Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 708192, 21 pages
doi:10.1155/2012/708192

Research Article

Positive Solutions for Nonlinear Fractional Differential Equations with Boundary Conditions Involving Riemann-Stieltjes Integrals

Jiqiang Jiang,[1] Lishan Liu,[1] and Yonghong Wu[2]

[1] *School of Mathematical Sciences, Qufu Normal University, Shandong, Qufu 273165, China*
[2] *Department of Mathematics and Statistics, Curtin University of Technology, Perth, WA 6845, Australia*

Correspondence should be addressed to Jiqiang Jiang, qfjjq@163.com
and Lishan Liu, lls@mail.qfnu.edu.cn

Received 10 March 2012; Accepted 23 July 2012

Academic Editor: Dumitru Baleanu

We consider the existence of positive solutions for a class of nonlinear integral boundary value problems for fractional differential equations. By using some fixed point theorems, the existence and multiplicity results of positive solutions are obtained. The results obtained in this paper improve and generalize some well-known results.

1. Introduction

This paper is concerned with the existence of positive solutions to the following boundary value problem (BVP) for fractional differential equation:

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f\left(t, u(t), u'(t), \dots, u^{(n-2)}(t)\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(n-2)}(1) = \beta\left[u^{(n-2)}\right], \end{gathered} \tag{1.1}$$

where $\beta[v] = \int_0^1 v(t)dA(t)$ is a linear functional on $C[0,1]$ given by a Riemann-Stieltjes integral with A representing a suitable function of bounded variation, D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $n-1 < \alpha \le n$, $n \ge 2$, $f : [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}^+$ satisfies the Carathéodory type conditions, $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$.

Fractional differential equations arise in the modeling and control of many real-world systems and processes particularly in the fields of physics, chemistry, aerodynamics, electrodynamics of complex media, and polymer rheology. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Hence, intensive research has been carried out worldwide to study the existence of solutions of nonlinear fractional differential equations (see [1–25]). For example, by means of a mixed monotone method, Zhang [11] studied a unique positive solution for the singular boundary value problem

$$\begin{gathered} D_{0+}^{\alpha}u(t) + q(t)f\left(t, u(t), u'(t), \dots, u^{(n-2)}(t)\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{gathered} \tag{1.2}$$

where $\alpha \in (n-1, n]$, $n \geq 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f = g + h$ is nonlinear, and g and h have different monotone properties.

Recently, nonlocal boundary value problems for fractional differential equations were investigated intensively [13–23]. In [14], Bai concerned the existence and uniqueness of a positive solution for the following nonlocal problem:

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad \beta u(\eta) = u(1), \end{gathered} \tag{1.3}$$

where $1 < \alpha \leq 2$, $0 < \beta\eta^{\alpha-1} < 1$, $0 < \eta < 1$, D_{0+}^{α} is the standard Riemann-Liouville differentiation. The function f is continuous on $[0,1] \times \mathbb{R}^{+}$.

In [20], El-Shahed and Nieto investigated the existence of nontrivial solutions for the following nonlinear m-point boundary value problem of fractional type:

$$\begin{gathered} {}_{R}D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in [0,1], \ \alpha \in (n-1, n], \ n \in \mathbb{N}, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{gathered} \tag{1.4}$$

where $n \geq 2$, $a_i > 0$ $(i = 1, 2, \dots, m-2)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Also the authors considered the analogous problem using the Caputo fractional derivative:

$$\begin{gathered} {}_{C}D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in [0,1], \ \alpha \in (n-1, n], \ n \in \mathbb{N}, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i). \end{gathered} \tag{1.5}$$

Under certain growth conditions on the nonlinearity, several sufficient conditions for the existence of nontrivial solution are obtained by using the Leray-Schauder nonlinear alternative.

Inspired by the work of the above papers, the aim of this paper is to establish the existence and multiplicity of positive solutions of the BVP (1.1). We discuss the boundary value problem with the Riemann-Stieltjes integral boundary conditions, that is, the BVP (1.1), which includes fractional order two-point, three-point, multipoint, and nonlocal boundary value problems as special cases. Moreover, the $\beta[\cdot]$ in (1.1) is a linear function on $C[0,1]$ denoting the Riemann-Stieltjes integral; the A in the Riemann-Stieltjes integral is of bounded variation, namely, dA can be a signed measure. By using the Krasnosel'skii fixed point theorem, the Leray-Schauder nonlinear alternative and the Leggett-Williams fixed point theorem, some existence and multiplicity results of positive solutions are obtained.

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence and multiplicity of positive solutions of the BVP (1.1) are established by using some fixed point theorems. In Section 4, we give four examples to demonstrate the application of our theoretical results.

2. Basic Definitions and Preliminaries

We begin this section with some preliminaries of fractional calculus. Let $\alpha > 0$ and $n = [\alpha] + 1$, where $[\alpha]$ is the largest integer smaller than or equal to α. For a function $f : (0, +\infty) \to R$, we define the fractional integral of order α of f as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \tag{2.1}$$

provided the integral exists. The fractional derivative of order $\alpha > 0$ of a continuous function f is defined by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0, +\infty)$. We recall the following properties [26, 27] which are useful for the sequel. For $\alpha > 0$, $\beta > 0$, we have

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t), \qquad D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t). \tag{2.3}$$

As an example, we can choose a function f such that f, $D_{0+}^{\alpha} f \in C(0, +\infty) \cap L_{\mathrm{loc}}^1(0, +\infty)$.

For $\alpha > 0$, the general solution of the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ with $u \in C(0,1) \cap L(0,1)$ is given by

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \cdots + c_n t^{\alpha-n}, \tag{2.4}$$

where $c_i \in R$ $(i = 1, 2, \dots, n)$. Hence for $u \in C(0,1) \cap L(0,1)$, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \cdots + c_n t^{\alpha-n}. \tag{2.5}$$

Set

$$G_0(t,s) = \frac{1}{\Gamma(\alpha-n+2)} \begin{cases} [t(1-s)]^{\alpha-n+1} - (t-s)^{\alpha-n+1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-n+1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.6}$$

Lemma 2.1 (see [11]). *Let* $y \in C_r[0,1] (C_r[0,1] = \{y \in C[0,1],\ t^r y \in C[0,1],\ 0 \le r < 1\})$. *Then the boundary value problem,*

$$\begin{gathered} D_{0+}^{\alpha-n+2} v(t) = y(t), \quad 0 < t < 1,\ n-1 < \alpha \le n,\ n \ge 2, \\ v(0) = 0, \qquad v(1) = 0, \end{gathered} \tag{2.7}$$

has a unique solution

$$v(t) = \int_0^1 G_0(t,s) y(s)\, ds. \tag{2.8}$$

Lemma 2.2 (see [11]). *The function* $G_0(t,s)$ *defined by* (2.6) *satisfies the following properties:*

