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Research Article

A Classification of a Totally Umbilical Slant Submanifold of Cosymplectic Manifolds

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We study slant submanifolds of a cosymplectic manifold. It is shown that a totally umbilical slant submanifold M of a cosymplectic manifold \overline{M} is either an anti-invariant submanifold or a 1–dimensional submanifold. We show that every totally umbilical proper slant submanifold of a cosymplectic manifold is totally geodesic.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by Chen as a natural generalization of both holomorphic and totally real submanifolds [1, 2]. Since then, many research papers have appeared concerning the existence of these submanifolds as well as on the geometry of the existent slant submanifolds in different known spaces (cf. [3, 4]). The slant submanifolds of an almost contact metric manifold were defined and studied by Lotta [4]. Later on, these submanifolds were studied by Cabrerizo et al. in the setting of Sasakian manifolds [3].

Recently, Şahin proved that a totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic [5]. Our aim in the present paper is to investigate slant submanifolds in contact manifolds. Thus, we study slant submanifolds of a cosymplectic manifold. We have shown that a totally umbilical slant submanifold M of a cosymplectic manifold \overline{M} is either an anti-invariant submanifold or the dim M=1 or the mean curvature vector $H \in \Gamma(\mu)$, and then we have obtained an interesting result for a totally umbilical proper slant submanifold of a cosymplectic manifold.

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2. Preliminaries

Let \overline{M} be a (2n + 1)-dimensional manifold with (1, 1) tensor field ϕ satisfying [6]:

$$\phi^2 = -I + \eta \otimes \xi,\tag{2.1}$$

where I is the identity transformation, ξ a vector field, and η a 1-form on \overline{M} satisfying $\phi \xi = \eta \circ \phi = 0$ and $\eta(\xi) = 1$. Then \overline{M} is said to have an almost contact structure. There always exists a Riemannian metric g on \overline{M} such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y, on \overline{M} . From (2.2), it is easy to observe that

$$g(\phi X, Y) + g(X, \phi Y) = 0. \tag{2.3}$$

The fundamental 2-form Φ is defined as: $\Phi(X,Y)=g(X,\phi Y)$. If $[\phi,\phi]+d\eta\otimes\xi=0$, then the almost contact structure is said to be normal, where $[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$. If $\Phi=d\eta$, the almost contact structure is a contact structure. A normal almost contact structure such that Φ is closed and $d\eta=0$ is called *cosymplectic structure*. It is well known [7] that the cosymplectic structure is characterized by

$$(\overline{\nabla}_X \phi) Y = 0, \qquad (\overline{\nabla}_X \eta) Y = 0,$$
 (2.4)

for all vector fields X, Y, on \overline{M} , where $\overline{\nabla}$ is the Levi-Civita connection of g. From the formula $\overline{\nabla}_X \phi = 0$, it follows that $\overline{\nabla}_X \xi = 0$.

Let M be submanifold of an almost contact metric manifold \overline{M} with induced metric g and let ∇ and ∇^{\perp} be the induced connections on the tangent bundle TM and the normal bundle TM of M, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$ -module of smooth sections of a vector bundle TM over M, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.6}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively for the immersion of M into \overline{M} . They are related as

$$g(h(X,Y),N) = g(A_N X,Y), \tag{2.7}$$

where g denotes the Riemannian metric on \overline{M} as well as the one induced on M [8].

