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Research Article

On the Nonlinear Instability of Traveling Waves for a Sixth-Order Parabolic Equation

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We study the instability of the traveling waves of a sixth-order parabolic equation which arises naturally as a continuum model for the formation of quantum dots and their faceting. We prove that some traveling wave solutions are nonlinear unstable under H^4 perturbations. These traveling wave solutions converge to a constant as $x \to \infty$.

1. Introduction

In this paper, we consider the following sixth-order parabolic equation

$$\frac{\partial u}{\partial t} = D^6 u + D^4\left(u - u^3\right) + g(u), \quad (x,t) \in R \times (0,T), \tag{1.1}$$

where $g(u) = a(1-u^2)$, $a > 0$.

Equation (1.1) arises naturally as a continuum model for the formation of quantum dots and their faceting; see [1]. Here $u(x,t)$ denotes the surface slope. The high-order derivatives are a result of the additional regularization energy which is required to form an edge between two-plane surfaces with different orientations.

During the past years, only a few works have been devoted to the sixth-order parabolic equation [2–7]. Barrett et al. [2] considered the above equation with $m = 2$. A finite element method is presented which proves to be well posed and convergent. Numerical experiments illustrate the theory.

Recently, Jüngel and Milišić [5] studied the sixth-order nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \left[u\left(\frac{1}{u}(u(\ln u)_{xx})_{xx} + \frac{1}{2}((\ln u)_{xx})^2\right)_x\right]_x. \tag{1.2}$$

They proved the global-in-time existence of weak nonnegative solutions in one space dimension with periodic boundary conditions.

Evans et al. [3, 4] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[|u|^n \nabla \Delta^2 u\right] - \Delta\left(|u|^{p-1}u\right), \quad n > 0,\ p > 1. \tag{1.3}$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n+1+4/N$ for $n \in (0, 5/4)$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{(-N/(nN+6))} f_k(y)$, $y = x/((T-t)^{(1/(nN+6))})$, where $T > 0$ is the blow-up time.

Korzec et al. [8] considered the sixth-order equation

$$u_t - \nu u u_x - \left(u - u^3 + \varepsilon^2 u_{xx}\right)_{xxxx} = 0. \tag{1.4}$$

New type of stationary solutions is derived by an extension of the method of matched asymptotic expansion.

In this paper, we study instability of the traveling waves of (1.1). Our main result is as follows.

Theorem 1.1. *All the traveling waves $\varphi(x - ct)$ of (1.1) satisfying $\varphi \in L^\infty(R)$, $\varphi^{(n)} \in L^\infty(R) \cap L^2(R)$ $(n = 1, 2, \ldots, 6)$ are nonlinearly unstable in the space $H^4(R)$, where $\varphi^{(n)}$ denotes nth derivative of φ.*

The stability and instability of special solutions for the higher-order parabolic equation are very important in the applied fields. Carlen et al. [9] proved the nonlinear stability of fronts for the Cahn-Hilliard, under L^1 perturbations. Gao and Liu [10] prove that it is nonlinearly unstable under H^2 perturbations, for some traveling wave solution of the convective Cahn-Hilliard equation. The relevant equations have also been studied in [11, 12]. The main difficulties for treating (1.1) are caused by the principal part and the lack of the Lyapunov functional. Our proof is based on the principle of linearization. We invoke a general theorem that asserts that linearized instability implies nonlinear instability.

This paper is organized as follows. In the next section, we find an exact traveling wave solution for (1.1). In Section 3, we give the proof of our main result.

2. Exact Traveling Wave Solutions

In this section, we construct an exact traveling wave which satisfies all conditions of Theorem 1.1.

If $\varphi(x-ct) = \varphi(z)$ is a traveling wave solution of (1.1), then φ satisfies the ordinary differential equation

$$-c\varphi' = \varphi^{(6)} + (1-3\varphi^2)\varphi^{(4)} - 36\varphi'^2\varphi'' - 18\varphi\varphi''^2 - 24\varphi\varphi'\varphi''' + a(1-\varphi^2). \tag{2.1}$$

Let $\varphi' = \partial\varphi/\partial z = k(1-\varphi^2)$. Then

$$\begin{aligned}
\varphi'' &= \frac{\partial}{\partial z}\left(k\left(1-\varphi^2\right)\right) = -2k^2\varphi\left(1-\varphi^2\right),\\
\varphi''' &= \frac{\partial}{\partial z}\left(-2k^2\varphi\left(1-\varphi^2\right)\right) = 2k^3\left(-1+3\varphi^2\right)\left(1-\varphi^2\right),\\
\varphi^{(4)} &= \frac{\partial}{\partial z}\left(2k^3\left(-1+3\varphi^2\right)\left(1-\varphi^2\right)\right) = 2k^4\left(8\varphi-12\varphi^3\right)\left(1-\varphi^2\right),\\
\varphi^{(5)} &= \frac{\partial}{\partial z}\left(2k^4\left(8\varphi-12\varphi^3\right)\left(1-\varphi^2\right)\right) = 8k^5\left(1-\varphi^2\right)\left(2-15\varphi^2+15\varphi^4\right),\\
\varphi^{(6)} &= \frac{\partial}{\partial z}\left(8k^5\left(1-\varphi^2\right)\left(2-15\varphi^2+15\varphi^4\right)\right) = 16k^6\varphi\left(1-\varphi^2\right)\left(60\varphi^2-45\varphi^4-17\right).
\end{aligned} \tag{2.2}$$

Substituting the above equations into (2.1), we have

$$-ck-a = \left(360k^4-720k^6\right)\varphi^5 + \left(960k^6-480k^4\right)\varphi^3 + \left(136k^4-272k^6\right)\varphi. \tag{2.3}$$

Then comparing the order of φ, we obtain

$$\begin{aligned}
-ck &= a,\\
360k^4 - 720k^6 &= 0,\\
960k^6 - 480k^4 &= 0,\\
136k^4 - 272k^6 &= 0.
\end{aligned} \tag{2.4}$$

A simple calculation shows that $k = 1/\sqrt{2}$, $c = -\sqrt{2}a$. Hence, we get

$$\varphi' = \frac{1}{\sqrt{2}}(1-\varphi^2), \tag{2.5}$$

that is,

$$\frac{1}{2}\ln\frac{1+\varphi}{1-\varphi} = \frac{1}{\sqrt{2}}z, \tag{2.6}$$

that is,

$$\varphi(z) = \frac{e^{(1/\sqrt{2})z} - e^{-(1/\sqrt{2})z}}{e^{(1/\sqrt{2})z} + e^{-(1/\sqrt{2})z}} = \tanh \frac{1}{\sqrt{2}} z. \tag{2.7}$$

We easily proved that

$$\lim_{z \to +\infty} \varphi(z) = 1, \qquad \lim_{z \to +\infty} \varphi(z) = -1 \tag{2.8}$$

and $\varphi(z)$ satisfies the conditions of the Theorem 1.1.