(i) $G_0(t,s) \ge 0$, $G_0(t,s) \le G_0(s,s)$ *for all* $t, s \in [0,1]$;

(ii) *there exist a positive function* $\rho \in C(0,1)$ *and* $0 < \xi < \eta < 1$ *such that*

$$\min_{t \in [\xi,\eta]} G_0(t,s) \ge \rho(s) G_0(s,s), \quad s \in (0,1), \tag{2.9}$$

where

$$\rho(s) = \begin{cases} \dfrac{[\eta(1-s)]^{\alpha-n+1} - (\eta-s)^{\alpha-n+1}}{[s(1-s)]^{\alpha-n+1}}, & s \in (0,r], \\ \left(\dfrac{\xi}{s}\right)^{\alpha-n+1}, & s \in [r,1). \end{cases} \tag{2.10}$$

By (2.4), the unique solution of the problem

$$D_{0+}^{\alpha-n+2}v(t) = 0, \quad 0 < t < 1, \ n-1 < \alpha \leq n, \ n \geq 2,$$
$$v(0) = 0, \qquad v(1) = \beta[v] \tag{2.11}$$

is $\gamma(t) = t^{\alpha-n+1}$, with $\beta[v]$ replaced by 1. As in [21], the Green's function for boundary value problem (2.11) is given by

$$G(t,s) = \frac{\gamma(t)}{1-\beta[\gamma]}\mathcal{G}(s) + G_0(t,s), \tag{2.12}$$

where $\mathcal{G}(s) := \int_0^1 G_0(t,s)dA(t)$.

Lemma 2.3. *Let* $0 \leq \beta[\gamma] < 1$ *and* $\mathcal{G}(s) \geq 0$ *for* $s \in [0,1]$, *the Green function* $G(t,s)$ *defined by* (2.12) *has the following properties:*

(i) $G(t,s) \geq 0$, $G(t,s) \leq (1+\beta[1]/(1-\beta[\gamma]))G_0(s,s)$ *for all* $t,s \in [0,1]$;

(ii) $\min_{t\in[\xi,\eta]}G(t,s) \geq \min_{t\in[\xi,\eta]}G_0(t,s) \geq \rho(s)G_0(s,s)$, $s \in (0,1)$.

Proof. By Lemma 2.2, it is easy to prove this lemma, so we omit it. □

Let $X = C[0,1]$. It follows that $(X, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined by the supernorm $\|x\| = \sup_{t\in[0,1]}|x(t)|$. $P = \{x \in X : x(t) \geq 0, t \in [0,1]\}$. Clearly P is a cone of X. Now, in the following, we give the assumptions to be used throughout the rest of this paper.

(H_1) A is a function of bounded variation, $\mathcal{G}(s) \geq 0$ for $s \in [0,1]$ and $0 \leq \beta[\gamma] < 1$.

(H_2) $f : [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}^+$ satisfies the following conditions of Carathéodory type:

(i) $f(\cdot, x)$ is Lebesgue measurable for each fixed $x \in \mathbb{R}^{n-1}$;

(ii) $f(t,\cdot)$ is continuous for a.e. $t \in [0,1]$.

In order to overcome the difficulty due to the dependance of f on derivatives, we consider the following modified problem:

$$D_{0+}^{\alpha-n+2}v(t) + f\left(t, I_{0+}^{n-2}v(t), I_{0+}^{n-3}v(t), \ldots, I_{0+}^{1}v(t), v(t)\right) = 0, \quad 0 < t < 1,$$
$$v(0) = 0, \qquad v(1) = \beta[v], \tag{2.13}$$

where $n-1 < \alpha \leq n$, $n \geq 2$.

Lemma 2.4. *The nonlocal fractional order boundary value problem* (1.1) *has a positive solution if and only if the nonlinear fractional integrodifferential equation* (2.13) *has a positive solution.*

Proof. If u is a positive solution of the fractional order boundary value problem (1.1), let $v(t) = D_{0+}^{n-2}u(t)$. Then from the boundary value conditions of (1.1) and the definition of the Riemann-Liouville fractional integral and derivative, we have

$$\begin{aligned}
v(t) &= D_{0+}^{n-2}u(t) = u^{(n-2)}(t),\\
I_{0+}^{1}v(t) &= I_{0+}^{1}u^{(n-2)}(t) = \frac{1}{\Gamma(1)}\int_0^t u^{(n-2)}(s)\,ds = u^{(n-3)}(t),\\
I_{0+}^{2}v(t) &= I_{0+}^{2}u^{(n-2)}(t) = \frac{1}{\Gamma(2)}\int_0^t (t-s)u^{(n-2)}(s)\,ds = u^{(n-4)}(t),\\
&\ \ \vdots\\
I_{0+}^{n-2}v(t) &= I_{0+}^{n-2}u^{(n-2)}(t) = \frac{1}{\Gamma(n-2)}\int_0^t (t-s)^{n-3}u^{(n-2)}(s)\,ds = u(t),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
D_{0+}^{\alpha-n+2}v(t) &= \frac{1}{\Gamma(2n-\alpha-2)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{2n-\alpha-3}u^{(n-2)}(s)\,ds\\
&= \frac{1}{\Gamma(2n-\alpha-3)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{2n-\alpha-4}u^{(n-3)}(s)\,ds\\
&= \cdots\\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}u(s)\,ds\\
&= D_{0+}^{\alpha}u(t),
\end{aligned} \tag{2.15}$$

which imply that $v(0) = u^{(n-2)}(0) = 0$, $v(1) = u^{(n-2)}(1) = \beta[u^{(n-2)}] = \beta[v]$. Thus $v(t)$ is a positive solution of the nonlinear fractional integrodifferential equation (2.13).

