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Research Article

Some Generalizations of Ulam-Hyers Stability Functional Equations to Riesz Algebras

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Badora (2002) proved the following stability result. Let ε and δ be nonnegative real numbers, then for every mapping f of a ring $\mathcal{R}$ onto a Banach algebra $\mathcal{B}$ satisfying $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon$ and $\|f(x \cdot y)-f(x)f(y)\| \leq \delta$ for all $x, y \in \mathcal{R}$, there exists a unique ring homomorphism $h : \mathcal{R} \to \mathcal{B}$ such that $\|f(x)-h(x)\| \leq \varepsilon, x \in \mathcal{R}$. Moreover, $b \cdot (f(x)-h(x)) = 0$, $(f(x)-h(x)) \cdot b = 0$, for all $x \in \mathcal{R}$ and all b from the algebra generated by $h(\mathcal{R})$. In this paper, we generalize Badora's stability result above on ring homomorphisms for Riesz algebras with extended norms.

1. Introduction

The approximation of solution of the Cauchy's equation $f(x+y) = f(x) + f(y)$ lying near to some solution has received a lot of attention from mathematicians in the areas of modern analysis and applied mathematics. Any solution f of this equation is called an *additive function*. Let E and E' be Banach spaces, and let δ be a positive number. A function f of E into E' is called δ-*additive* if $\|f(x+y)-f(x)-f(y)\| < \delta$ for all $x, y \in E$. In the 1940s, Ulam [1] proposed the following *stability problem* of this equation. Does there exist for each $\varepsilon > 0$ a $\delta > 0$ such that, to each δ-additive function f of E into E' there, corresponds an additive function l of E into E' satisfying the inequality $\|f(x)-l(x)\| \leq \varepsilon$ for each $x \in E$? In 1941, Hyers [2] answered this question in the affirmative way and showed that δ may be taken equal to ε. The answer of Hyers is presented in a great number of articles and books. There are several definitions and critics of the notion of this stability in the literature (see, e.g., [3, 4]). In 1949, Bourgin [5] generalized Hyers' results to the ring homomorphisms and proved the following.

Theorem 1.1. *Let ε and δ be nonnegative real numbers. Then every mapping f of a Banach algebra $\mathcal{A}$ with an identity element onto a Banach algebra $\mathcal{B}$ with an identity element satisfying*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \varepsilon, \\ \|f(x\cdot y) - f(x)f(y)\| &\leq \delta, \end{aligned} \tag{1.1}$$

for all $x, y \in \mathcal{A}$, is a ring homomorphism of $\mathcal{A}$ onto $\mathcal{B}$, that is, $f(x+y) = f(x) + f(y)$ and $f(x\cdot y) = f(x)f(y)$ for all $x, y \in \mathcal{A}$.

Finally, Badora [6] proved the following theorem on the Bourgin's result related to stability problem (Theorem 1.1) without additional assumptions.

Theorem 1.2. *Let $\mathcal{R}$ be a ring, let $\mathcal{B}$ be a Banach algebra, and let ε and δ be nonnegative real numbers. Assume that $f : \mathcal{R} \to \mathcal{B}$ satisfies (1.1) for all $x, y \in E$. Then there exists a unique ring homomorphism $h : \mathcal{R} \to \mathcal{B}$ such that*

$$\|f(x) - h(x)\| \leq \varepsilon, \quad x \in \mathcal{R}. \tag{1.2}$$

Moreover,

$$b \cdot \big(f(x) - h(x)\big) = 0, \qquad \big(f(x) - h(x)\big) \cdot b = 0, \tag{1.3}$$

for all $x \in \mathcal{R}$ and all b from the algebra generated by $h(R)$.

The present paper is in essence a revised and extended compilation of Hyers' result and Theorem 1.2 to the Riesz algebras with extended norms. After outlining the basic information on Riesz space theory, we present the main definitions and facts concerning approximate Riesz algebra (with an extended norm)-valued ring homomorphisms.

2. Preliminaries

A real Banach space $F = (F, +, \cdot, \|\cdot\|)$ endowed with a (partial) order $\leq$ is called a *Banach lattice* whenever

(1) the order $\leq$ agrees with the linear operations, that is, $x \leq y \Rightarrow \alpha x + z \leq \alpha y + z$ for all $z \in F$ and $0 \leq \alpha \in \mathbb{R}$;

(2) the order $\leq$ makes F a lattice, that is, for all $x, y \in F$, the supremum $x \vee y$ and infimum $x \wedge y$ exist in F (hence, the modulus $|x| := x \vee (-x)$ exists for each $x \in F$);

(3) the norm $\|\cdot\|$ is monotonous with respect to the order $\leq$, that is, for all $x, y \in F$, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ (hence, $\|x\| = \||x|\|$ for all $x \in F$).