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX,\tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX . Similarly for any $N \in \Gamma(T^{\perp}M)$, we write

$$\phi N = tN + fN,\tag{2.9}$$

where tN is the tangential component and tN is the normal component of ϕN . If we denote the orthogonal complementary distribution of F(TM) in $T^{\perp}M$ by μ , then we have the direct sum

$$T^{\perp}M = F(TM) \oplus \mu. \tag{2.10}$$

We can see that μ is an invariant subbundle with respect to ϕ . Furthermore, the covariant derivatives of the tensor fields P and F are defined as

$$(\overline{\nabla}_X P) Y = \nabla_X P Y - P \nabla_X Y,$$

$$(\overline{\nabla}_X F) Y = \nabla_X^{\perp} F Y - F \nabla_X Y,$$

$$(2.11)$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold \overline{M} is called *totally umbilical* if

$$h(X,Y) = g(X,Y)H, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is given by

$$H = \sum_{i=1}^{m} h(e_i, e_i), \tag{2.13}$$

where m is the dimension of M and $\{e_1, e_2, \dots, e_m\}$ is the local orthonormal frame on M. A submanifold M is said to be *totally geodesic* if h(X, Y) = 0 for each $X, Y \in \Gamma(TM)$ and is *minimal* if H = 0 on M.

3. Slant Submanifolds

Throughout the section, we assume that M is a slant submanifold of a cosymplectic manifold \overline{M} . We always consider such submanifold tangent to the structure vector field ξ . For each

nonzero vector X tangent to M at x, we denote by $0 \le \theta(X) \le \pi/2$, the angle between ϕX and $T_x M$, known as the Wirtinger angle of X. If the Wirtinger angle $\theta(X)$ is constant, that is, independent of the choice of $x \in M$ and $X \in T_x M - \{\xi\}$, then M is said to be a slant submanifold [4]. In this case the constant angle θ is called slant angle of the slant submanifold. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant submanifold. If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM is decomposed as

$$TM = \mathfrak{D} \oplus \langle \xi \rangle, \tag{3.1}$$

where $\langle \xi \rangle$ denotes the distribution spanned by the structure vector field ξ and \mathfrak{D} is the complementary distribution of $\langle \xi \rangle$ in TM, known as the *slant distribution*.

We recall the following result for a slant submanifold.

Theorem 3.1 (see [3]). Let M be a submanifold of an almost contact metric manifold \overline{M} , such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that

$$P^2 = \lambda \left(-I + \eta \otimes \xi \right). \tag{3.2}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2 \theta$.

The following relations are straightforward consequence of (3.2):

$$g(PX, PY) = \cos^2\theta \left[g(X, Y) - \eta(X)\eta(Y) \right], \tag{3.3}$$

$$g(FX, FY) = \sin^2\theta \left[g(X, Y) - \eta(X)\eta(Y) \right], \tag{3.4}$$

for any X, Y tangent to M.

Now, we prove the following.

Theorem 3.2. Let M be a totally umbilical slant submanifold of a cosymplectic manifold \overline{M} . Then at least one of the following statements is ture:

- (i) M is an anti-invariant submanifold;
- (ii) M is a 1-dimensional submanifold;
- (iii) If M is a proper slant submanifold, then $H \in \Gamma(\mu)$,

where H is the mean curvature vector of the submanifold M.

Proof. Let M be a totally umbilical slant submanifold of a cosymplectic manifold \overline{M} , then for any $X, Y \in \Gamma(TM)$, we have

$$h(PX, PY) = g(PX, PY)H. \tag{3.5}$$

From (2.5) and (3.3), we deduce that

$$\overline{\nabla}_{PX}PX - \nabla_{PX}PX = \cos^2\theta \{g(X, X) - \eta(X)\eta(X)\}H. \tag{3.6}$$

Using (2.8) and the fact that \overline{M} is cosymplectic we obtain that

$$\phi \overline{\nabla}_{PX} X - \overline{\nabla}_{PX} FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \tag{3.7}$$

Then from (2.5) and (2.6), we get

$$\phi \nabla_{PX} X + \phi h(X, PX) + A_{FX} PX - \nabla_{PX}^{\perp} FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \tag{3.8}$$

Thus by (2.8) and (2.12), we obtain

$$P\nabla_{PX}X + F\nabla_{PX}X + g(PX, X)\phi H + A_{FX}PX - \nabla_{PX}^{\perp}FX - \nabla_{PX}PX = \cos^{2}\theta \left\{ ||X||^{2} - \eta^{2}(X) \right\} H.$$
(3.9)