3. Proof of The Result

To prove the Theorem 1.1, we first consider an evolution equation

$$\frac{\partial u}{\partial t} = Lu + F(u), \tag{3.1}$$

where L is a linear operator that generates a strongly continuous semigroup e^{tL} on a Banach space X, and F is a strongly continuous operator such that $F(0) = 0$. In [13], authors considered the whole problem only on space X, that is, the nonlinear operator maps X to X. However, many equations posses nonlinear terms that include derivatives and therefore, F maps into a large Banach space Z. Hence, they again got the following lemma.

Lemma 3.1 (see [14]). *Assume the following.*

(i) *X, Z are two Banach spaces with $X \subset Z$ and $\|u\|_Z \leq c_1 \|u\|_X$ for $u \in X$.*

(ii) *L generates a strongly continuous semigroup e^{tL} on the space Z, and the semigroup e^{tL} maps Z into X for $t > 0$ and $\int_0^1 \|e^{tL}\|_{Z \to X} dt = C_4 < \infty$.*

(iii) *The spectrum of L on X meets the right half-plane, $\{Re\lambda > 0\}$.*

(iv) *$F : X \to Z$ is continuous and $\exists\ \rho_0 > 0, C_3 > 0, \alpha > 1$ such that $\|F(u)\|_Z < C_0 \|u\|_X^{\alpha}$, for $\|u\|_X < \rho_0$.*

Then the zero solution of (3.1) *is nonlinearly unstable in the space* X.

In this paper, we are going to use Lemma 3.1 for the proof of Theorem 1.1.

Definition 3.2. A traveling wave solution $\varphi(x - ct)$ of (1.1) is said to be nonlinearly unstable in the space X, if there exist positive ε_0 and C_0, a sequence $\{u_n\}$ of solutions of (1.1) and a sequence of time $t_n > 0$ such that $\|u_n(0) - \varphi(x)\|_X \to 0$ but $\|u_n(t_n) - \varphi(\cdot - ct_n)\|_X \geq \varepsilon_0$.

If $\varphi(x-ct) \in H^4(R)$ is a traveling wave solution of (1.1), then letting $\omega(x,t) = u(x,t) - \varphi(x-ct)$, we have

$$\begin{aligned}(\omega+\varphi)_t &= \partial_x^6(\omega+\varphi) + \partial_x^4\left[(\omega+\varphi) - (\omega+\varphi)^3\right] + a\left[1 - (\omega+\varphi)^2\right] \\ &= \partial_x^6\omega + \varphi^{(6)} + \partial_x^4\left(\omega + \varphi - \omega^3 - \varphi^3 - 3\omega^2\varphi - 3\omega\varphi^2\right) \\ &\quad + a\left(1 - \omega^2 - \varphi^2 - 2\omega\varphi\right),\end{aligned} \tag{3.2}$$

that is,

$$\omega_t = \partial_x^6\omega + \partial_x^4\left(\omega - \omega^3 - 3\omega^2\varphi - 3\omega\varphi^2\right) + a\left(-\omega^2 - 2\omega\varphi\right), \tag{3.3}$$

that is,

$$\begin{aligned}&\omega_t - \partial_x^6\omega - \left(1 - 3\varphi^2\right)\partial_x^4\omega + 24\varphi\varphi'\partial_x^3\omega + \left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2\omega \\ &\quad + \left(24\varphi\varphi''' + 72\varphi'\varphi''\right)\partial_x\omega + \left(18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi\right)\omega \\ &= F(\omega),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}F(\omega) = &\left(3\varphi^{(4)} - a\right)\omega^2 + 24\varphi'''\omega\partial_x\omega + 36\varphi''(\partial_x\omega)^2 + 72\varphi'\partial_x\omega\partial_x^2\omega \\ &+ 36(\partial_x\omega)^2\partial_x^2\omega + (24\varphi + 24\omega)\partial_x\omega\partial_x^3\omega + 36\varphi''\omega\partial_x^2\omega \\ &+ (18\varphi + 18\omega)\left(\partial_x^2\omega\right)^2 + 24\varphi'\omega\partial_x^3\omega + 6\varphi\omega\partial_x^4\omega + 3\omega^2\partial_x^4\omega,\end{aligned} \tag{3.5}$$

with the initial value

$$\omega(x,0) = \omega_0(x) \equiv u_0(x) - \varphi(x). \tag{3.6}$$

So the stability of traveling wave solutions of (1.1) is translated into the stability of the zero solution of (3.4). In order to prove Theorem 1.1, taking $Z = L^2(R)$, $X = H^4(R)$, we need to prove that the four conditions of Lemma 3.1 are satisfied by the associated equation (3.4). The condition (i) is satisfied by our choice of Z and X.

Denote the linear partial differential operator in (3.4) by $L = (\partial_x^6 + \partial_x^4) - [3\varphi^2\partial_x^4 + 24\varphi\varphi'\partial_x^3 + (36\varphi\varphi'' + 36\varphi'^2)\partial_x^2 + (24\varphi\varphi''' + 72\varphi'\varphi'')\partial_x + (18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi)] = L_0 - [3\varphi^2\partial_x^4 + 24\varphi\varphi'\partial_x^3 + (36\varphi\varphi'' + 36\varphi'^2)\partial_x^2 + (24\varphi\varphi''' + 72\varphi'\varphi'')\partial_x + (18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi)]$ with $L_0 = \partial_x^6 + \partial_x^4$. Then (3.4) may be rewritten in the form of (3.1)

$$\omega_t = L\omega + F(\omega). \tag{3.7}$$

Note the F maps $H^4(R)$ into $L^2(R)$, using the Sobolev embedding theorem, we have

$$\|F(\omega)\|_{L^2} \le C\|\omega\|_{H^4}^2, \quad C > 0, \text{ for } \|\omega\|_{H^4} < 1. \tag{3.8}$$

So, the condition (iv) is satisfied.

To prove condition (ii) in Lemma 3.1, we need the following two lemmas.

