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Research Article

Univalence Criteria for Two Integral Operators

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We study the univalence conditions for two integral operators to be univalent in the open unit disk. Many known univalence conditions are written to prove our main results.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}, \tag{1.2}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0. \tag{1.3}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.

In [1], for $0 < b \leq 1$, Silverman considered the class:

$$G_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right|, \quad z \in \mathbb{U} \right\}. \tag{1.4}$$

Here, in our present investigation, we consider two general families of integral operators:

$$I(f_1,\dots,f_n;\ g_1,\dots,g_n)(z)=\left(\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{(\alpha_i-1)/M_i}(g_i'(t))^{\gamma_i}\,dt\right)^{1/\beta}, \tag{1.5}$$

$\alpha_i,\gamma_i\in\mathbb{C};\ \beta\in\mathbb{C}\setminus\{0\};\ f_i,g_i\in\mathcal{A},\ M_i\geq 1$ for all $i\in\{1,2,\dots,n\}$;

$$J(f_1,\dots,f_n;g_1,\dots,g_n)(z)=\left(\left(1+\sum_{i=1}^{n}\alpha_i\right)\int_0^z\prod_{i=1}^{n}(f_i(t))^{\alpha_i}(g_i'(t))^{\gamma_i}\,dt\right)^{1/(1+\sum_{i=1}^{n}\alpha_i)}, \tag{1.6}$$

$\alpha_i,\gamma_i\in\mathbb{C};\ f_i,g_i\in\mathcal{A}$ for all $i\in\{1,2,\dots,n\}$.

Many authors have studied the problem of integral operators which preserve the class $\mathcal{S}$ (see, e.g., [2–5]).

In the present paper, we study the univalence conditions involving the general families of integral operators defined by (1.5) and (1.6).

In the proof of our main results (Theorem 2.1 and Theorem 3.1), we need the following univalence criterion. The univalence criterion, asserted by Theorem 1.1, is a generalization of Ahlfor's and Becker's univalence criterion; it was proven by Pescar [6].

Theorem 1.1 (see Pescar [6]). *Let $\beta\in\mathbb{C}$ with $\operatorname{Re}\beta>0$, $c\in\mathbb{C}$ with $|c|\leq 1$, $c\neq -1$. If $f\in\mathcal{A}$ satisfies*

$$\left|c|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right|\leq 1, \tag{1.7}$$

for all $z\in\mathbb{U}$, then the integral operator,

$$F_\beta(z)=\left(\beta\int_0^z t^{\beta-1}f'(t)\,dt\right)^{1/\beta}, \tag{1.8}$$

is in the class $\mathcal{S}$.

Finally, in the present investigation, one also needs the familiar Schwarz Lemma (see, for details, [7]).

Lemma 1.2 ((General Schwarz Lemma) (see [7])). *Let the function f be regular in the disk $\mathbb{U}_R=\{z\in\mathbb{C}:|z|<R\}$, with $|f(z)|<M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z=0$, then*

$$|f(z)|\leq\frac{M}{R^m}|z|^m\quad(z\in\mathbb{U}_R). \tag{1.9}$$

The equality can hold only if

$$f(z)=e^{i\theta}\frac{M}{R^m}z^m, \tag{1.10}$$

where θ is constant.

2. Univalence Conditions for $I(f_1,\dots,f_n;g_1,\dots,g_n)(z)$

Theorem 2.1. *Let $M_i \geq 1$ ($i \in \{1,2,\dots,n\}$) and β, α_i, γ_i be complex numbers with $\operatorname{Re}\beta \geq 0$ and*

$$\operatorname{Re}\beta \geq \sum_{i=1}^{n}\left[3|\alpha_i - 1| + |\gamma_i|(2b_i + 1)\right], \tag{2.1}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\left[3|\alpha_i - 1| + |\gamma_i|(2b_i + 1)\right]. \tag{2.2}$$

If for all $i \in \{1,2,\dots,n\}$, $f_i \in \mathcal{A}$ satisfy the conditions:

$$|f_i(z)| \leq M_i \quad (z \in \mathbb{U}), \qquad \left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| \leq \frac{2M_i - 1}{M_i} \quad (z \in \mathbb{U}), \tag{2.3}$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left|\frac{z g_i'(z)}{g_i(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{2.4}$$

then the integral operator $I(f_1,\dots,f_n;g_1,\dots,g_n)(z)$ defined by (1.5) is in the class $\mathcal{S}$.