Equating the normal components, we get

$$F\nabla_{PX}X - \nabla_{PX}^{\perp}FX = \cos^{2}\theta \left\{ \|X\|^{2} - \eta^{2}(X) \right\} H.$$
 (3.10)

On the other hand, from (3.4), we have

$$g(FX, FX) = \sin^2\theta \left\{ g(X, X) - \eta(X)\eta(X) \right\},\tag{3.11}$$

for any $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation with respect to PX, we obtain

$$2g(\overline{\nabla}_{PX}FX, FX) = 2\sin^2\theta g(\overline{\nabla}_{PX}X, X) - 2\sin^2\theta \eta(X)g(\overline{\nabla}_{PX}X, \xi) - 2\sin^2\theta \eta(X)g(X, \overline{\nabla}_{PX}\xi). \tag{3.12}$$

Using the property of metric connection $\overline{\nabla}$, the last two terms of the right-hand side are cancelling each other, thus we have

$$g(\overline{\nabla}_{PX}FX, FX) = \sin^2\theta g(\overline{\nabla}_{PX}X, X). \tag{3.13}$$

Then by (2.5) and (2.6), we derive

$$g(\nabla_{PX}^{\perp}FX, FX) = \sin^2\theta g(\nabla_{PX}X, X). \tag{3.14}$$

Now, taking the inner product in (3.10) with FX, for any $X \in \Gamma(TM)$, then

$$g(F\nabla_{PX}X, FX) - g(\nabla_{PX}^{\perp}FX, FX) = \cos^{2}\theta \{||X||^{2} - \eta^{2}(X)\}g(H, FX).$$
 (3.15)

Then from (3.4) and (3.14), we obtain

$$-\sin^{2}\theta \eta(X)\eta(\nabla_{PX}X) = \cos^{2}\theta \Big\{ \|X\|^{2} - \eta^{2}(X) \Big\} g(H, FX),$$
(3.16)

or

$$-\sin^{2}\theta \eta(X)g(\nabla_{PX}X,\xi) = \cos^{2}\theta \left\{ ||X||^{2} - \eta^{2}(X) \right\} g(H,FX). \tag{3.17}$$

Using (2.5), we derive

$$-\sin^2\theta \eta(X)g(\overline{\nabla}_{PX}X,\xi) = \cos^2\theta \{||X||^2 - \eta^2(X)\}g(H,FX). \tag{3.18}$$

Since $\overline{\nabla}$ is the metric connection, then the above equation can be written as

$$\sin^2\theta \eta(X)g(X,\overline{\nabla}_{PX}\xi) = \cos^2\theta \{||X||^2 - \eta^2(X)\}g(H,FX). \tag{3.19}$$

As \overline{M} is cosymplectic thus using the fact that $\overline{\nabla}_{PX}\xi = 0$, the left hand side of the above equation vanishes identically, then

$$\cos^{2}\theta \left\{ \|X\|^{2} - \eta^{2}(X) \right\} g(H, FX) = 0.$$
 (3.20)

Thus from (3.20), it follows that either $\theta = \pi/2$ or $X = \xi$ or $H \in \Gamma(\mu)$, where μ is the invariant normal subbundle orthogonal to FTM. This completes the proof.

Theorem 3.3. Every totally umbilical proper slant submanifold M of a cosymplectic manifold \overline{M} is totally geodesic, provided $\nabla_X^{\perp} H \in \Gamma(\mu)$, for any $X \in TM$.