Recall that a (partially) ordered vector space F satisfying (1) and (2) above is called a *Riesz space*. $C(K)$ the spaces of real valued continuous functions on a compact Hausdorff space K, l_p-spaces, c the spaces of convergent sequences, and c_0 the spaces of sequences converging to zero are natural examples of Riesz spaces under the pointwise ordering. A Riesz space F is called *Archimedean* if $0 \leq u, v \in F$, and $nu \leq v$ for each $n \in \mathbb{N}$ imply $u = 0$. Throughout the present paper, all the Riesz spaces are assumed to be Archimedean. A subset S in a Riesz space F is said to be *solid* if it follows from $|u| \leq |v|$ in F and $v \in S$ that $u \in S$.

A solid linear subspace of a Riesz space F is called an *ideal*. Every subset D of a Riesz space F is included in a smallest ideal F_D, called *ideal generated by* D. A *principal ideal* of a Riesz space F is any ideal generated by a singleton $\{u\}$. This ideal will be denoted by I_u. It is easy to see that

$$I_u = \{v \in F : \lambda \geq 0 \text{ such that } |v| \leq \lambda|u|\}. \tag{2.1}$$

We assume that u is a fixed positive element in the Riesz space F. First of all, we present the following definition.

Definition 2.1. (1) It is said that the sequence (x_n) in F converges u-uniformly to the element $x \in F$ whenever, for every $\varepsilon > 0$, there exists n_0 such that $|x_{n_0+k} - x| \leq \varepsilon u$ holds for each k.

(2) It is said that the sequence (x_n) in F converges *relatively uniformly* to x whenever x_n converges u-uniformly to x for some $0 \leq u \in F$.

When dealing with relative uniform convergence in an Archimedean Riesz space F, it is natural to associate with every positive element $u \in F$ an extended norm $\|\cdot\|_u$ in F by the formula

$$\|x\|_u = \inf\{\lambda \geq 0 : |x| \leq \lambda u\} \quad (x \in F). \tag{2.2}$$

Note that $\|x\|_u < \infty$ if and only if $x \in I_u$, the ideal generated by u. Also $|x| \leq \delta u$ if and only if $\|x\|_u \leq \delta$.

The sequence (x_n) in F is called an extended u-normed Cauchy sequence, if for every $\varepsilon > 0$ there exists k such that $\|x_{n+k} - x_{m+k}\|_u < \varepsilon$ for all m, n. If every extended u-normed Cauchy sequence is convergent in F, then F is called an extended u-normed Banach lattice.

A Riesz space F is called a *Riesz algebra* or a *lattice-ordered algebra* if there exists in F an associative multiplication with the usual algebra properties such that $uv \geq 0$ for all $0 \leq u, v \in F$.

For more detailed information about Riesz spaces, the reader can consult the book "*Riesz Spaces*" by Luxemburg and Zaanen [7].

3. Main Results

We begin with the following theorem concerning stability of the functional equation $HofoG = f$. For a function $G : K \to K$, let us denote by $G^0(x) = x$ for $x \in K$, G^2 the composition of G by itself and in general let $G^i = Go(G^{i-1})$ for $i = 1, 2, \ldots$.

The theorem can easily be obtained from [8] or [9]. We give the proof here for the benefit of the reader.

Theorem 3.1. *Let (Y, d) be a complete metric space, K a nonempty set and $\lambda \in [0, \infty)$ such that $G : K \to K$ and $H : Y \to Y$ are two given functions. Assume that $f : K \to Y$ is a function satisfying*

$$d\big(HofoG(x), f(x)\big) \leq h(x), \tag{3.1}$$

for each $x \in K$ and for some function $h : K \to [0, \infty)$. If the function $H : Y \to Y$ satisfies the inequality

$$d(H(u), H(v)) \leq \lambda d(u, v), \quad u, v \in Y, \tag{3.2}$$

and the series

$$\sum_{i=0}^{\infty} \lambda^i h\left(G^i(x)\right) \tag{3.3}$$

is convergent for each $x \in K$, then for each integer n, one has

(1)

$$d\left(H^{n+1} o f o G^{n+1}(x), H^n o f o G^n(x)\right) \leq \lambda^n h(G^n(x)), \quad x \in K, \tag{3.4}$$