Proof. We begin by setting

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{(\alpha_i - 1)/M_i} (g_i'(t))^{\gamma_i}\, dt, \tag{2.5}$$

and then we calculate for $h(z)$ the derivates of the first and second orders.

From (2.5), we obtain

$$\begin{aligned} h'(z) &= \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{(\alpha_i - 1)/M_i} (g_i'(z))^{\gamma_i}, \\ \frac{z h''(z)}{h'(z)} &= \sum_{i=1}^{n}\left[\frac{\alpha_i - 1}{M_i}\left(\frac{z f_i'(z)}{f_i(z)} - 1\right) + \gamma_i \frac{z g_i''(z)}{g_i'(z)}\right] \\ &= \sum_{i=1}^{n}\left[\frac{\alpha_i - 1}{M_i}\left(\frac{z f_i'(z)}{f_i(z)} - 1\right)\right. \\ &\quad \left. + \gamma_i\left(\frac{z g_i''(z)}{g_i'(z)} - \frac{z g_i'(z)}{g_i(z)} + 1\right) + \gamma_i\left(\frac{z g_i'(z)}{g_i(z)} - 1\right)\right]. \end{aligned} \tag{2.6}$$

Thus, we have

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\left[\frac{|\alpha_i-1|}{M_i}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)\right.\\ &\qquad \left.+|\gamma_i|\left|\frac{zg_i''(z)}{g_i'(z)}-\frac{zg_i'(z)}{g_i(z)}+1\right|+|\gamma_i|\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right]\\ &\leq \sum_{i=1}^{n}\left[\frac{|\alpha_i-1|}{M_i}\left(\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|+1\right)\left|\frac{f_i(z)}{z}\right|+1\right)\right.\\ &\qquad \left.+|\gamma_i|\left|\frac{zg_i''(z)}{g_i'(z)}-\frac{zg_i'(z)}{g_i(z)}+1\right|+|\gamma_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right].\end{aligned} \tag{2.7}$$

From the hypothesis (2.3) of Theorem 2.1, we have

$$\begin{aligned}&\left|f_i(z)\right|\leq M_i \quad (z\in\mathbb{U};\ M_i\geq 1),\\ &\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|\leq\frac{2M_i-1}{M_i} \quad (z\in\mathbb{U};\ M_i\geq 1)\end{aligned} \tag{2.8}$$

for all $i\in\{1,2,\ldots,n\}$.

By applying the General Schwarz Lemma, we thus obtain

$$\left|f_i(z)\right|\leq M_i|z| \quad (z\in\mathbb{U};\ i\in\{1,2,\ldots,n\}). \tag{2.9}$$

Since $g_i\in G_{b_i}$, $0<b_i\leq 1$ for all $i\in\{1,2,\ldots,n\}$, from (1.4), (2.4), we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\left[3|\alpha_i-1|+|\gamma_i|b_i\left|\frac{zg_i'(z)}{g_i(z)}\right|+|\gamma_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]\\ &\leq \sum_{i=1}^{n}\left[3|\alpha_i-1|+|\gamma_i|b_i\left(\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+1\right)+|\gamma_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]\\ &\leq \sum_{i=1}^{n}\left[3|\alpha_i-1|+|\gamma_i|b_i\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\gamma_i|b_i+|\gamma_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right]\\ &\leq \sum_{i=1}^{n}\left[3|\alpha_i-1|+(|\gamma_i|b_i+|\gamma_i|)\left|\frac{zg_i'(z)}{g_i(z)}-1\right|+|\gamma_i|b_i\right]\\ &\leq \sum_{i=1}^{n}\left[3|\alpha_i-1|+|\gamma_i|(2b_i+1)\right],\end{aligned} \tag{2.10}$$

which readily shows that

$$\begin{aligned}\left|c|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| &\leq |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[3|\alpha_i-1|+\left|\gamma_i\right|(2b_i+1)\right]\\ &\leq |c|+\frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\left[3|\alpha_i-1|+\left|\gamma_i\right|(2b_i+1)\right]\\ &\leq 1,\end{aligned} \tag{2.11}$$

where we have also used the hypothesis (2.2) of Theorem 2.1.