On the other hand, if v is a positive solution of the nonlinear fractional integrodifferential equation (2.13), let $u(t) = I_{0+}^{n-2}v(t)$, then by (2.3) and the definition of the Riemann-Liouville fractional derivative, we have

$$\begin{aligned}
u'(t) &= D_{0+}^{1}u(t) = D_{0+}^{1}I_{0+}^{n-2}v(t) = D_{0+}^{1}I_{0+}^{1}I_{0+}^{n-3}v(t) = I_{0+}^{n-3}v(t),\\
u''(t) &= D_{0+}^{2}u(t) = D_{0+}^{2}I_{0+}^{n-2}v(t) = D_{0+}^{2}I_{0+}^{2}I_{0+}^{n-4}v(t) = I_{0+}^{n-4}v(t),\\
&\ \ \vdots\\
u^{(n-3)}(t) &= D_{0+}^{n-3}u(t) = D_{0+}^{n-3}I_{0+}^{n-2}v(t) = D_{0+}^{n-3}I_{0+}^{n-3}I_{0+}^{1}v(t) = I_{0+}^{1}v(t),\\
u^{(n-2)}(t) &= D_{0+}^{n-2}u(t) = D_{0+}^{n-2}I_{0+}^{n-2}v(t) = v(t),
\end{aligned}$$

$$\begin{aligned}
D_{0+}^{\alpha}u(t) &= \frac{d^n}{dt^n} I_{0+}^{n-\alpha}u(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} I_{0+}^{n-2}v(t) = \frac{d^n}{dt^n} I_{0+}^{2n-\alpha-2}v(t) = D_{0+}^{\alpha-n+2}v(t) \\
&= -f\left(t, I_{0+}^{n-2}v(t), I_{0+}^{n-3}v(t), \dots, I_{0+}^{1}v(t), v(t)\right) \\
&= -f\left(t, u(t), u'(t), \dots, u^{(n-3)}(t), u^{(n-2)}(t)\right), \quad 0 < t < 1,
\end{aligned} \tag{2.16}$$

which imply that $u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0$, $u^{(n-2)}(0) = v(0) = 0$, $u^{(n-2)}(1) = v(1) = \beta[v] = \beta[u^{(n-2)}]$. Moreover, it follows from the monotonicity and property of I_{0+}^{n-2} that $I_{0+}^{n-2}v(t) \in C([0,1], \mathbb{R}^+)$. Consequently, $u(t) = I_{0+}^{n-2}v(t)$ is a positive solution of the fractional order boundary value problem (1.1). □

By Lemma 2.4, we will concentrate our study on (2.13). We here define an operator $T : P \to P$ by

$$Tv(t) = \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) ds, \quad t \in [0,1]. \tag{2.17}$$

Clearly, v is a fixed point of T in P, and so v is a positive solution of BVP (2.13).

In order to prove our main results, we need the following lemmas.

Lemma 2.5 (see [28]). *Let X be a real Banach space, Ω be a bounded open subset of X, where $\theta \in \Omega$, $T : \overline{\Omega} \to X$ is a completely continuous operator. Then, either there exist $x \in \partial\Omega$, $\mu \in (0,1)$ such that $\mu T(x) = x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Lemma 2.6 (see [29]). *Let X be a real Banach space, P be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets of X with $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tx\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$, $x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7 (see [30, 31]). *Let P be a cone in a real Banach space X, $P_c = \{x \in P : \|x\| < c\}$, φ be a nonnegative continuous concave functional on P such that $\varphi(x) \le \|x\|$ for all $x \in \overline{P}_c$, and $P(\varphi, b, d) = \{x \in P : b \le \varphi(x), \|x\| \le d\}$. Suppose that $T : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exist positive constants $0 < a < b < d \le c$ such that*

(C_1) $\{x \in P(\varphi, b, d) : \varphi(x) > b\} \ne \phi$ *and* $\varphi(Tx) > b$ *for* $x \in P(\varphi, b, d)$,

(C_2) $\|Tx\| < a$ *for* $x \in \overline{P}_a$,

(C_3) $\varphi(Tx) > b$ *for* $x \in P(\varphi, b, c)$ *with* $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, and x_3 satisfying

$$\|x_1\| < a, \quad b < \varphi(x_2), \quad a < \|x_3\| \quad \text{with} \quad \varphi(x_3) < b. \tag{2.18}$$

Remark 2.8. If $d = c$, then condition (C_1) of Lemma 2.7 implies condition (C_3) of Lemma 2.7.

For notational convenience, we introduce the following constants:

$$L_1 = \int_{\xi}^{\eta} \rho(s) G_0(s,s)\, ds, \qquad L_2 = \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) \int_0^1 G_0(s,s)\, ds, \tag{2.19}$$

and a nonnegative continuous concave functional φ on the cone P defined by

$$\varphi(v) = \min_{\xi \le t \le \eta} |v(t)|. \tag{2.20}$$

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *Assume that (H_1) and (H_2) hold and there exist nonnegative functions $h_1, h_2, \ldots, h_{n-1} \in L[0,1]$ such that*

$$\left| f(t, x_1, x_2, \ldots, x_{n-1}) - f(t, y_1, y_2, \ldots, y_{n-1}) \right| \le \sum_{i=1}^{n-1} h_i(t) \left| x_i - y_i \right|, \tag{3.1}$$

for almost every $t \in [0,1]$ and all $(x_1, x_2, \ldots, x_{n-1}),\ (y_1, y_2, \ldots, y_{n-1}) \in R^{n-1}$.

If

$$0 < \int_0^1 G_0(s,s) \sum_{i=1}^{n-1} h_i(s)\, ds < \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}, \tag{3.2}$$

then BVP (1.1) *has a unique positive solution.*

Proof. We will show that T is a contraction mapping. For any $v_1, v_2 \in P$ and $1 \le i \le n-2$, by the definition of fractional integral, we obtain

$$\begin{aligned} \left| I_{0+}^{i} v_1(t) - I_{0+}^{i} v_2(t) \right| &= \left| \frac{1}{(i-1)!} \int_0^t (t-s)^{i-1} (v_1(s) - v_2(s))\, ds \right| \\ &\le \frac{1}{(i-1)!} \int_0^t (t-s)^{i-1} |v_1(s) - v_2(s)|\, ds \\ &\le \frac{1}{(i-1)!} \int_0^t (t-s)^{i-1}\, ds \|v_1 - v_2\| \\ &= \frac{1}{i!} t^i \|v_1 - v_2\| \le \|v_1 - v_2\|. \end{aligned} \tag{3.3}$$

So, for any $v_1, v_2 \in P$, by (3.3) and Lemma 2.3, we have

$$
\begin{aligned}
|Tv_1(t) - Tv_2(t)| &= \left| \int_0^1 G(t,s) \Big[f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) \right. \\
&\qquad\qquad \left. - f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) \Big] ds \right| \\
&\le \int_0^1 G(t,s) \Big| f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) \\
&\qquad\qquad - f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) \Big| ds \\
&\le \int_0^1 \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) G_0(s,s) \sum_{i=1}^{n-1} h_i(s) \left| I_{0+}^{n-1-i}v_1(s) - I_{0+}^{n-1-i}v_2(s) \right| ds \\
&\le \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) \int_0^1 G_0(s,s) \sum_{i=1}^{n-1} h_i(s)\, ds \|v_1 - v_2\|.
\end{aligned}
\tag{3.4}
$$