Lemma 3.3. *Let $L_0 = \partial_x^6 + \partial_x^4$. Then*

$$\left\|e^{tL_0}\right\|_{H^m \to H^m} \le e^{(4/27)t}, \quad \text{for } m \in R^+,\ 0 \le t < \infty, \tag{3.9}$$

$$\left\|e^{tL_0}\right\|_{L^2 \to H^4} \le a(t) \equiv 5t^{-2/3}, \quad \text{for } 0 < t \le 1. \tag{3.10}$$

Proof. We write $u(x,t) = e^{tL_0}u_0(x)$. By Fourier transformation

$$\begin{aligned} \widehat{u}(\xi,t) &= e^{-t(\xi^6-\xi^4)}\widehat{u}_0(\xi), \\ \|u\|_{H^m}^2 &\equiv \int_{-\infty}^{\infty}\left(1+\xi^2\right)^m |\widehat{u}(\xi,t)|^2 d\xi \\ &= \int_{-\infty}^{\infty}\left(1+\xi^2\right)^m e^{-2t(\xi^6-\xi^4)}|\widehat{u}_0(\xi)|^2 d\xi \\ &\le \sup_{\xi\in R} e^{-2t(\xi^6-\xi^4)}\int_{-\infty}^{\infty}\left(1+\xi^2\right)^m |\widehat{u}_0(\xi)|^2 d\xi \\ &= e^{(8/27)t}\|u_0\|_{H^m}^2. \end{aligned} \tag{3.11}$$

Hence,

$$\left\|e^{tL_0}\right\|_{H^m \to H^m} \le e^{(4/27)t}. \tag{3.12}$$

On the other hand, letting $s = \xi^2$, we have

$$\|u\|_{H^4}^2 \le \sup_{s\in R} f(s) \int_{-\infty}^{\infty} |\widehat{u}_0(\xi)|^2 d\xi, \tag{3.13}$$

with $f(s) = (1+s)^4 e^{-2t(s^3-s^2)}$, $t > 0$. Elementary computation shows that

$$\sup_{s>0} f(s) \le \left(\frac{4}{3} + \frac{1}{6}t^{-4/3}\right)e^{(8/27)t}. \tag{3.14}$$

Thus,

$$\|u(x,t)\|_{H^4} \le \left(\frac{4}{3}+\frac{1}{6}t^{-4/3}\right)^{1/2} e^{(4/27)t}\|u_0\|_{L^2},$$
$$\left\|e^{tL_0}\right\|_{L^2\to H^4} \le \left(\frac{4}{3}+\frac{1}{6}t^{-4/3}\right)^{1/2} e^{(4/27)t} \le 5t^{-2/3}, \quad \text{for } 0<t\le 1, \tag{3.15}$$

since $e^{(4/27)t} \le e^{4/27} < 2$. Thus, Lemma 3.3 has been proved. □

Lemma 3.4. *Let* $L = (\partial_x^6+\partial_x^4) - [3\varphi^2\partial_x^4 + 24\varphi\varphi'\partial_x^3 + (36\varphi\varphi'' + 36\varphi'^2)\partial_x^2 + (24\varphi\varphi''' + 72\varphi'\varphi'')\partial_x + (18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi)] = L_0 - [3\varphi^2\partial_x^4 + 24\varphi\varphi'\partial_x^3 + (36\varphi\varphi'' + 36\varphi'^2)\partial_x^2 + (24\varphi\varphi''' + 72\varphi'\varphi'')\partial_x + (18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi)]$ *with* $L_0 = \partial_x^6 + \partial_x^4$, $\varphi^{(i)} \in L^\infty(R), i = 0,1,2,3,4$. *Then*

$$\left\|e^{tL}\right\|_{L^2\to H^4} \le C_1 t^{-2/3}, \quad \textit{for } 0<t\le 1, \tag{3.16}$$

$$\left\|e^{tL}\right\|_{H^4\to H^4} \le C_2 < \infty, \quad \textit{for } 0<t\le 1. \tag{3.17}$$

Proof. Consider the initial value problem

$$u_t = Lu = L_0 u - 3\varphi^2\partial_x^4 u - 24\varphi\varphi'\partial_x^3 u - \left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2 u$$
$$- (24\varphi\varphi''' + 72\varphi'\varphi'')\partial_x u - \left(18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi\right)u, \tag{3.18}$$
$$u(x,0) = u_0(x).$$

Then $u(x,t) = e^{tL}u_0(x)$, $t \ge 0$, $x \in R$, thus

$$u(x,t) = e^{tL_0}u_0 - \int_0^t e^{(t-\tau)L_0}\Big[3\varphi^2\partial_x^4 u + 24\varphi\varphi'\partial_x^3 u + \left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2 u$$
$$+ (24\varphi\varphi''' + 72\varphi'\varphi'')\partial_x u \tag{3.19}$$
$$+\left(18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi\right)u\Big]d\tau.$$

Denote $A = \|\varphi\|_{L^\infty}$, $B = \|\varphi'\|_{L^\infty}$, $C = \|\varphi''\|_{L^\infty}$, $D = \|\varphi'''\|_{L^\infty}$, $E = \|\varphi^{(4)}\|_{L^\infty}$ and

$$M = 3A^2 + 24AB + 36AC + 36B^2 + 24AD + 72BC + 18C^2 + 24BD + 6AE + 2aA. \tag{3.20}$$

Then, we have

$$
\begin{aligned}
\|u(t)\|_{H^4} \le{}& \left\|e^{tL_0}\right\|_{H^4\to H^4}\|u_0\|_{H^4} + \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{H^4\to H^4} 3\|\varphi\|_{L^\infty}^2 \left\|\partial_x^4 u\right\|_{L^2} d\tau \\
&+ 24\int_0^t \left\|e^{(t-\tau)L_0}\right\|_{H^4\to H^4} \|\varphi\|_{L^\infty}\|\varphi'\|_{L^\infty}\left\|\partial_x^3 u\right\|_{L^2} d\tau \\
&+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{H^4\to H^4}\left(36\|\varphi\|_{L^\infty}\|\varphi''\|_{L^\infty} + 36\left\|\varphi'^2\right\|_{L^\infty}\right)\left\|\partial_x^2 u\right\|_{L^2} d\tau \\
&+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{H^4\to H^4}(24\|\varphi\|_{L^\infty}\|\varphi'''\|_{L^\infty} + 72\|\varphi'\|_{L^\infty}\|\varphi''\|_{L^\infty})\|\partial_x u\|_{L^2} d\tau \\
&+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{H^4\to H^4}\Big(18\|\varphi''\|_{L^\infty}^2 + 24\|\varphi'\|_{L^\infty}\|\varphi'''\|_{L^\infty} \\
&\qquad + 6\|\varphi\|_{L^\infty}\left\|\varphi^{(4)}\right\|_{L^\infty} + 2a\|\varphi\|_{L^\infty}\Big)\|u\|_{L^2} d\tau \\
\le{}& e^{(4/27)t}\|u_0\|_{H^4} + M\int_0^t e^{(4/27)(t-\tau)}\|u(\tau)\|_{H^4} d\tau,
\end{aligned} \tag{3.21}
$$

where we use $u(t)$ to denote $u(\cdot,t)$.