(2) $H^n o f o G^n(x), (x \in K)$ is a Cauchy sequence. $F(x) = \lim_{n\to\infty} H^n o f o G^n(x)$ exists for every $x \in K$, and $F : K \to Y$ is the unique function satisfying $HoFoG = F$ and the inequality

$$d(F(x), f(x)) \leq \sum_{i=0}^{\infty} \lambda^i h\left(G^i(x)\right). \tag{3.5}$$

Proof. (1) Replacing x by $G(x)$ in (3.1), we get

$$d\left(H o f o G^2(x), f(G(x))\right) \leq h(G(x)). \tag{3.6}$$

Then by (3.2), we obtain

$$d\left(H^2 o f o G^2(x), H o f o G(x)\right) \leq \lambda d\left(H o f o G^2(x), f o G(x)\right) \leq \lambda h(G(x)). \tag{3.7}$$

The proof follows by induction.

(2) Let $m > n$, then

$$\begin{aligned} d(H^n o f o G^n(x), H^m o f o G^m(x)) &\leq \sum_{i=n}^{m-1} d\left(H^{i+1} o f o G^{i+1}(x), H^i o f o G^i(x)\right) \\ &\leq \sum_{i=n}^{m-1} \lambda^i h\left(G^i(x)\right), \end{aligned} \tag{3.8}$$

thus $H^n o f o G^n(x)$ is a Cauchy sequence for each $x \in K$ and it is convergent as (Y, d) is complete. Let $F(x) = \lim_{n\to\infty} H^n o f o G^n(x)$ for each $x \in K$.

By using (3.4), we get

$$\begin{aligned} d(H^n o f o G^n(x), f(x)) &\leq \sum_{i=1}^{n} d\left(H^i o f o G^i(x), H^{i-1} o f o G^{i-1}(x)\right) \\ &\leq \sum_{i=1}^{n} \lambda^{i-1} h\left(G^{i-1}(x)\right) = \sum_{i=0}^{n} \lambda^i h\left(G^i(x)\right), \end{aligned} \tag{3.9}$$

taking the limit as n goes to infinity, then we obtain (3.5). By continuity of H, we have

$$\begin{aligned} HoFoG(x) = H\left[\lim_{n\to\infty} H^n o f o G^n(G(x))\right] &= \lim_{n\to\infty} HoH^n o f o G^n(G(x)) \\ &= \lim_{n\to\infty} H^{n+1} o f o G^{n+1}(x) = F(x). \end{aligned} \tag{3.10}$$

Suppose that another function $\overline{F}: K \to Y$ satisfies $Ho\overline{F}oG(x) = \overline{F}$ and (3.5). By induction it is easy to show that $H^n oFoG^n(x) = F$ and $H^n o\overline{F}oG^n(x) = \overline{F}$. Hence for $x \in K$,

$$\begin{aligned} d\left(\overline{F}(x), F(x)\right) &= d\left(H^n o\overline{F}oG^n(x), H^n oFoG^n(x)\right) \le \lambda^n d\left(\overline{F}oG^n(x), FoG^n(x)\right) \\ &\le \lambda^n d\left(\overline{F}oG^n(x), foG^n(x)\right) + \lambda^n d\left(foG^n(x), FoG^n(x)\right) \\ &\le 2\sum_{i=n}^{\infty}\lambda^n h\left(G^i(x)\right). \end{aligned} \tag{3.11}$$

Since for every $x \in K$, $\sum_{i=n}^{\infty} h(G^i(x)) \to 0$ with $n \to \infty$, this completes the proof. □

Let Y be a linear space over either complex or real numbers. The operation of addition of elements $x, y \in Y$ will be denoted, as usual, by $x + y$. The operation of multiplication of an element $x \in Y$ by a scalar t will be denoted by tx. Suppose that in the linear space Y, we are given a metric d. The space (Y, d) is called a *metric linear space* if the operations of addition and multiplication by numbers are continuous with respect to the metric d. A metric linear space (Y, d) is called *complete* if every Cauchy sequence (x_n) converges to an element $x_0 \in Y$, that is, $\lim_{n\to\infty} d(x_n, x_0) = 0$.

We now give the following corollary in [9] which will be useful in the sequel.