This implies that

$$
\|Tv_1 - Tv_2\| \le \kappa \|v_1 - v_2\|, \tag{3.5}
$$

where $\kappa = (1 + \beta[1]/(1-\beta[\gamma])) \int_0^1 G_0(s,s) \sum_{i=1}^{n-1} h_i(s)\, ds \in (0,1)$. By the Banach contraction mapping principle, we deduce that T has a unique fixed point v^*. Thus, by Lemma 2.4, $u^*(t) = I_{0+}^{n-2}v^*(t)$ is a unique positive solution of BVP (1.1). □

Lemma 3.2. *Assume that* (H_1) *and* (H_2) *hold and the following conditions are satisfied.*

(H_3) *There exist nonnegative real-valued functions* $q, p_1, p_2, \dots, p_{n-1} \in L[0,1]$ *such that*

$$
f(t, x_1, x_2, \dots, x_{n-1}) \le q(t) + \sum_{i=1}^{n-1} p_i(t)|x_i|, \tag{3.6}
$$

for almost every $t \in [0,1]$ *and all* $(x_1, x_2, \dots, x_{n-1}) \in R^{n-1}$.

Then $T : P \to P$ *is a completely continuous operator.*

Proof. For any $v \in P$, as $G(t,s) \ge 0$ for all $t, s \in [0,1]$, we have $Tv(t) \ge 0$, so $T(P) \subset P$. Let $D \subset P$ be any bounded set. Then there exists a constant $L > 0$ such that $\|v\| \le L$ for any $v \in D$. Moreover for any $v \in D$, $s \in [0,1]$, $v(s) \le \|v\| \le L$. Proceeding as for (3.3), we obtain

$$
\left| I_{0+}^{n-1-i}v(s) \right| = I_{0+}^{n-1-i}v(s) \le \|v\| \le L, \quad i = 1, 2, \dots, n-1. \tag{3.7}
$$

Thus,

$$
\begin{aligned}
|Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\
&\leq \int_0^1 \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) G_0(s,s) \left[q(s) + \sum_{i=1}^{n-1} p_i(s) I_{0+}^{n-1-i} v(s)\right] ds \\
&\leq \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) \int_0^1 G_0(s,s) \left[q(s) + \sum_{i=1}^{n-1} p_i(s) \|v\|\right] ds \\
&\leq \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right) (L+1) \int_0^1 G_0(s,s) \left[q(s) + \sum_{i=1}^{n-1} p_i(s)\right] ds \\
&< +\infty.
\end{aligned}
\tag{3.8}
$$

Therefore, $T(D)$ is uniformly bounded.

Now we show that $T(D)$ is equicontinuous on $[0,1]$. Since $G(t,s)$ is continuous on $[0,1] \times [0,1]$, $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$. Hence, for any $\varepsilon > 0$, there exists a constant $\delta_0 > 0$ such that for any $s \in [0,1]$, $t, t' \in [0,1]$, when $|t - t'| < \delta_0$, it holds

$$
\left|G(t,s) - G(t',s)\right| < \left[1 + (L+1) \int_0^1 \left(q(s) + \sum_{i=1}^{n-1} p_i(s)\right) ds\right]^{-1} \varepsilon. \tag{3.9}
$$

Consequently, for any $t, t' \in [0,1]$ and $|t - t'| < \delta_0$, we have

$$
\begin{aligned}
\left|Tv(t) - Tv(t')\right| &\leq \int_0^1 \left|G(t,s) - G(t',s)\right| f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\
&\leq \int_0^1 \left|G(t,s) - G(t',s)\right| \left[q(s) + \sum_{i=1}^{n-1} p_i(s) \|v\|\right] ds \\
&\leq (L+1) \int_0^1 \left|G(t,s) - G(t',s)\right| \left[q(s) + \sum_{i=1}^{n-1} p_i(s)\right] ds \\
&< \varepsilon.
\end{aligned}
\tag{3.10}
$$

This implies that $T(D)$ is equicontinuous. Thus according to the Ascoli-Arzela Theorem, $T(D)$ is a relatively compact set.

In the end, we show that $T : P \to P$ is continuous. Assume that $v_m, v_0 \in P$ $(m = 1, 2, \ldots)$, $v_m \to v_0$ $(m \to +\infty)$, then

$$
|v_m(t) - v_0(t)| \leq \|v_m - v_0\| \longrightarrow 0, \tag{3.11}
$$

and $\|v_m\| \le L$ $(m = 0, 1, 2, \ldots)$, where L is a positive constant. Keeping in mind that f satisfies Carathéodory conditions on $[0,1] \times \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
&\lim_{m \to +\infty} f\left(t, I_{0+}^{n-2} v_m(t), I_{0+}^{n-3} v_m(t), \ldots, I_{0+}^{1} v_m(t), v_m(t)\right) \\
&\quad = f\left(t, I_{0+}^{n-2} v_0(t), I_{0+}^{n-3} v_0(t), \ldots, I_{0+}^{1} v_0(t), v_0(t)\right), \quad \text{for a.e. } t \in [0,1].
\end{aligned} \tag{3.12}$$

Proceeding as for (3.3), for $m \in \mathbb{N}$ we obtain

$$\left| I_{0+}^{n-1-i} v_m(s) \right| = I_{0+}^{n-1-i} v_m(s) \le \|v_m\| \le L \quad i = 1, 2, \ldots, n-1. \tag{3.13}$$

This together with (3.6),

$$0 \le f\left(t, I_{0+}^{n-2} v_m(t), I_{0+}^{n-3} v_m(t), \ldots, I_{0+}^{1} v_m(t), v_m(t)\right) \le q(t) + L \sum_{i=1}^{n-1} p_i(t). \tag{3.14}$$

The Lebesgue dominated convergence theorem gives

$$\begin{aligned}
&\lim_{m \to +\infty} \int_0^1 \left| f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \\
&\quad \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right| ds = 0.
\end{aligned} \tag{3.15}$$

Now we deduce from (3.15), Lemma 2.3

$$\begin{aligned}
&|Tv_m(t) - Tv_0(t)| \\
&\quad = \left| \int_0^1 G(t,s) \left[f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \right. \\
&\qquad \left. \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right] ds \right| \\
&\quad \le \left(1 + \frac{\beta[1]}{1 - \beta[\gamma]}\right) \int_0^1 G_0(s,s) \left| f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \\
&\qquad \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right| ds \\
&\quad \le \left(1 + \frac{\beta[1]}{1 - \beta[\gamma]}\right) \max_{s \in [0,1]} G_0(s,s) \int_0^1 \left| f\left(s, I_{0+}^{n-2} v_m(s), I_{0+}^{n-3} v_m(s), \ldots, I_{0+}^{1} v_m(s), v_m(s)\right) \right. \\
&\qquad \left. - f\left(s, I_{0+}^{n-2} v_0(s), I_{0+}^{n-3} v_0(s), \ldots, I_{0+}^{1} v_0(s), v_0(s)\right) \right| ds
\end{aligned} \tag{3.16}$$

that $\|Tv_m - Tv_0\| \to 0$, as $m \to +\infty$. So $T : P \to P$ is continuous. Therefore $T : P \to P$ is completely continuous. □

Remark 3.3. If $f : [0,1] \times \mathbb{R}^{n-1}$ is continuous, by similar argument as above, we can show that T is completely continuous.