By iteration,

$$
\begin{aligned}
\|u(t)\|_{H^4} &\le e^{(4/27)t}\|u_0\|_{H^4} + M\int_0^t e^{(4/27)(t-\tau)}\left[e^{(4/27)\tau}\|u_0\|_{H^4} + M\int_0^\tau e^{(4/27)(\tau-s)}\|u(s)\|_{H^4} ds\right] d\tau \\
&= e^{(4/27)t}\|u_0\|_{H^4} + M\int_0^t e^{(4/27)t}\|u_0\|_{H^4} d\tau + M^2\int_0^t\int_0^\tau e^{(4/27)(t-s)}\|u(s)\|_{H^4} ds\, d\tau \\
&\le e^{(4/27)t}\|u_0\|_{H^4} + Mte^{(4/27)t}\|u_0\|_{H^4} + M^2\int_0^t\left[\int_s^t e^{(4/27)(t-s)}\|u(s)\|_{H^4} d\tau\right] ds \\
&\le e^{(4/27)t}\|u_0\|_{H^4} + Mte^{(4/27)t}\|u_0\|_{H^4} + M^2te^{(4/27)t}\int_0^t \|u(s)\|_{H^4} ds \\
&\le e^{4/27}\|u_0\|_{H^4} + Me^{4/27}\|u_0\|_{H^4} + e^{4/27}M^2\int_0^t \|u(s)\|_{H^4} ds, \quad \text{for } 0 < s \le \tau \le t \le 1.
\end{aligned} \tag{3.22}
$$

Let $v(t) = \int_0^t \|u(s)\|_{H^4} ds$. Then

$$
\frac{dv(t)}{dt} \le \left(e^{4/27} + e^{4/27}M\right)\|u_0\|_{H^4} + e^{4/27}M^2 v(t), \quad \text{for } 0 < t \le 1. \tag{3.23}
$$

Multiplying both sides of the above inequality by $e^{-e^{(4/27)}M^2t}$, we have

$$\frac{d\left(e^{-e^{4/27}M^2t}v(t)\right)}{dt} \le e^{-e^{4/27}M^2t}e^{4/27}(1+M)\|u_0\|_{H^4}, \quad \text{where } 0<t\le 1. \tag{3.24}$$

Integrating the above inequality with respect to t over $(0,t)$, we obtain

$$e^{-e^{4/27}M^2t}v(t) \le \int_0^t e^{-e^{4/27}M^2s}e^{4/27}(1+M)\|u_0\|_{H^4}ds, \tag{3.25}$$

that is,

$$v(t) \le e^{e^{4/27}M^2t}\int_0^t e^{-e^{4/27}M^2s}e^{4/27}(1+M)\|u_0\|_{H^4}ds. \tag{3.26}$$

Observing that $v(t) = \int_0^t \|u(s)\|_{H^4}ds$ is bounded and substituting the above inequality into (3.22), we get

$$\begin{aligned}\|u(t)\|_{H^4} &\le e^{4/27}\|u_0\|_{H^4} + Me^{4/27}\|u_0\|_{H^4} + e^{4/27}M^2\int_0^t \|u(s)\|_{H^4}ds \\ &\le c_2 < \infty, \quad \text{for } 0<t\le 1,\ c_2>0,\end{aligned} \tag{3.27}$$

thus (3.17) has been proven.

Next, we prove the inequality (3.16). Clearly, we have

$$\begin{aligned}\|u(t)\|_{H^4} \le{}& \left\|e^{tL_0}\right\|_{L^2\to H^4}\|u_0\|_{L^2} + \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{L^2\to H^4}3\|\varphi\|_{L^\infty}^2\left\|\partial_x^4 u\right\|_{L^2}d\tau \\ &+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{L^2\to H^4}24\|\varphi\|_{L^\infty}\|\varphi'\|_{L^\infty}\left\|\partial_x^3 u\right\|_{L^2}d\tau \\ &+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{L^2\to H^4}\left(36\|\varphi\|_{L^\infty}\|\varphi''\|_{L^\infty} + 36\left\|\varphi'^2\right\|_{L^\infty}\right)\left\|\partial_x^2 u\right\|_{L^2}d\tau \\ &+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{L^2\to H^4}(24\|\varphi\|_{L^\infty}\|\varphi'''\|_{L^\infty} \\ &\qquad +72\|\varphi'\|_{L^\infty}\|\varphi''\|_{L^\infty})\|\partial_x u\|_{L^2}d\tau \\ &+ \int_0^t \left\|e^{(t-\tau)L_0}\right\|_{L^2\to H^4}\Big(18\|\varphi''\|_{L^\infty}^2 + 24\|\varphi'\|_{L^\infty}\|\varphi'''\|_{L^\infty} \\ &\qquad +6\|\varphi\|_{L^\infty}\left\|\varphi^{(4)}\right\|_{L^\infty} + 2a\|\varphi\|_{L^\infty}\Big)\|u\|_{L^2}d\tau \\ \le{}& a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)\|u(\tau)\|_{H^4}d\tau,\end{aligned} \tag{3.28}$$

where $a(t)$ is defined in Lemma 3.3, and we use $u(t)$ to denote $u(\cdot,t)$.

By iteration,

$$\begin{aligned}
\|u(t)\|_{H^4} &\leq a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)\left[a(\tau)\|u_o\|_{L^2} + M\int_0^\tau a(\tau-s)\|u(s)\|_{H^4}ds\right]d\tau \\
&= a(t)\|u_0\|_{L^2} + M\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}d\tau \\
&\quad + M^2\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^4}dsd\tau.
\end{aligned} \tag{3.29}$$

The second term on the right of (3.29) is

$$\begin{aligned}
M\int_0^t a(t-\tau)a(\tau)\|u_0\|_{L^2}d\tau &= M\|u_0\|_{L^2}\int_0^t 5(t-\tau)^{-2/3}5\tau^{-2/3}d\tau \\
&= 25M\|u_0\|_{L^2}\int_0^t t^{-4/3}\left(1-\frac{\tau}{t}\right)^{-2/3}\left(\frac{\tau}{t}\right)^{-2/3}d\tau \\
&= 25MC_3t^{-1/3}\|u_0\|_{L^2}, \qquad 0 < t < 1,
\end{aligned} \tag{3.30}$$

where $C_3 = \int_0^1 (1-r)^{-1/4}r^{-1/4}dr$. By exchanging the order of integration, we get from the third term on the right side of (3.29),

$$\int_0^t\int_0^\tau a(t-\tau)a(\tau-s)\|u(s)\|_{H^4}dsd\tau = \int_0^t\left[\int_s^t a(t-\tau)a(\tau-s)d\tau\right]\|u(s)\|_{H^4}ds, \tag{3.31}$$

then

$$\begin{aligned}
\int_s^t a(t-\tau)a(\tau-s)d\tau &= 25\int_s^t (t-\tau)^{-2/3}(\tau-s)^{-2/3}d\tau \\
&= 25C_3(t-s)^{-1/3}, \quad 0 < s \leq t \leq 1.
\end{aligned} \tag{3.32}$$