Corollary 3.2. *Let (Y, d) a complete metric linear space and K be a linear space. Suppose that there exists $\xi, \eta \in [0, \infty)$ such that $\xi\eta < 1$,*

$$d\left(\frac{1}{2}x, \frac{1}{2}y\right) \le \xi d(x, y) \quad \text{for } x, y \in Y, \tag{3.12}$$

$$\chi(2x, 2y) \le \eta\chi(x, y) \quad \text{for } x, y \in K, \tag{3.13}$$

where $\chi: K \times K \to [0, \infty)$. Let $\varphi: K \to Y$ satisfy

$$d(\varphi(x + y), \varphi(x) + \varphi(y)) \le \chi(x, y) \quad \text{for } x, y \in K. \tag{3.14}$$

Then there is a unique solution $F: K \to Y$ of $F(x + y) = F(x) + F(y)$ with

$$d(\varphi(x), F(x)) \le \frac{\xi\chi(x, x)}{1 - \xi\eta} \quad \text{for } x \in K. \tag{3.15}$$

Proof. From (3.12) and (3.14), we get

$$d\left(\frac{1}{2}\varphi(2x), \varphi(x)\right) \le \xi d(\varphi(2x), 2\varphi(x)) \le \xi\chi(x, x) \tag{3.16}$$

for $x \in K$. By using Theorem 3.1 with $f = \varphi, H(z) = (1/2)z, G(x) = 2x, \lambda = \xi$, and $h(x) = \xi\chi(x,x)$, the limit function $F(x)$ exists for each $x \in K$ and

$$d(f(x), F(x)) \leq \sum_{i=0}^{\infty} \xi^i h\Big(G^i(x)\Big). \tag{3.17}$$

As $G^i(x) = 2^i x$ and $h(G^i(x)) \leq \xi\eta^i\chi(x,x)$ for every $x \in K$, we get

$$d(\varphi(x), F(x)) = d(f(x), F(x)) \leq \xi\chi(x,x)\sum_{i=0}^{\infty}(\xi\eta)^i = \frac{\xi\chi(x,x)}{1-\xi\eta}. \tag{3.18}$$

Next, by (3.14), for every $x, y \in K$ we have

$$d\left(\frac{1}{2^n}\varphi(2^n(x+y)), \frac{1}{2^n}\varphi(2^n x) + \frac{1}{2^n}\varphi(2^n x)\right) \leq \eta^n \chi(x,y) \tag{3.19}$$

for $n \in \mathbb{N}$, so letting $n \to \infty$ we obtain $F(x+y) = F(x) + F(y)$.

Suppose $F_0 : K \to Y$ is also a solution of $F(x+y) = F(x) + F(y)$ and

$$d(\varphi(x), F_0(x)) \leq \sum_{i=0}^{\infty} \xi^i h\Big(G^i(x)\Big), \quad \text{for every} \quad x \in K. \tag{3.20}$$

Then $HoF_0oG = F_0$, whence, by Theorem 3.1, we have $F = F_0$ which implies the uniqueness of F. □

The following theorem is an extended application of Hyers' result to the Riesz spaces.

Theorem 3.3. *Let E a linear space, F be a Riesz space equipped with an extended norm $\|\cdot\|_u$ such that the space $(F, \|\cdot\|_u)$ is complete. If, for some $\delta > 0$, a map $f : E \to (F, \|\cdot\|_u)$ is δ-additive, then limit $l(x) = lim_{n\to\infty} f(2^n x)/2^n$ exists for each $x \in E$. $l(x)$ is the unique additive function satisfying the inequality $\|f(x) - l(x)\|_u \leq \delta$ for all $x \in E$.*

Now, if F is a Banach space or extended u-normed Banach lattice, then we can take $d(x,y) = \|x-y\|$ or $d(x,y) = \|x-y\|_u$, $\chi(x,x) = \varepsilon$, $\xi = 1/2$, and $\eta = 1$. We may obtain the classical Hyers' result [2] and Theorem 3.3 with such $\chi(x,x)$, ξ and η by using Corollary 3.2.

Finally, we give the following theorem which is an extended application of Badora's result (Theorem 1.2) to Riesz algebras with extended norms. For a proof, we use Theorem 3.3 and the similar techniques of Badora [6] with suitable modifications.