Theorem 3.4. *Assume that* (H_1)*–*(H_3) *hold. If*

$$\int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)\, ds < \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}, \tag{3.17}$$

then BVP (1.1) *has at least one positive solution.*

Proof. Let

$$\Omega = \{v \in P : \|v\| < r\}, \quad \text{where } r = \frac{(1+\beta[1]/(1-\beta[\gamma]))\int_0^1 G_0(s,s)q(s)ds}{1-(1+\beta[1]/(1-\beta[\gamma]))\int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)ds}, \tag{3.18}$$

we have $\Omega \subset P$. From Lemma 3.2, we know that $T : \Omega \to P$ is completely continuous. If there exists $v \in \partial\Omega$, $\mu \in (0,1)$ such that

$$v = \mu T v, \tag{3.19}$$

then by (H_3) and (3.19), we have

$$\begin{aligned} v(t) = \mu T v(t) &= \mu \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right)ds \\ &\le \mu \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right)ds \\ &\le \mu\left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)\left[\int_0^1 G_0(s,s)q(s)\, ds + \int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)ds\|v\|\right], \end{aligned} \tag{3.20}$$

which implies that

$$\begin{aligned} \|v\| &\le \mu\left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)\left[\int_0^1 G_0(s,s)q(s)\, ds + r\int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)ds\right] \\ &< \left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)\left[\int_0^1 G_0(s,s)q(s)ds + r\int_0^1 G_0(s,s)\sum_{i=1}^{n-1} p_i(s)ds\right] = r. \end{aligned} \tag{3.21}$$

This means that $v \notin \partial\Omega$. By Lemma 2.5, T has a fixed point $\widehat{v} \in \overline{\Omega}$. By Lemma 2.4, BVP (1.1) has at least one positive solution $\widehat{u}(t) = I_{0+}^{n-2}\widehat{v}(t)$. □

Theorem 3.5. *Assume that* (H_1)*–*(H_3) *hold. If there exist two positive constants* $r_1 < r_2$ *such that*

(i) $f(t, x_1, x_2, \ldots, x_{n-1}) \le L_2^{-1} r_2$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0, r_2] \times \cdots \times [0, r_2]$,

(ii) $f(t, x_1, x_2, \ldots, x_{n-1}) \ge L_1^{-1} r_1$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0, r_1] \times \cdots \times [0, r_1]$,

where L_1, L_2 *are defined by* (2.19), *then BVP* (1.1) *has at least one positive solution.*

Proof. Let $\Omega_2 = \{v \in P : \|v\| < r_2\}$. For any $v \in \partial\Omega_2$, we have $\|v\| = r_2$ and $0 \le v(t) \le r_2$ for every $t \in [0,1]$. Similar to (3.7), for $0 \le v(s) \le r_2$, we have

$$0 \le \left| I_{0+}^{n-1-i} v(s) \right| = I_{0+}^{n-1-i} v(s) \le \|v\| \le r_2, \quad i = 1, 2, \ldots, n-1. \tag{3.22}$$

It follows from condition (i) that

$$f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) \le L_2^{-1} r_2, \quad \text{for } (s, v) \in [0,1] \times [0, r_2]. \tag{3.23}$$

Thus, for any $v \in \partial\Omega_2$, by (3.23) and Lemma 2.3, we have

$$\begin{aligned} |Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) ds \\ &\le \int_0^1 G(t,s) L_2^{-1} r_2 \, ds \le \left(1 + \beta[1]/(1 - \beta[\gamma])\right) L_2^{-1} r_2 \int_0^1 G_0(s,s) ds \\ &= r_2 = \|v\|, \quad t \in [0,1], \end{aligned} \tag{3.24}$$

which means that

$$\|Tv\| \le \|v\|, \quad v \in \partial\Omega_2. \tag{3.25}$$

On the other hand, let $\Omega_1 = \{v \in P : \|v\| < r_1\}$. For any $v \in \partial\Omega_1$, we have $\|v\| = r_1$ and $0 \le v(t) \le r_1$ for every $t \in [0,1]$. Similar to (3.23), from condition (ii), we can get

$$f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) \ge L_1^{-1} r_1, \quad \text{for } (s, v) \in [0,1] \times [0, r_1]. \tag{3.26}$$

Hence for any $t \in [\xi, \eta]$, $v \in \partial\Omega_1$, by (3.26) and Lemma 2.3 we have

$$\begin{aligned} |Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) ds \\ &\ge \int_0^1 G(t,s) L_1^{-1} r_1 \, ds \ge L_1^{-1} r_1 \int_\xi^\eta G(t,s) ds \\ &\ge L_1^{-1} r_1 \int_\xi^\eta \rho(s) G_0(s,s) ds \\ &= r_1 = \|v\|. \end{aligned} \tag{3.27}$$

Thus we get

$$\|Tv\| \geq \|v\|, \quad v \in \partial\Omega_1. \tag{3.28}$$

By (3.25), (3.28), and Lemma 2.6, T has a fixed point $\tilde{v} \in \overline{\Omega}_2 \setminus \Omega_1$ such that $r_1 \leq \|\tilde{v}\| \leq r_2$. By Lemma 2.4, BVP (1.1) has at least one positive solution $\tilde{u}(t) = I_{0+}^{n-2}\tilde{v}(t)$. □

Theorem 3.6. *Assume that (H_1)–(H_3) hold. If there exist constants* $0 < a < b < c$ *such that*

(I) $f(t, x_1, x_2, \ldots, x_{n-1}) < L_2^{-1}a$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0,a] \times \cdots \times [0,a]$,

(II) $f(t, x_1, x_2, \ldots, x_{n-1}) \leq L_2^{-1}c$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [0,1] \times [0,c] \times \cdots \times [0,c]$,

(III) $f(t, x_1, x_2, \ldots, x_{n-1}) \geq L_1^{-1}b$, *for* $(t, x_1, x_2, \ldots, x_{n-1}) \in [\xi, \eta] \times [(b/(n-2)!)\xi^{n-2}, c] \times [(b/(n-3)!)\xi^{n-3}, c] \times \cdots \times [b\xi, c] \times [b, c]$,

where L_1, L_2 *are defined by* (2.19), *then BVP* (1.1) *has at least three positive solutions* u_1, u_2, *and* u_3 *satisfying*

$$\begin{gathered} \left\|D_{0+}^{n-2}u_1\right\| < a, \qquad b < \varphi\left(D_{0+}^{n-2}u_2\right) < \left\|D_{0+}^{n-2}u_2\right\| \leq c, \\ a < \left\|D_{0+}^{n-2}u_3\right\|, \qquad \varphi\left(D_{0+}^{n-2}u_3\right) < b. \end{gathered} \tag{3.29}$$

Proof. We will show that all conditions of Lemma 2.7 are satisfied.