Therefore (3.28)–(3.32) imply

$$\begin{aligned}
\|u(t)\|_{H^4} \leq{} & \left[a(t) + 25C_3Mt^{-1/3}\right]\|u_0\|_{L^2} \\
& + 25C_3M^2\int_0^t (t-s)^{-1/3}\|u(s)\|_{H^4}ds, \quad 0 < t \leq 1.
\end{aligned} \tag{3.33}$$

From (3.17), we know $\|u(t)\|_{H^4} \leq C_2\|u_0\|_{H^4}, 0 < t \leq 1$. Then

$$\begin{aligned}
\|u(t)\|_{H^4} \leq{} & \left[a(t) + 25C_3Mt^{-1/3}\right]\|u_0\|_{L^2} \\
& + \frac{75}{2}C_3C_2M^2\|u_0\|_{H^4}t^{2/3}, \quad 0 < t \leq 1.
\end{aligned} \tag{3.34}$$

Therefore, there exists a t^*, such that

$$\|u(t)\|_{H^4} \leq C_1 t^{-2/3} \|u_0\|_{L^2}, \quad 0 < t \leq t^* \leq 1, \ C_1 > 0. \tag{3.35}$$

So, we proved the inequality (3.16).

Hence (3.17) is proven and proof of Lemma 3.4 is finished. □

By Lemma 3.4, the condition (ii) is proved.

We now proceed to verify condition (iii) of Lemma 3.1. Observing that if $u(x,t)$ satisfies

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^6 u}{\partial x^6} + \frac{\partial^4 u}{\partial x^4} - 3\varphi^2 \frac{\partial^4 u}{\partial x^4} - 24\varphi\varphi' \frac{\partial^3 u}{\partial x^3} - \left(36\varphi\varphi'' + 36\varphi'^2\right) \frac{\partial^2 u}{\partial x^2} \\ &\quad - \left(24\varphi\varphi''' + 72\varphi'\varphi''\right) \frac{\partial u}{\partial x} - \left(18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi\right) u,\end{aligned} \tag{3.36}$$

then $u(x, s+t)$ also satisfies the above equation. By uniqueness of solution, we know that L generates a strongly continuous semigroup on the Banach space $H^4(R)$ (see [15] p.344). By Fourier transformation, the essential spectrum of L_0 on $H^4(R)$ is

$$\sigma(L_0) \supset \left\{-\xi^6 + \xi^4 \mid \xi \in R\right\}. \tag{3.37}$$

The curve $\lambda = -\xi^6 + \xi^4$ meets the vertical lines $Re\lambda = \alpha$ for $-\infty < \alpha \leq 4/27$ because $-\infty < -\xi^6 + \xi^4 \leq 4/27$.

We now prove that the same curve belongs to the essential spectrum of L.

Lemma 3.5. *The essential spectrum of L on $H^4(R)$ contains that of L_0.*

Proof. Let $\xi \in R$ and let $\lambda = P(\xi) = -\xi^6 + \xi^4$. Following Schechter [16], $\lambda \in \sigma(L)$ if there exists a sequence $\{\xi_n\} \subset H^4(R)$ with

$$\|\xi_n\|_{H^4} = 1, \qquad \|(L-\lambda)\xi_n\|_{H^4} \longrightarrow 0, \tag{3.38}$$

and $\{\xi_n\}$ does not have a strongly convergent subsequence in $H^4(R)$. Here we use the definition $\lambda \notin \sigma(L)$ if and only if $L - \lambda$ is Fredholm with index zero. Now let $\xi_0 \not\equiv 0$ be a C^∞ function with compact support in $(0, \infty)$. Define

$$\xi_n(x) = \frac{c_n e^{i\xi x} \xi_0(x/n)}{\sqrt{n}}, \quad n = 1, 2, \ldots, \tag{3.39}$$

where c_n is chosen so that $\|\xi_n\|_{H^4} = 1$. In fact,

$$\|\xi_n\|_{L^2} = c_n \|\xi_0\|_{L^2}, \qquad 1 = \|\xi_n\|_{H^4} \leq k c_n, \tag{3.40}$$

for some positive constant k. Hence $c_n \geq 1/k > 0$. Since $\|\xi_n\|_{L^\infty} \to 0$ but $\|\xi_n\|_{L^2}$ is bounded away from zero, $\{\xi_n\}$ can have no convergent subsequence in $L^2(R)$.

It remains to show that $\|(L-\lambda)\xi_n\|_{H^4} \to 0$. We write

$$L-\lambda = L_0-\lambda-\Big[3\varphi^2\partial_x^4+24\varphi\varphi'\partial_x^3+\Big(36\varphi\varphi''+36\varphi'^2\Big)\partial_x^2 \\ +(24\varphi\varphi'''+72\varphi'\varphi'')\partial_x+\Big(18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\Big)\Big]. \tag{3.41}$$

A simple calculation shows that

$$\begin{aligned}
(L_0-\lambda)\xi_n(x) &= c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s)}(x/n)}{s!n^{(1/2)+s}},\\
\partial(L_0-\lambda)\xi_n(x) &= i\xi(L_0-\lambda)\xi_n(x)+c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+1)}(x/n)}{s!n^{(3/2)+s}},\\
\partial^2(L_0-\lambda)\xi_n(x) &= -\xi^2(L_0-\lambda)\xi_n(x)+2i\xi c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+1)}(x/n)}{s!n^{(3/2)+s}}\\
&\quad + c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+2)}(x/n)}{s!n^{(5/2)+s}},\\
\partial^3(L_0-\lambda)\xi_n(x) &= -i\xi^3(L_0-\lambda)\xi_n(x)-3\xi^2 c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+1)}(x/n)}{s!n^{(3/2)+s}}\\
&\quad +3i\xi c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+2)}(x/n)}{s!n^{(5/2)+s}}\\
&\quad + c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+3)}(x/n)}{s!n^{(7/2)+s}},\\
\partial^4(L_0-\lambda)\xi_n(x) &= \xi^4(L_0-\lambda)\xi_n(x)-4i\xi^3 c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+1)}(x/n)}{s!n^{(3/2)+s}}\\
&\quad -6\xi^2 c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+2)}(x/n)}{s!n^{(5/2)+s}}\\
&\quad +4i\xi c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+3)}(x/n)}{s!n^{(7/2)+s}}\\
&\quad + c_n e^{i\xi x}\sum_{1\le s\le 6}\frac{(-i)^s P^{(s)}(\xi)\xi_0^{(s+4)}(x/n)}{s!n^{(9/2)+s}}.
\end{aligned} \tag{3.42}$$