Theorem 3.4. *Let E be a linear algebra, and let F be a Riesz algebra with an extended norm $\|\cdot\|_u$ such that $(F, \|\cdot\|_u)$ is complete. Also, let $\|\cdot\|_v$ be another extended norm in F weaker than $\|\cdot\|_u$ such that whenever*

(1) *$x_n \to x$ and $x_n \cdot y \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$;*

(2) *$y_n \to y$ and $x \cdot y_n \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$.*

Let ε and δ be nonnegative real numbers. Assume that a map $f : E \to F$ satisfies

$$\|f(x+y) - f(x) - f(y)\|_u \leq \varepsilon, \tag{3.21}$$

$$\|f(x \cdot y) - f(x)f(y)\|_v \leq \delta \tag{3.22}$$

for all $x, y \in E$. Then there exists a unique ring homomorphism $h : E \to F$ such that $\|f(x) - h(x)\|_u \leq \varepsilon$, $x \in E$. Moreover,

$$b \cdot (f(x) - h(x)) = 0, \qquad (f(x) - h(x)) \cdot b = 0, \tag{3.23}$$

for all $x \in E$ and all b from the algebra generated by $h(E)$.

Proof. From Theorem 3.3, it follows that there exists a unique additive function $h : E \to F$ such that

$$\|f(x) - h(x)\|_u \leq \varepsilon, \quad (x \in E). \tag{3.24}$$

Hence, it is enough to show that h is a multiplicative function. Using the additivity of h, it follows that

$$\left\|\frac{1}{n}f(nx) - h(x)\right\|_u \leq \frac{1}{n}\varepsilon, \quad (x \in E, n \in \mathbb{N}), \tag{3.25}$$

which means that

$$h(x) = \lim_{n \to \infty} \frac{1}{n} f(nx), \quad (x \in E), \tag{3.26}$$

with respect to $\|\cdot\|_u$ norm.

Let

$$r(x, y) = f(x \cdot y) - f(x)f(y), \quad (x, y \in E). \tag{3.27}$$

Then using inequality (3.22), we get

$$\lim_{n \to \infty} \frac{1}{n} r(nx, y) = 0, \quad (x, y \in E), \tag{3.28}$$

with respect to $\|\cdot\|_v$ norm.

Applying (3.26) and (3.28), we have

$$\begin{aligned} h(x \cdot y) &= \lim_{n \to \infty} \frac{1}{n} f(n(x \cdot y)) = \lim_{n \to \infty} \frac{1}{n} f((nx) \cdot y) \\ &= \lim_{n \to \infty} \frac{1}{n} (f(nx)f(y) + r(nx, y)) = h(x)f(y), \end{aligned} \tag{3.29}$$

for all $x, y \in E$, since $\|\cdot\|_v$ is weaker than $\|\cdot\|_u$. Hence, we get the following functional equation:

$$h(x \cdot y) = h(x)f(y), \quad (x, y \in E). \tag{3.30}$$

From this equation and the additivity of h, we have

$$\begin{aligned} h(x)f(ny) &= h(x\cdot(ny)) = h((nx)\cdot y) = h(nx)f(y) \\ &= nh(x)f(y), \quad (x,y\in E, n\in\mathbb{N}). \end{aligned} \tag{3.31}$$

Therefore,

$$h(x)\frac{1}{n}f(ny) = h(x)f(y), \quad (x,y\in E, n\in\mathbb{N}). \tag{3.32}$$

Sending n to infinity, by (3.26), we see that

$$h(x)h(y) = h(x)f(y), \quad (x,y\in E). \tag{3.33}$$

Combining this equation with (3.30), we see that h is a multiplicative function.

Moreover, from (3.22) we get

$$\lim_{n\to\infty}\frac{1}{n}r(x,ny) = 0, \quad (x,y\in E), \tag{3.34}$$

with respect to $\|\cdot\|_v$ norm.

Thus, by (3.26) and the fact that $\|\cdot\|_v$ is weaker than $\|\cdot\|_u$, we get that

$$\begin{aligned} h(x\cdot y) &= \lim_{n\to\infty}\frac{1}{n}f(n(x\cdot y)) = \lim_{n\to\infty}\frac{1}{n}f(x\cdot(ny)) \\ &= \lim_{n\to\infty}\left(f(x)\frac{1}{n}f(ny) + \frac{1}{n}r(x,ny)\right) = f(x)h(y), \end{aligned} \tag{3.35}$$

for all $x,y\in E$. Hence, by (3.33),

$$f(x)h(y) = h(x\cdot y) = h(x)h(y) = h(x)f(y), \quad (x,y\in E), \tag{3.36}$$

so that

$$h(x)(f(y)-h(y)) = 0, \qquad (f(x)-h(x))h(y) = 0, \quad (x,y\in E) \tag{3.37}$$

which completes the proof. □

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