} [u(s) - w(s)]^* \right| + \left| I^{\mu_{n-1}-\mu_1} [u(s) - w(s)]^* \right| + \cdots \\
&\qquad + \left| I^{\mu_{n-1}-\mu_{n-2}} [u(s) - w(s)]^* \right| + \left| [u(s) - w(s)]^* \right| \varepsilon \\
&\quad \le \left(\sum_{i=1}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)} + \frac{1}{\Gamma(\mu_{n-1})} + 1 \right) \|u\| \varepsilon = \sigma \|u\| \varepsilon.
\end{aligned} \tag{3.29}$$

Take

$$R = \frac{\lambda\tau\theta \int_0^1 \big[\beta(s) + \alpha(s)\big] ds + \lambda\tau \int_0^1 \alpha(s) ds}{1 - \lambda\tau\sigma\varepsilon \int_0^1 \beta(s) ds} + N, \tag{3.30}$$

where

$$\theta = \max_{\sum_{i=1}^{n} |u_i| \le N} h(u_1, u_2, \dots, u_n) + 1. \tag{3.31}$$

Then $R > N > r$.

Now let $\Omega_2 = \{u \in P : \|u\| < R\}$ and $\partial\Omega_2 = \{u \in P : \|u\| = R\}$. Then, for any $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}(Tu)(t) \\
&= \lambda \max_{t\in[0,1]} \int_0^1 K(t,s)\big[f\big(s, I^{\mu_{n-1}}[u(s)-w(s)]^*, I^{\mu_{n-1}-\mu_1}[u(s)-w(s)]^*, \dots, \\
&\qquad\qquad I^{\mu_{n-1}-\mu_{n-2}}[u(s)-w(s)]^*, [u(s)-w(s)]^*\big) + \alpha(s)\big]ds \\
&\le \lambda\tau \int_0^1 \big[\beta(s) h\big(I^{\mu_{n-1}}[u(s)-w(s)]^*, I^{\mu_{n-1}-\mu_1}[u(s)-w(s)]^*, \dots, \\
&\qquad\qquad I^{\mu_{n-1}-\mu_{n-2}}[u(s)-w(s)]^*, [u(s)-w(s)]^*\big) + \alpha(s)\big]ds \\
&\le \lambda\tau \left(\max_{\sum_{i=1}^n |u_i| \le N} h(u_1, u_2, \dots, u_n) + 1\right) \int_0^1 [\beta(s) + \alpha(s)]ds + \lambda\tau \int_0^1 [\beta(s)\sigma\varepsilon\|u\| + \alpha(s)]ds \\
&\le \lambda\tau\theta \int_0^1 [\beta(s) + \alpha(s)]ds + \lambda\tau \int_0^1 \alpha(s)ds + \lambda\tau\sigma\varepsilon \int_0^1 \beta(s)dsR \le R = \|u\|,
\end{aligned} \tag{3.32}$$

which implies that

$$\|Tu\| \le \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.33}$$

By Lemma 2.9, T has at least a fixed point $u \in (P \cap \overline{\Omega}_2) \setminus \Omega_1$ such that $r \le \|u\| \le R$.

It follows from $r = 4\lambda\eta$ that

$$u(t) - w(t) \ge \frac{1}{2}\|u\|\gamma(t) - \lambda\eta\gamma(t) = \lambda\eta\gamma(t) > 0, \quad t \in (0,1). \tag{3.34}$$

Let $x(t) = I^{\mu_{n-1}}[u(t) - w(t)]$, then

$$x(t) > 0, \quad t \in (0,1). \tag{3.35}$$

By Lemma 2.6, we know that the differential equation (1.1) has at least a positive solutions x. □

Example 3.3. Consider the existence of positive solutions for the nonlinear higher order fractional differential equation with four-point boundary condition

$$\begin{gathered}
-\mathfrak{D}_t^{5/2}x(t) = \left[|x(t)| + \left|\mathfrak{D}_t^{1/4}x(t)\right| + \left|\mathfrak{D}_t^{3/8}x(t)\right|\right]^{1/2} + \ln t, \\
\mathfrak{D}_t^{1/4}x(0) = \mathfrak{D}_t^{3/8}x(0) = \mathfrak{D}_t^{11/8}x(0) = 0, \qquad \mathfrak{D}_t^{3/8}x(1) = 2\mathfrak{D}_t^{3/8}x\left(\frac{1}{2}\right) - \mathfrak{D}_t^{3/8}x\left(\frac{3}{4}\right).
\end{gathered} \tag{3.36}$$

Then there exists $\lambda^* > 0$ such that the higher order four-point boundary value problem (1.1) has at least one positive solution for any $\lambda \in (\lambda^*, +\infty)$.

Proof. Let

$$f(t,x,y,z) = (|x| + |y| + |z|)^{1/2} + \ln t, \quad t \in (0,1) \times \mathbb{R}^3, \tag{3.37}$$

then

$$\begin{gathered} -|\ln t| = \ln t \le f(t,x,y,z) = (|x| + |y| + |z|)^{1/2} + \ln t \le (|x| + |y| + |z|)^{1/2}, \\ \alpha(\mathrm{t}) = |\ln t|, \qquad \beta(t) = 1, \qquad h(x,y,z) = (|x| + |y| + |z|)^{1/2}, \\ \liminf_{|x|+|y|+|z| \to +\infty} \min_{t \in [1/4,3/4]} f(t,x,y,z) = +\infty, \qquad \lim_{|x|+|y|+|z| \to +\infty} \frac{h(x,y,z)}{|x| + |y| + |z|} = 0. \end{gathered} \tag{3.38}$$

Clearly, $\alpha, \beta \in L^1[(0,1),(0,+\infty)]$, and

$$0 < \sum_{j=1}^{m-2} a_j \xi_j^{\mu-\mu_{n-1}-1} = 1 - 2\left(\frac{1}{2}\right)^{9/8} + \left(\frac{3}{4}\right)^{9/8} = 0.8065 < 1. \tag{3.39}$$

By Theorem 3.2, there exists $\lambda^* > 0$ such that the higher order multipoint boundary value problem (3.36) has at least one positive solution for any $\lambda \in (\lambda^*, +\infty)$. □

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