Finally, by applying Theorem 1.1, we conclude that the integral operator $I(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$ defined by (1.5) is in the class $\mathcal{S}$. This evidently completes the proof of Theorem 2.1. □

Setting $\alpha_1=\alpha_2=\cdots=\alpha_n=1$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $M_i\geq 1$ $(i\in\{1,2,\ \ldots,n\})$ and β, γ_i be complex numbers with $\operatorname{Re}\beta\geq 0$ and*

$$\operatorname{Re}\beta\geq\sum_{i=1}^{n}\left[\left|\gamma_i\right|(2b_i+1)\right] \tag{2.12}$$

and let $c\in\mathbb{C}$ be such that

$$|c|\leq 1-\frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\left[\left|\gamma_i\right|(2b_i+1)\right]. \tag{2.13}$$

If for all $i\in\{1,2,\ldots,n\}$, $g_i\in G_{b_i}$, $0<b_i\leq 1$ with

$$\left|\frac{zg_i'(z)}{g_i(z)}-1\right|<1\quad(z\in\mathbb{U}), \tag{2.14}$$

then the integral operator,

$$I(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)=\left(\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(g_i'(t)\right)^{\gamma_i}dt\right)^{1/\beta}, \tag{2.15}$$

is in the class $\mathcal{S}$.

Setting $\gamma_i=1$ for all $i\in\{1,2,\ldots,n\}$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $M_i\geq 1$ $(i\in\{1,2,\ldots,n\})$ and β, α_i be complex numbers with $\operatorname{Re}\beta\geq 0$ and*

$$\operatorname{Re}\beta\geq\sum_{i=1}^{n}\left[3|\alpha_i-1|+(2b_i+1)\right], \tag{2.16}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}[3|\alpha_i - 1| + (2b_i + 1)]. \tag{2.17}$$

If for all $i \in \{1, 2, \ldots, n\}$, $f_i \in \mathcal{A}$ *satisfy the conditions:*

$$\left|f_i(z)\right| \leq M_i \quad (z \in \mathbb{U}), \qquad \left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| \leq \frac{2M_i - 1}{M_i} \quad (z \in \mathbb{U}) \tag{2.18}$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ *with*

$$\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{2.19}$$

then the integral operator

$$I(f_1, \ldots, f_n; g_1, \ldots, g_n)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{(\alpha_i - 1)/M_i} (g_i'(t))\, dt\right)^{1/\beta} \tag{2.20}$$

is in the class $\mathcal{S}$.

Setting $n = 1$ in Theorem 2.1, we have the following result.

Corollary 2.4. *Let* $M \geq 1$ *and* β, α, γ *be complex numbers with* $\operatorname{Re}\beta \geq 0$ *and*

$$\operatorname{Re}\beta \geq \left[3|\alpha - 1| + \left|\gamma\right|(2b + 1)\right], \tag{2.21}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}\left[3|\alpha - 1| + \left|\gamma\right|(2b + 1)\right]. \tag{2.22}$$

If the function $f \in \mathcal{A}$ *satisfies the conditions:*

$$\left|f(z)\right| \leq M \quad (z \in \mathbb{U}), \qquad \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq \frac{2M - 1}{M} \quad (z \in \mathbb{U}), \tag{2.23}$$

and $g \in G_b$, $0 < b \leq 1$ *with*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{2.24}$$

then the integral operator,

$$I(f;g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{(\alpha-1)/M} (g'(t))^{\gamma}\, dt\right)^{1/\beta}, \tag{2.25}$$

is in the class $\mathcal{S}$.