Thus,

$$\begin{aligned}
&\|(L_0-\lambda)\xi_n(x)\|_{H^4}\\
&\quad\le\left(1+|\xi|+|\xi|^2+|\xi|^3+|\xi|^4\right)\sum_{1\le s\le 6}\frac{\left|P^{(s)}(\xi)\right|c_n\left\|\xi_0^{(s)}(x/n)\right\|_{L^2}}{s!n^{(1/2)+s}}\\
&\qquad+\left(1+2|\xi|+3|\xi|^2+4|\xi|^3\right)\sum_{1\le s\le 6}\frac{\left|P^{(s)}(\xi)\right|c_n\left\|\xi_0^{(s+1)}(x/n)\right\|_{L^2}}{s!n^{(3/2)+s}}\\
&\qquad+\left(1+3|\xi|+6|\xi|^2\right)\sum_{1\le s\le 6}\frac{\left|P^{(s)}(\xi)\right|c_n\left\|\xi_0^{(s+2)}(x/n)\right\|_{L^2}}{s!n^{(5/2)+s}}\\
&\qquad+(1+4|\xi|)\sum_{1\le s\le 6}\frac{\left|P^{(s)}(\xi)\right|c_n\left\|\xi_0^{(s+3)}(x/n)\right\|_{L^2}}{s!n^{(7/2)+s}}\\
&\qquad+\sum_{1\le s\le 6}\frac{\left|P^{(s)}(\xi)\right|c_n\left\|\xi_0^{(s+4)}(x/n)\right\|_{L^2}}{s!n^{(9/2)+s}}\longrightarrow 0,\quad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{3.43}$$

Moreover, for any positive integer m, $\|\partial_x^m\xi_n\|_{L^\infty}\to 0$, as $n\to\infty$, we have

$$\begin{aligned}
&\left\|3\varphi^2\partial_x^4\xi_n\right\|_{L^2}^2\le\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|3\varphi^2\right\|_{L^2}^2\longrightarrow 0,\\
&\left\|\partial_x\left[3\varphi^2\partial_x^4\xi_n\right]\right\|_{L^2}^2\le\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\left\|3\varphi^2\right\|_{L^2}^2+\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\|6\varphi\varphi'\|_{L^2}^2\longrightarrow 0,\\
&\left\|\partial_x^2\left[3\varphi^2\partial_x^4\xi_n\right]\right\|_{L^2}^2\le\left\|\partial_x^6\xi_n\right\|_{L^\infty}^2\left\|3\varphi^2\right\|_{L^2}^2+2\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\|6\varphi\varphi'\|_{L^2}^2+\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|6\varphi'^2+6\varphi\varphi''\right\|_{L^2}^2\longrightarrow 0,\\
&\left\|\partial_x^3\left[3\varphi^2\partial_x^4\xi_n\right]\right\|_{L^2}^2\le\left\|\partial_x^7\xi_n\right\|_{L^\infty}^2\left\|3\varphi^2\right\|_{L^2}^2+3\left\|\partial_x^6\xi_n\right\|_{L^\infty}^2\|6\varphi\varphi'\|_{L^2}^2\\
&\qquad+3\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\left\|6\varphi'^2+6\varphi\varphi''\right\|_{L^2}^2+\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\|18\varphi'\varphi''+6\varphi\varphi'''\|_{L^2}^2\longrightarrow 0,\\
&\left\|\partial_x^4\left[3\varphi^2\partial_x^4\xi_n\right]\right\|_{L^2}^2\le\left\|\partial_x^8\xi_n\right\|_{L^\infty}^2\left\|3\varphi^2\right\|_{L^2}^2+4\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\|18\varphi'\varphi''+6\varphi\varphi'''\|_{L^2}^2\\
&\qquad+4\left\|\partial_x^7\xi_n\right\|_{L^\infty}^2\|6\varphi\varphi'\|_{L^2}^2+6\left\|\partial_x^6\xi_n\right\|_{L^\infty}^2\left\|6\varphi'^2+6\varphi\varphi''\right\|_{L^2}^2\\
&\qquad+\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}\right\|_{L^2}^2\longrightarrow 0.
\end{aligned}\tag{3.44}$$

From the assumptions on φ, we obtain

$$\begin{aligned}
&\left\|24\varphi\varphi'\partial_x^3\xi_n\right\|_{L^2}^2\le\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\|24\varphi\varphi'\|_{L^2}^2\longrightarrow 0,\\
&\left\|\partial_x\left[24\varphi\varphi'\partial_x^3\xi_n\right]\right\|_{L^2}^2\le\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\|24\varphi\varphi'\|_{L^2}^2+\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|24\varphi'^2+24\varphi\varphi''\right\|_{L^2}^2\longrightarrow 0,
\end{aligned}$$

$$\begin{aligned}
\left\|\partial_x^2\left[24\varphi\varphi'\partial_x^3\xi_n\right]\right\|_{L^2}^2 \le{}& \left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\|24\varphi\varphi'\|_{L^2}^2 + 2\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|24\varphi'^2 + 24\varphi\varphi''\right\|_{L^2}^2 \\
&+ \left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\|72\varphi'\varphi'' + 24\varphi\varphi'''\|_{L^2}^2 \longrightarrow 0, \\
\left\|\partial_x^3\left[24\varphi\varphi'\partial_x^3\xi_n\right]\right\|_{L^2}^2 \le{}& \left\|\partial_x^6\xi_n\right\|_{L^\infty}^2\|24\varphi\varphi'\|_{L^2}^2 + 3\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\left\|24\varphi'^2 + 24\varphi\varphi''\right\|_{L^2}^2 \\
&+ 3\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\|72\varphi'\varphi'' + 24\varphi\varphi'''\|_{L^2}^2 \\
&+ \left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|72\varphi''^2 + 96\varphi'\varphi''' + 24\varphi\varphi^{(4)}\right\|_{L^2}^2 \longrightarrow 0, \\
\left\|\partial_x^4\left[24\varphi\varphi'\partial_x^3\xi_n\right]\right\|_{L^2}^2 \le{}& \left\|\partial_x^7\xi_n\right\|_{L^\infty}^2\|24\varphi\varphi'\|_{L^2}^2 + 4\left\|\partial_x^6\xi_n\right\|_{L^\infty}^2\left\|24\varphi'^2 + 24\varphi\varphi''\right\|_{L^2}^2 \\
&+ 6\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\|72\varphi'\varphi'' + 24\varphi\varphi'''\|_{L^2}^2 \\
&+ 4\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|72\varphi''^2 + 96\varphi'\varphi''' + 24\varphi\varphi^{(4)}\right\|_{L^2}^2 \\
&+ \left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|240\varphi''\varphi''' + 120\varphi'\varphi^{(4)} + 24\varphi\varphi^{(5)}\right\|_{L^2}^2 \longrightarrow 0.
\end{aligned} \tag{3.45}$$