First, if $v \in \overline{P}_c$, then $\|v\| \leq c$. So we have $0 \leq v(t) \leq c$, $t \in [0,1]$. Similar to (3.23), it follows from condition (II) that

$$f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) \leq L_2^{-1}c, \quad \text{for } (s, v) \in [0,1] \times [0,c]. \tag{3.30}$$

Thus, for any $v \in \overline{P}_c$, by (3.30), we have

$$\begin{aligned} |Tv(t)| &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \ldots, I_{0+}^{1}v(s), v(s)\right) ds \\ &\leq \int_0^1 G(t,s) L_2^{-1} c \, ds \leq \left(1 + \frac{\beta[1]}{1 - \beta[\gamma]}\right) L_2^{-1} c \int_0^1 G_0(s,s) ds \\ &= c, \end{aligned} \tag{3.31}$$

which means that $\|Tv\| \leq c$, $v \in \overline{P}_c$. Therefore, $T : \overline{P}_c \to \overline{P}_c$. By Lemma 3.2, we know that $T : \overline{P}_c \to \overline{P}_c$ is completely continuous.

Next, similar to (3.30) and (3.31), it follows from condition (I) that if $v \in \overline{P}_a$ then $\|Tv\| < a$. So the condition (C_2) of Lemma 2.7 holds.

Now, we take $v(t) = (b+c)/2$, $t \in [0,1]$. It is easy to see that $v(t) = (b+c)/2 \in P(\varphi, b, c)$, and so

$$\varphi(v) = \min_{\xi \le t \le \eta} |v(t)| = \frac{b+c}{2} > b, \tag{3.32}$$

where $\varphi(v)$ is defined by (2.20). This proves that $\{v \in P(\varphi, b, c) : \varphi(v) > b\} \neq \phi$.

On the other hand, if $v \in P(\varphi, b, c)$, then $b \le v(t) \le c$, $t \in [\xi, \eta]$. By the definition of fractional integral, for any $t \in [\xi, \eta]$, $1 \le i \le n-2$, we obtain

$$\begin{aligned} \frac{b}{i!}\xi^i &\le \frac{b}{(i-1)!}\int_0^t (t-s)^{i-1}\,ds \le I_{0+}^i v(t) = \frac{1}{(i-1)!}\int_0^t (t-s)^{i-1} v(s)ds \\ &\le \frac{c}{(i-1)!}\int_0^t (t-s)^{i-1}\,ds \\ &\le \frac{c}{i!}\eta^i \le c. \end{aligned} \tag{3.33}$$

It follows from (3.33) and condition (III) that

$$f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) \ge L_1^{-1}b, \quad \text{for } s \in [\xi, \eta],\ v \in P(\varphi, b, c). \tag{3.34}$$

Hence we have

$$\begin{aligned} \varphi(Tv) &= \min_{\xi \le t \le \eta} |Tv(t)| \\ &= \min_{\xi \le t \le \eta} \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) ds \\ &\ge L_1^{-1} b \min_{\xi \le t \le \eta} \int_0^1 G(t,s)ds > L_1^{-1} b \int_\xi^\eta \rho(s) G_0(s,s) ds = b, \end{aligned} \tag{3.35}$$

which implies that $\varphi(Tv) > b$, for $v \in P(\varphi, b, c)$. This shows that condition (C_1) of Lemma 2.7 is also satisfied.

By Lemma 2.7 and Remark 2.8, BVP (2.13) has at least three positive solutions v_1, v_2, and v_3 such that $\|v_1\| < a$, $b < \varphi(v_2) < \|v_2\| \le c$, and $a < \|v_3\|$, $\varphi(v_3) < b$. By Lemma 2.4, BVP (1.1) has at least three positive solutions $u_i(t) = I_{0+}^{n-2}v_i(t)$, $(i = 1, 2, 3)$. By (2.3), we have $D_{0+}^{n-2}u_i(t) = D_{0+}^{n-2}I_{0+}^{n-2}v_i(t) = v_i$, $i = 1, 2, 3$. So u_1, u_2, u_3 are three positive solutions of BVP (1.1) satisfying

$$\begin{gathered} \left\|D_{0+}^{n-2}u_1\right\| < a, \qquad b < \varphi\left(D_{0+}^{n-2}u_2\right) < \left\|D_{0+}^{n-2}u_2\right\| \le c, \\ a < \left\|D_{0+}^{n-2}u_3\right\|, \qquad \varphi\left(D_{0+}^{n-2}u_3\right) < b. \end{gathered} \tag{3.36}$$

The proof of Theorem 3.6 is completed. □

4. Examples

Example 4.1. Consider the following problem:

$$D_{0+}^{7/2}u(t)+\frac{(1-t)^3e^tu}{(1+e^t)(1+u)}+\frac{1}{2}t^2\sin^2u'+\frac{1}{4}tu''=0,\quad 0<t<1,$$
$$u(0)=u'(0)=u''(0)=0,\qquad u''(1)=\beta[u''].\tag{4.1}$$

Let $\beta[u'']=(1/2)\ u''(1/2)$. Then

$$G_0(t,s)=\frac{1}{\Gamma(3/2)}\begin{cases}[t(1-s)]^{1/2}-(t-s)^{1/2}, & 0\le s\le t\le 1,\\ [t(1-s)]^{1/2}, & 0\le t\le s\le 1.\end{cases}\tag{4.2}$$

$$\mathcal{G}(s)=\frac{1}{2}G_0\left(\frac{1}{2},s\right)\ge 0,\qquad \beta[1]=\int_0^1 dA(t)=\frac{1}{2},\qquad \beta[\gamma]=\int_0^1 t^{1/2}dA(t)=\frac{\sqrt{2}}{4}<1.\tag{4.3}$$

Let

$$f(t,x,y,z)=\frac{(1-t)^3e^tx}{(1+e^t)(1+x)}+\frac{1}{2}t^2\sin^2y+\frac{1}{4}tz,$$
$$h_1(t)=\frac{(1-t)^3e^t}{1+e^t},\qquad h_2(t)=\frac{1}{2}t^2,\qquad h_3(t)=\frac{1}{4}t.\tag{4.4}$$