3. Univalence Conditions for $J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$

Theorem 3.1. *Let* $M_i \geq 1$ *(*$i \in \{1,2,\ldots,n\}$*) and* β, α_i, γ_i *be complex numbers,* $\beta = (1 + \sum_{i=1}^{n} \alpha_i)$, $\operatorname{Re}\beta \geq 0$ *and*

$$\operatorname{Re}\beta \geq \sum_{i=1}^{n} \left[|\alpha_i| + |\gamma_i|(b_i+1)(2M_i+1) + |\gamma_i| b_i\right], \tag{3.1}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta} \sum_{i=1}^{n} \left[|\alpha_i| + |\gamma_i|(b_i+1)(2M_i+1) + |\gamma_i| b_i\right]. \tag{3.2}$$

If for all $i \in \{1,2,\ldots,n\}$, $f_i \in \mathcal{A}$ *satisfy the condition:*

$$\left|\frac{z f_i'(z)}{f_i(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{3.3}$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ *with*

$$\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{3.4}$$

$$|g_i(z)| \leq M_i \quad (z \in \mathbb{U};\ i \in \{1,2,\ldots,n\}), \tag{3.5}$$

then the integral operator $J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$ *defined by* (1.6) *is in the class* $\mathcal{S}$.

Proof. We begin by observing that the integral operator $J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$ defined by (1.6) can be rewritten as follows:

$$J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z) = \left(\left(1 + \sum_{i=1}^{n} \alpha_i\right) \int_0^z t^{\sum_{i=1}^{n}\alpha_i} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (g_i'(t))^{\gamma_i}\, dt\right)^{1/(1+\sum_{i=1}^{n}\alpha_i)}, \tag{3.6}$$

where $f_i \in \mathcal{A}$ for all $i \in \{1,2,\ldots,n\}$.

Defining the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (g_i'(t))^{\gamma_i}\, dt, \tag{3.7}$$

we take the same steps as in the proof of Theorem 2.1, and we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n}\left[|\alpha_i|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left(|\gamma_i|b_i + |\gamma_i|\right)\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + |\gamma_i|b_i\right]. \tag{3.8}$$

Thus, we have

$$\begin{aligned}&\left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \\ &\quad\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\alpha_i| + \left(|\gamma_i|b_i + |\gamma_i|\right)\left(\left|\frac{z^2 g_i'(z)}{g_i^2(z)}\right|\left|\frac{g_i(z)}{z}\right| + 1\right) + |\gamma_i|b_i\right].\end{aligned} \tag{3.9}$$

Furthermore, from the hypothesis (3.4) of Theorem 3.1, we have

$$\begin{aligned}&\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| < 1 \quad (z \in \mathbb{U}),\\ &\left|g_i(z)\right| \leq M_i \quad (z \in \mathbb{U};\ i \in \{1,2,\ldots,n\}).\end{aligned} \tag{3.10}$$

By applying the General Schwarz Lemma, we obtain

$$\left|g_i(z)\right| \leq M_i|z| \quad (z \in \mathbb{U};\ i \in \{1,2,\ldots,n\}). \tag{3.11}$$

So, we obtain

$$\begin{aligned}&\left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \\ &\quad\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\alpha_i| + \left(|\gamma_i|b_i + |\gamma_i|\right)\left(\left(\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| + 1\right)M_i + 1\right) + |\gamma_i|b_i\right] \\ &\quad\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\alpha_i| + \left(|\gamma_i|b_i + |\gamma_i|\right)(2M_i + 1) + |\gamma_i|b_i\right] \\ &\quad\leq |c| + \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\left[|\alpha_i| + \left(|\gamma_i|b_i + |\gamma_i|\right)(2M_i + 1) + |\gamma_i|b_i\right] \\ &\quad\leq 1.\end{aligned} \tag{3.12}$$

Finally, by applying Theorem 1.1, we conclude that the integral operator $J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$ defined by (1.6) is in the class $\mathcal{S}$. This evidently completes the proof of Theorem 3.1. □