Similarly, we have

$$\begin{aligned}
\left\|\left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2\xi_n\right\|_{L^2}^2 \le{}& \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\left\|36\varphi\varphi'' + 36\varphi'^2\right\|_{L^2}^2 \longrightarrow 0, \\
\left\|\partial_x\left[\left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2\xi_n\right]\right\|_{L^2}^2 \le{}& \left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|36\varphi\varphi'' + 36\varphi'^2\right\|_{L^2}^2 \\
&+ \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\|108\varphi'\varphi'' + 36\varphi\varphi'''\|_{L^2}^2 \longrightarrow 0, \\
\left\|\partial_x^2\left[\left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2\xi_n\right]\right\|_{L^2}^2 \le{}& \left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|36\varphi\varphi'' + 36\varphi'^2\right\|_{L^2}^2 \\
&+ 2\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\|108\varphi'\varphi'' + 36\varphi\varphi'''\|_{L^2}^2 \\
&+ \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\left\|108\varphi''^2 + 144\varphi'\varphi''' + 36\varphi\varphi^{(4)}\right\|_{L^2}^2 \longrightarrow 0, \\
\left\|\partial_x^3\left[\left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2\xi_n\right]\right\|_{L^2}^2& \\
\le{}& \left\|\partial_x^5\xi_n\right\|_{L^\infty}^2\left\|36\varphi\varphi'' + 36\varphi'^2\right\|_{L^2}^2 + 3\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\|108\varphi'\varphi'' + 36\varphi\varphi'''\|_{L^2}^2 \\
&+ 3\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|108\varphi''^2 + 144\varphi'\varphi''' + 36\varphi\varphi^{(4)}\right\|_{L^2}^2 \\
&+ \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\left\|360\varphi''\varphi''' + 180\varphi'\varphi^{(4)} + 36\varphi\varphi^{(5)}\right\|_{L^2}^2 \longrightarrow 0,
\end{aligned}$$

$$
\begin{aligned}
&\left\|\partial_x^4\left[\left(36\varphi\varphi'' + 36\varphi'^2\right)\partial_x^2\xi_n\right]\right\|_{L^2}^2 \\
&\quad \le \left\|\partial_x^6\xi_n\right\|_{L^\infty}^2 \left\|36\varphi\varphi'' + 36\varphi'^2\right\|_{L^2}^2 \\
&\qquad + 4\left\|\partial_x^5\xi_n\right\|_{L^\infty}^2 \left\|108\varphi'\varphi'' + 36\varphi\varphi'''\right\|_{L^2}^2 \\
&\qquad + 6\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2 \left\|108\varphi''^2 + 144\varphi'\varphi''' + 36\varphi\varphi^{(4)}\right\|_{L^2}^2 \\
&\qquad + 4\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2 \left\|360\varphi''\varphi''' + 180\varphi'\varphi^{(4)} + 36\varphi\varphi^{(5)}\right\|_{L^2}^2 \\
&\qquad + \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2 \left\|360\varphi'''^2 + 540\varphi''\varphi^{(4)} + 216\varphi'\varphi^{(5)} + 36\varphi\varphi^{(6)}\right\|_{L^2}^2 \longrightarrow 0, \\
&\left\|\left(24\varphi\varphi''' + 72\varphi'\varphi''\right)\partial_x\xi_n\right\|_{L^2}^2 \le \left\|\partial_x\xi_n\right\|_{L^\infty}^2 \left\|24\varphi\varphi''' + 72\varphi'\varphi''\right\|_{L^2}^2 \longrightarrow 0, \\
&\left\|\partial_x\left[\left(24\varphi\varphi''' + 72\varphi'\varphi''\right)\partial_x\xi_n\right]\right\|_{L^2}^2 \le \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2 \left\|24\varphi\varphi''' + 72\varphi'\varphi''\right\|_{L^2}^2 \\
&\qquad + \left\|\partial_x\xi_n\right\|_{L^\infty}^2 \left\|96\varphi'\varphi''' + 72\varphi''^2 + 24\varphi\varphi^{(4)}\right\|_{L^2}^2 \longrightarrow 0, \\
&\left\|\partial_x^2\left[\left(24\varphi\varphi''' + 72\varphi'\varphi''\right)\partial_x\xi_n\right]\right\|_{L^2}^2 \le \left\|\partial_x^3\xi_n\right\|_{L^\infty}^2 \left\|24\varphi\varphi''' + 72\varphi'\varphi''\right\|_{L^2}^2 \\
&\qquad + 2\left\|\partial_x^2\xi_n\right\|_{L^\infty}^2 \left\|96\varphi'\varphi''' + 72\varphi''^2 + 24\varphi\varphi^{(4)}\right\|_{L^2}^2 \\
&\qquad + \left\|\partial_x\xi_n\right\|_{L^\infty}^2 \left\|240\varphi''\varphi''' + 120\varphi'\varphi^{(4)} + 24\varphi\varphi^{(5)}\right\|_{L^2}^2 \longrightarrow 0, \\
&\left\|\partial_x^3\left[\left(24\varphi\varphi''' + 72\varphi'\varphi''\right)\partial_x\xi_n\right]\right\|_{L^2}^2 \\
&\quad \le \left\|\partial_x^4\xi_n\right\|_{L^\infty}^2 \left\|24\varphi\varphi''' + 72\varphi'\varphi''\right\|_{L^2}^2 \\
&\qquad + 3\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2 \left\|96\varphi'\varphi''' + 72\varphi''^2 + 24\varphi\varphi^{(4)}\right\|_{L^2}^2 \\
&\qquad + 3\left\|\partial_x^2\xi_n\right\|_{L^\infty}^2 \left\|240\varphi''\varphi''' + 120\varphi'\varphi^{(4)} + 24\varphi\varphi^{(5)}\right\|_{L^2}^2 \\
&\qquad + \left\|\partial_x\xi_n\right\|_{L^\infty}^2 \left\|240\varphi'''^2 + 360\varphi''\varphi^{(4)} + 144\varphi'\varphi^{(5)} + 24\varphi\varphi^{(6)}\right\|_{L^2}^2 \longrightarrow 0, \\
&\left\|\partial_x^4\left[\left(24\varphi\varphi''' + 72\varphi'\varphi''\right)\partial_x\xi_n\right]\right\|_{L^2}^2 \\
&\quad \le \left\|\partial_x^5\xi_n\right\|_{L^\infty}^2 \left\|24\varphi\varphi''' + 72\varphi'\varphi''\right\|_{L^2}^2 + 4\left\|\partial_x^4\xi_n\right\|_{L^\infty}^2 \left\|96\varphi'\varphi''' + 72\varphi''^2 + 24\varphi\varphi^{(4)}\right\|_{L^2}^2 \\
&\qquad + 6\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2 \left\|240\varphi''\varphi''' + 120\varphi'\varphi^{(4)} + 24\varphi\varphi^{(5)}\right\|_{L^2}^2 \\
&\qquad + 4\left\|\partial_x^2\xi_n\right\|_{L^\infty}^2 \left\|240\varphi'''^2 + 360\varphi''\varphi^{(4)} + 144\varphi'\varphi^{(5)} + 24\varphi\varphi^{(6)}\right\|_{L^2}^2 \\
&\qquad + \left\|\partial_x\xi_n\right\|_{L^\infty}^2 \left\|840\varphi'''\varphi^{(4)} + 504\varphi''\varphi^{(5)} + 168\varphi'\varphi^{(6)} + 24\varphi\varphi^{(7)}\right\|_{L^2}^2 \longrightarrow 0.
\end{aligned}
\tag{3.46}
$$