Proof. As \overline{M} is cosymplectic, then we have

$$\overline{\nabla}_{U}\phi V = \phi \overline{\nabla}_{U}V, \tag{3.21}$$

for any $U, V \in \Gamma(T\overline{M})$. Using this fact and formulae (2.5) and (2.8) we obtain that

$$\overline{\nabla}_X P Y + \overline{\nabla}_X F Y = P \nabla_X Y + F \nabla_X Y + \phi h(X, Y), \tag{3.22}$$

for any $X, Y \in \Gamma(TM)$. Then from (2.5), (2.6) and (2.12), we get

$$\nabla_X PY + h(X, PY) - A_{FY}X + \nabla_X^{\perp} FY = P\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H. \tag{3.23}$$

Taking the inner product in (3.23) with ϕH and using the fact that $H \in \Gamma(\mu)$ (by Theorem 3.2), we obtain

$$g(h(X, PY), \phi H) + g(\nabla_X^{\perp} FY, \phi H) = g(X, Y)g(\phi H, \phi H). \tag{3.24}$$

Then from (2.2) and (2.12), we derive

$$g(X, PY)g(H, \phi H) + g(\nabla_X^{\perp} FY, \phi H) = g(X, Y)g(H, H). \tag{3.25}$$

That is,

$$g\left(\nabla_X^{\perp} F Y, \phi H\right) = g(X, Y) \|H\|^2. \tag{3.26}$$

Now, we consider

$$\overline{\nabla}_X \phi H = \phi \overline{\nabla}_X H, \tag{3.27}$$

for any $X \in \Gamma(TM)$. From (2.6), we obtain

$$-A_{\phi H}X + \nabla_X^{\perp}\phi H = \phi \left(-A_HX + \nabla_X^{\perp}H\right). \tag{3.28}$$

Thus, on using (2.8), (2.9), we get

$$-A_{\phi H}X + \nabla_X^{\perp}\phi H = -PA_HX - FA_HX + t\nabla_X^{\perp}H + f\nabla_X^{\perp}H. \tag{3.29}$$

Taking the inner product with FY, for any $Y \in \Gamma(TM)$, then

$$g\left(\nabla_X^{\perp}\phi H, FY\right) = -g(FA_HX, FY) + g\left(f\nabla_X^{\perp}H, FY\right). \tag{3.30}$$

Since $f \nabla_X^{\perp} H \in \Gamma(\mu)$, then by (3.4) the above equation takes the form

$$g\left(\nabla_{X}^{\perp}\phi H, FY\right) = -\sin^{2}\theta \left\{g(A_{H}X, Y) - \eta(A_{H}X)\eta(Y)\right\}. \tag{3.31}$$

Using (2.6), (2.7), and (2.12), we get

$$g(\overline{\nabla}_X \phi H, FY) = -\sin^2\theta \{g(X, Y) - \eta(X)\eta(Y)\} ||H||^2. \tag{3.32}$$

The above equation can be written as

$$g(\overline{\nabla}_X FY, \phi H) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} \|H\|^2. \tag{3.33}$$

Again using the fact that $H \in \Gamma(\mu)$, then by (2.6), we obtain

$$g\left(\nabla_X^{\perp} F Y, \phi H\right) = \sin^2 \theta \left\{g(X, Y) - \eta(X)\eta(Y)\right\} \|H\|^2. \tag{3.34}$$

From (3.26) and (3.34), we derive

$$\left\{\cos^{2}\theta g(X,Y) + \sin^{2}\theta \eta(X)\eta(Y)\right\} \|H\|^{2} = 0.$$
 (3.35)

Thus, (3.35) implies either H = 0 or $\theta = \tan^{-1}(\sqrt{-g(X,Y)/\eta(X)\eta(Y)})$, which is not possible, because the slant angle $\theta \in (0, \pi/2)$. Hence, M is totally geodesic in \overline{M} .

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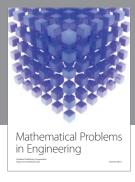
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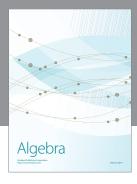
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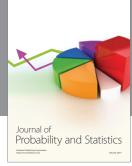
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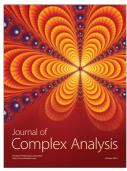




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