Then f is a nonnegative continuous function on $[0,1]\times(\mathbb{R}^+)^3$ and, for any (t,x_1,y_1,z_1) and $(t,x_2,y_2,z_2)\in[0,1]\times(\mathbb{R}^+)^3$, satisfies

$$\left|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\right|\le h_1(t)|x_1-x_2|+h_2(t)\left|y_1-y_2\right|+h_3(t)|z_1-z_2|.\tag{4.5}$$

So we have

$$\begin{aligned}\int_0^1 G_0(s,s)\sum_{i=1}^3 h_i(s)\,ds&\le\frac{1}{\Gamma(3/2)}\int_0^1(s(1-s))^{1/2}\left((1-s)^3+s^2+s\right)ds\\&=\frac{B(3/2,9/2)+B(7/2,3/2)+B(5/2,3/2)}{\Gamma(3/2)}=\frac{33}{128}\sqrt{\pi}\approx 0.4569608\\&<\left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}=\frac{4-\sqrt{2}}{6-\sqrt{2}}\approx 0.5638901,\end{aligned}\tag{4.6}$$

where $B(\cdot,\cdot)$ denotes a Beta function. So all conditions of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, BVP (4.1) has at least one positive solution.

Example 4.2. Consider the following problem:

$$D_{0+}^{5/2}u(t)+\frac{1}{2}\left(t-t^2\right)\ln(1+u)+\frac{1}{2}t^2u'+t^3+\sin t=0,\quad 0<t<1,$$
$$u(0)=u'(0)=0,\qquad u'(1)=\beta\left[u'\right],\tag{4.7}$$

where $\beta[u']=\int_0^1 u'(s)dA(s)$ with

$$A(s)=\begin{cases}0, & s\in\left[0,\frac{1}{4}\right),\\ 2, & s\in\left[\frac{1}{4},\frac{9}{16}\right),\\ 1, & s\in\left[\frac{9}{16},1\right].\end{cases}\tag{4.8}$$

Set

$$f(t,x,y)=\frac{1}{2}\left(t-t^2\right)\ln(1+x)+\frac{1}{2}t^2y+t^3+\sin t,\qquad p_1(t)=\frac{1}{2}\left(t-t^2\right),$$
$$p_2(t)=\frac{1}{2}t^2,\qquad q(t)=t^3+1.\tag{4.9}$$

Then $f:[0,1]\times[0,+\infty)\times[0,+\infty)\longrightarrow[0,+\infty)$ is continuous and

$$f(t,x,y)\le p_1(t)x+p_2(t)y+q(t).\tag{4.10}$$

As in [21], $\beta[\gamma]=\int_0^1\gamma(t)dA(t)=2\sqrt{1/4}+(-1)\times\sqrt{9/16}=1/4<1$, $\mathcal{G}(s)=\int_0^1 G_0(t,s)dA(t)\ge 0$,

$$\begin{aligned}\int_0^1 G_0(s,s)(p_1(s)+p_2(s))\,ds&\le\frac{1}{\Gamma(3/2)}\int_0^1(s(1-s))^{1/2}\left(\left(s-s^2\right)+s^2\right)ds\\ &=\frac{B(5/2,5/2)+B(7/2,3/2)}{\Gamma(3/2)}=\frac{1}{8}\sqrt{\pi}\approx 0.22155673\\ &<\left(1+\frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}=\frac{3}{7}\approx 0.42857143,\end{aligned}\tag{4.11}$$

where $B(\cdot,\cdot)$ denotes a Beta function and $G_0(t,s)$ is defined by (4.2). So all conditions of Theorem 3.4 are satisfied. Thus, by Theorem 3.4, BVP (4.7) has at least one positive solution.

Example 4.3. Consider the following problem:

$$D_{0+}^{5/2}u(t) + \frac{t^2}{5}\ln(1+u) + \frac{te^t u'}{10+10e^t} + \frac{\sin t}{20} + \frac{1}{2} = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \beta[u'], \tag{4.12}$$

where $\beta[u'] = \int_0^1 u'(s)dA(s)$ with $A(s)$ as given by (4.8). Set

$$f(t,x,y) = \frac{t^2}{5}\ln(1+x) + \frac{te^t y}{10+10e^t} + \frac{\sin t}{20} + \frac{1}{2}, \qquad p_1(t) = \frac{t^2}{5},$$
$$p_2(t) = \frac{te^t}{10+10e^t}, \qquad q(t) = \frac{\sin t}{20} + \frac{1}{2}. \tag{4.13}$$

Then $f : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous and

$$f(t,x,y) \le p_1(t)x + p_2(t)y + q(t). \tag{4.14}$$

By Example 4.2, $\beta[\gamma] = \int_0^1 \gamma(t)dA(t) = 1/4 < 1$, $\mathcal{G}(s) = \int_0^1 G_0(t,s)dA(t) \ge 0$, where $G_0(t,s)$ is defined by (4.2). As in [1, 3], we also take $\xi = 1/4$, $\eta = 3/4$, then

$$L_1^{-1} = \left(\int_{1/4}^{3/4} \rho(s)G_0(s,s)\,ds\right)^{-1} \approx 13.6649,$$
$$L_2^{-1} = \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}\left(\int_0^1 G_0(s,s)\,ds\right)^{-1} = \frac{12}{7\sqrt{\pi}} \approx 0.967182. \tag{4.15}$$

Choosing $r_1 = 1/30$, $r_2 = 1$, we have

$$f(t,x,y) \le 0.85 \le L_2^{-1}r_2, \quad \text{for } (t,x,y) \in [0,1]\times[0,1]\times[0,1],$$
$$f(t,x,y) \ge 0.5 \ge L_1^{-1}r_1, \quad \text{for } (t,x,y) \in [0,1]\times\left[0,\frac{1}{30}\right]\times\left[0,\frac{1}{30}\right]. \tag{4.16}$$

So all conditions of Theorem 3.5 are satisfied. Thus, by Theorem 3.5, BVP (4.12) has at least one positive solution.