In addition,

$$\begin{aligned}
&\left\|\left(18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right)\xi_n\right\|_{L^2}^2 \le \|\xi_n\|_{L^\infty}^2\left\|18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right\|_{L^2}^2 \longrightarrow 0,\\
&\left\|\partial_x\left[\left(18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right)\xi_n\right]\right\|_{L^2}^2\\
&\quad\le \|\partial_x\xi_n\|_{L^\infty}^2\left\|18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right\|_{L^2}^2\\
&\quad\quad+\|\xi_n\|_{L^\infty}^2\left\|60\varphi''\varphi'''+30\varphi'\varphi^{(4)}+6\varphi\varphi^{(5)}+2a\varphi'\right\|_{L^2}^2 \longrightarrow 0,\\
&\left\|\partial_x^2\left[\left(18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right)\xi_n\right]\right\|_{L^2}^2\\
&\quad\le \left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\left\|18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right\|_{L^2}^2\\
&\quad\quad+2\|\partial_x\xi_n\|_{L^\infty}^2\left\|60\varphi''\varphi'''+30\varphi'\varphi^{(4)}+6\varphi\varphi^{(5)}+2a\varphi'\right\|_{L^2}^2\\
&\quad\quad+\|\xi_n\|_{L^\infty}^2\left\|60\varphi'''^2+90\varphi''\varphi^{(4)}+36\varphi'\varphi^{(5)}+6\varphi\varphi^{(6)}+2a\varphi''\right\|_{L^2}^2 \longrightarrow 0,\\
&\left\|\partial_x^3\left[\left(18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right)\xi_n\right]\right\|_{L^2}^2\\
&\quad\le \left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right\|_{L^2}^2\\
&\quad\quad+3\left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\left\|60\varphi''\varphi'''+30\varphi'\varphi^{(4)}+6\varphi\varphi^{(5)}+2a\varphi'\right\|_{L^2}^2\\
&\quad\quad+3\|\partial_x\xi_n\|_{L^\infty}^2\left\|60\varphi'''^2+90\varphi''\varphi^{(4)}+36\varphi'\varphi^{(5)}+6\varphi\varphi^{(6)}+2a\varphi''\right\|_{L^2}^2\\
&\quad\quad+\|\xi_n\|_{L^\infty}^2\left\|210\varphi'''\varphi^{(4)}+126\varphi''\varphi^{(5)}+42\varphi'\varphi^{(6)}+6\varphi\varphi^{(7)}+2a\varphi'''\right\|_{L^2}^2 \longrightarrow 0,\\
&\left\|\partial_x^4\left[\left(18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right)\xi_n\right]\right\|_{L^2}^2\\
&\quad\le \left\|\partial_x^4\xi_n\right\|_{L^\infty}^2\left\|18\varphi''^2+24\varphi'\varphi'''+6\varphi\varphi^{(4)}+2a\varphi\right\|_{L^2}^2\\
&\quad\quad+4\left\|\partial_x^3\xi_n\right\|_{L^\infty}^2\left\|60\varphi''\varphi'''+30\varphi'\varphi^{(4)}+6\varphi\varphi^{(5)}+2a\varphi'\right\|_{L^2}^2\\
&\quad\quad+6\left\|\partial_x^2\xi_n\right\|_{L^\infty}^2\left\|60\varphi'''^2+90\varphi''\varphi^{(4)}+36\varphi'\varphi^{(5)}+6\varphi\varphi^{(6)}+2a\varphi''\right\|_{L^2}^2\\
&\quad\quad+4\|\partial_x\xi_n\|_{L^\infty}^2\left\|210\varphi'''\varphi^{(4)}+126\varphi''\varphi^{(5)}+42\varphi'\varphi^{(6)}+6\varphi\varphi^{(7)}+2a\varphi'''\right\|_{L^2}^2\\
&\quad\quad+\|\xi_n\|_{L^\infty}^2\left\|210\left(\varphi^{(4)}\right)^2+12\varphi'''\varphi^{(5)}+168\varphi''\varphi^{(6)}+48\varphi'\varphi^{(7)}+6\varphi\varphi^{(8)}+2a\varphi^{(4)}\right\|_{L^2}^2 \longrightarrow 0.
\end{aligned} \tag{3.47}$$

Thus,

$$\begin{aligned}&\Big\|3\varphi^2\partial_x^4\xi_n + 24\varphi\varphi'\partial_x^3\xi_n + \Big(36\varphi\varphi'' + 36\varphi'^2\Big)\partial_x^2\xi_n + \big(24\varphi\varphi''' + 72\varphi'\varphi''\big)\partial_x\xi_n \\ &\quad + \Big(18\varphi''^2 + 24\varphi'\varphi''' + 6\varphi\varphi^{(4)} + 2a\varphi\Big)\xi_n\Big\|_{H^4} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.48}$$

So from the estimates above,

$$\|(L-\lambda)\xi_n\|_{H^4} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.49}$$

The proof of Lemma 3.5 is completed. □

Therefore all the four conditions of Lemma 3.1 are satisfied by the linearized equation (3.4) and Theorem 1.1 has been proved.

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