Setting $\alpha_1=\alpha_2=\cdots=\alpha_n=1$ in Theorem 3.1, we have

Corollary 3.2. *Let $M_i\geq 1$ $(i\in\{1,2,\ldots,n\})$ and β, γ_i be complex numbers, $\beta=(1+n)$, $\operatorname{Re}\beta\geq 0$ and*

$$\operatorname{Re}\beta\geq\sum_{i=1}^{n}\left[1+\left|\gamma_i\right|(b_i+1)(2M_i+1)+\left|\gamma_i\right|b_i\right], \tag{3.13}$$

and let $c\in\mathbb{C}$ be such that

$$|c|\leq 1-\frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\left[1+\left|\gamma_i\right|(b_i+1)(2M_i+1)+\left|\gamma_i\right|b_i\right]. \tag{3.14}$$

If for all $i\in\{1,2,\ldots,n\}$, $f_i\in\mathcal{A}$ satisfy the condition:

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|<1\quad(z\in\mathbb{U}), \tag{3.15}$$

and $g_i\in G_{b_i}$, $0<b_i\leq 1$ with

$$\left|\frac{z^2g_i'(z)}{g_i^2(z)}-1\right|<1\quad(z\in\mathbb{U}),\qquad \left|g_i(z)\right|\leq M_i\quad(z\in\mathbb{U}), \tag{3.16}$$

then the integral operator,

$$J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)=\left((1+n)\int_0^z\prod_{i=1}^{n}\left(f_i(t)\right)\left(g_i'(t)\right)^{\gamma_i}dt\right)^{1/(1+n)}, \tag{3.17}$$

is in the class $\mathcal{S}$.

Setting $\gamma_i=1$ for all $i\in\{1,2,\ldots,n\}$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let $M_i\geq 1$ $(i\in\{1,2,\ldots,n\})$ and β, α_i be complex numbers, $\beta=(1+\sum_{i=1}^{n}\alpha_i)$, $\operatorname{Re}\beta\geq 0$ and*

$$\operatorname{Re}\beta\geq\sum_{i=1}^{n}\left[|\alpha_i|+(b_i+1)(2M_i+1)+b_i\right], \tag{3.18}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}[|\alpha_i| + (b_i+1)(2M_i+1) + b_i]. \tag{3.19}$$

If for all $i \in \{1,2,\ldots,n\}$, $f_i \in \mathcal{A}$ *satisfy the condition:*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{3.20}$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ *with*

$$\left|\frac{z^2 g_i'(z)}{g_i^2(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \qquad \left|g_i(z)\right| \leq M_i \quad (z \in \mathbb{U}), \tag{3.21}$$

then the integral operator,

$$J(f_1,\ldots,f_n;g_1,\ldots,g_n)(z) = \left(\left(1+\sum_{i=1}^{n}\alpha_i\right)\int_0^z \prod_{i=1}^{n}(f_i(t))^{\alpha_i}(g_i'(t))\,dt\right)^{1/(1+\sum_{i=1}^{n}\alpha_i)}, \tag{3.22}$$

is in the class $\mathcal{S}$.

Setting $n = 1$ in Theorem 3.1, we have the following result.

Corollary 3.4. *Let* $M \geq 1$ *and* β, α, γ *be complex numbers,* $\beta = (1+\alpha)$, $\operatorname{Re}\beta \geqq 0$ *and*

$$\operatorname{Re}\beta \geq [|\alpha| + |\gamma|(b+1)(2M+1) + |\gamma|b], \tag{3.23}$$

and let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}[|\alpha| + |\gamma|(b+1)(2M+1) + |\gamma|b]. \tag{3.24}$$

If one has that the function $f \in \mathcal{A}$ *satisfies the condition:*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 \quad (z \in \mathbb{U}), \tag{3.25}$$

and $g \in G_b$, $0 < b \leq 1$ *with*

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \qquad \left|g(z)\right| \leq M \quad (z \in \mathbb{U}), \tag{3.26}$$

then the integral operator,

$$J(f;g)(z) = \left((1+\alpha)\int_0^z (f(t))^{\alpha}(g'(t))^{\gamma}\,dt\right)^{1/(1+\alpha)}, \tag{3.27}$$

is in the class $\mathcal{S}$.

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