Example 4.4. Consider the following problem:

$$D_{0+}^{5/2}u(t) + f(t,u(t),u'(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \beta[u'], \tag{4.17}$$

where $\beta[u'] = \int_0^1 u'(s)dA(s)$ with $A(s)$ as given by (4.8). Set

$$f(t,x,y) = \begin{cases} \dfrac{t^2}{100}\ln(1+x) + 15y^2 + \dfrac{t}{1000}, & (t,x,y) \in [0,1]\times[0,1]\times[0,1], \\ \dfrac{t^2}{100}\ln(1+x) + \dfrac{29}{2} + \dfrac{1}{2}y + \dfrac{t}{1000}, & (t,x,y) \in [0,1]\times\left((\mathbb{R}^+)^2 \setminus [0,1]^2\right), \end{cases} \tag{4.18}$$

$$p_1(t) = \frac{t^2}{100}, \qquad p_2(t) = 15, \qquad q(t) = \frac{t}{1000} + \frac{29}{2}.$$

Then

$$f(t,x,y) \le p_1(t)x + p_2(t)y + q(t). \tag{4.19}$$

By Example 4.3, $\beta[\gamma] = \int_0^1 \gamma(t)dA(t) = 1/4 < 1$, $\mathcal{G}(s) = \int_0^1 G_0(t,s)dA(t) \ge 0$,

$$\begin{aligned} L_1^{-1} &= \left(\int_{1/4}^{3/4} \rho(s)G_0(s,s)ds\right)^{-1} \approx 13.6649, \\ L_2^{-1} &= \left(1 + \frac{\beta[1]}{1-\beta[\gamma]}\right)^{-1}\left(\int_0^1 G_0(s,s)ds\right)^{-1} = \frac{12}{7\sqrt{\pi}} \approx 0.967182. \end{aligned} \tag{4.20}$$

Choosing $a = 1/20$, $b = 1$, $c = 100$, we have

$$\begin{aligned} &f(t,x,y) \le 0.044845 < L_2^{-1}a \approx 0.048359, \quad \text{for } (t,x,y) \in [0,1]\times\left[0,\frac{1}{20}\right]\times\left[0,\frac{1}{20}\right], \\ &f(t,x,y) \ge 14.5025 \ge L_1^{-1}b \approx 13.6649, \quad \text{for } (t,x,y) \in \left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4},100\right]\times[1,100], \\ &f(t,x,y) \le 65.501 \le L_2^{-1}c \approx 96.7182, \quad \text{for } (t,x,y) \in [0,1]\times[0,100]\times[0,100]. \end{aligned} \tag{4.21}$$

So all conditions of Theorem 3.6 are satisfied. Thus, by Theorem 3.6, BVP (4.17) has at least three positive solutions u_1, u_2, and u_3, satisfying

$$\begin{aligned} &\max_{0\le t\le 1}\left|u_1'(t)\right| < \frac{1}{20}, \qquad 1 < \min_{1/4\le t\le 3/4}\left|u_2'(t)\right| < \max_{0\le t\le 1}\left|u_2'(t)\right| \le 100, \\ &\frac{1}{20} \le \max_{0\le t\le 1}\left|u_3'(t)\right| \le 100 \qquad \text{with } \min_{1/4\le t\le 3/4}\left|u_3'(t)\right| < 1. \end{aligned} \tag{4.22}$$

Acknowledgments

The first and second authors were supported financially by the National Natural Science Foundation of China (11071141, 11126231) and the Natural Science Foundation of Shandong Province of China (ZR2011AQ008, ZR2010AM017). The third author was supported financially by the Australia Research Council through an ARC Discovery Project grant.

References

[1] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[2] Z. Bai and T. Qiu, "Existence of positive solution for singular fractional differential equation," *Applied Mathematics and Computation*, vol. 215, no. 7, pp. 2761–2767, 2009.

[3] D. Băleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific Publishing, Hackensack, NJ, USA, 2012.

[4] D. Băleanu, O. G. Mustafa, and R. P. Agarwal, "Asymptotic integration of $(1 + \alpha)$-order fractional differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1492–1500, 2011.

[5] D. Băleanu, O. G. Mustafa, and R. P. Agarwal, "On L^p-solutions for a class of sequential fractional differential equations," *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 2074–2081, 2011.

[6] D. Băeanu, O. G. Mustafa, and D. O'Regan, "A Nagumo-like uniqueness theorem for fractional differential equations," *Journal of Physics A*, vol. 44, no. 39, Article ID 392003, 6 pages, 2011.

[7] D. Jiang and C. Yuan, "The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 2, pp. 710–719, 2010.

[8] S. Liang and J. Zhang, "Positive solutions for boundary value problems of nonlinear fractional differential equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5545–5550, 2009.

[9] X. Xu, D. Jiang, and C. Yuan, "Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 10, pp. 4676–4688, 2009.

[10] C. Yuan, "Multiple positive solutions for $(n - 1, 1)$-type semipositone conjugate boundary value problems of nonlinear fractional differential equations," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 36, pp. 1–12, 2010.

[11] S. Zhang, "Positive solutions to singular boundary value problem for nonlinear fractional differential equation," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1300–1309, 2010.

[12] Y. Zhao, S. Sun, Z. Han, and Q. Li, "The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 2086–2097, 2011.

[13] B. Ahmad and J. J. Nieto, "Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 494720, 9 pages, 2009.

[14] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.

[15] M. Benchohra, S. Hamani, and S. K. Ntouyas, "Boundary value problems for differential equations with fractional order and nonlocal conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2391–2396, 2009.

[16] C. S. Goodrich, "Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 191–202, 2011.

[17] G. M. N'Guérékata, "A Cauchy problem for some fractional abstract differential equation with non local conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 5, pp. 1873–1876, 2009.

[18] W. Jiang, "The existence of solutions to boundary value problems of fractional differential equations at resonance," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 5, pp. 1987–1994, 2011.

[19] C. F. Li, X. N. Luo, and Y. Zhou, "Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.

[20] M. El-Shahed and J. J. Nieto, "Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3438–3443, 2010.

[21] Y. Wang, L. Liu, and Y. Wu, "Positive solutions for a nonlocal fractional differential equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 11, pp. 3599–3605, 2011.

[22] X. Zhang, L. Liu, and Y. Wu, "The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8526–8536, 2012.

[23] X. Zhang, L. Liu, B. Wiwatanapataphee, and Y. Wu, "Positive solutions of Eigen-value problems for a class of fractional differential equations with derivatives," *Abstract and Applied Analysis*, vol. 2012, Article ID 512127, 16 pages, 2012.

[24] X. Zhang, L. Liu, and Y. Wu, "Multiple positive solutions of a singular fractional differential equation with negatively perturbed term," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1263–1274, 2012.

[25] Y. Wang, L. Liu, and Y. Wu, "Positive solutions of a fractional boundary value problem with changing sign nonlinearity," *Abstract and Applied Analysis*, vol. 2012, Article ID 149849, 12 pages, 2012.

[26] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[27] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.

[28] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[29] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1988.

[30] D. Guo, J. Sun, and Z. Liu, *Functional Method for Nonlinear Ordinary Differential Equation*, Shandong Science and Technology Press, Jinan, China, 1995.

[31] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.