Research Article

Schwarz-Pick Estimates for Holomorphic Mappings with Values in Homogeneous Ball

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Let B_X be the unit ball in a complex Banach space X. Assume B_X is homogeneous. The generalization of the Schwarz-Pick estimates of partial derivatives of arbitrary order is established for holomorphic mappings from the unit ball B^n to B_X associated with the Carathéodory metric, which extend the corresponding Chen and Liu, Dai et al. results.

1. Introduction

By the classical Pick's invariant form of Schwarz's lemma, a holomorphic function f(z) which is bounded by one in the unit disk $D \subset \mathbb{C}$ satisfies the following inequality

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2}$$
 (1.1)

at each point *z* of D. Ruscheweyh in [1] firstly obtained best-possible estimates of higher order derivatives of bounded holomorphic functions on the unit disk in 1985. Recently, a lot of attention (see Ghatage et al. [2], MacCluer et al. [3], Avkhadiev and Wirths [4], Ghatage and Zheng [5], Dai and Pan [6]) has been paid to the Schwarz-Pick estimates of high-order derivative estimates in one complex variable. The best result is given as follows:

$$\left| f^{(k)}(z) \right| \le \frac{k! \left(1 - \left| f(z) \right|^2 \right)}{\left(1 - \left| z \right|^2 \right)^k} (1 + |z|)^{k-1}, \quad z \in D, \ k \ge 1.$$
 (1.2)

It is natural to consider an extension of the above Schwarz-Pick estimates to higher dimensions. Anderson et al. [7] gave Schwarz-Pick estimates of derivatives of arbitrary order of functions in the Schur-Agler class on the unit polydisk and the unit ball of \mathbb{C}^n , respectively. Recently, Chen and Liu in [8] obtained estimates of high-order derivatives for all the bounded holomorphic functions on the unit ball of \mathbb{C}^n . Later, Dai et al. in [9, 10] generalized the high order Schwarz-Pick estimates for holomorphic mappings between unit balls in complex Hilbert space. Their main result is expressed as follows.

Theorem A. Suppose f(z) is holomorphic mapping from B^n to B^m . Then for any multiindex $k \ge 1$ and $\beta \in \mathbb{C}^n \setminus \{0\}$,

$$H_{f(z)}\left(D^{k}(f,z,\beta),D^{k}(f,z,\beta)\right) \leq k! \left(1 + \frac{(|\langle \beta,z\rangle|)}{((1-|z|^{2})|\beta|^{2} + |\langle \beta,z\rangle|^{2})^{1/2}}\right)^{k-1} (H_{z}(\beta,\beta))^{k},$$
(1.3)

where $D^k(f,z,\beta) = \sum_{|\alpha|=k} (k!/\alpha!) (\partial f^k(z)/\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_N^{\alpha_N}) \beta^{\alpha}$ and $H_z(\beta,\beta)$ is the Bergman metric on B^n .

In this paper, we will extend Theorem A to holomorphic mappings from the unit ball B^n to B_X associated with the Carathéodory metric. In particular, when $B_X = B^m$, our result coincides with Theorem A. Furthermore, our result shows that the high-order Schwarz-Pick estimates on the unit ball do depend on the geometric property of the image domain B_X .

Throughout this paper, the symbol X is used to denote a complex Banach space with norm $||\cdot||$, and $B_X = \{z \in X : ||z|| < 1\}$ is the unit ball in X. Let \mathbb{C}^n be the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$, where the symbol 'stands for the transpose of vector or matrix. The unit ball of \mathbb{C}^n is always written by B^n . It is well known that if f is a holomorphic mapping from B_X into X, then the following well-known expansion

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x) ((y-x)^n)$$
 (1.4)

holds for all y in some neighborhood of $x \in B_X$, where $D^n f(x)$ means the nth Fréchet derivative of f at the point x, and

$$D^{n} f(x) ((y-x)^{n}) = D^{n} f(x) (y-x, y-x, \dots, y-x).$$
 (1.5)

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^n X$ into X. For a domain $\Omega \in X$, a mapping $f: \Omega \to X$ is called to be biholomorphic if $f(\Omega)$ is a domain; the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. Let $\operatorname{Aut}(\Omega)$ denote the set of biholomorphic mappings of Ω onto itself. Ω is said to be homogeneous, if for each pair of points $x, y \in \Omega$, there is an $f \in \operatorname{Aut}(\Omega)$ such that f(x) = y.

In multiindex notation, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $z^{\alpha} = z_1^{\alpha_1} \dots z_n^{\alpha_n}$.

Let K(z,z) be the Bergman kernel function. Then the Bergman metric $H_z(\beta,\beta)$ can be defined as

$$H_z(\beta, \beta) = \sum_{j,k=1}^n \frac{\partial^2 \log K(z, z)}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k, \tag{1.6}$$

where $z \in \Omega$, $u = (u_1, u_2, ..., u_n) \in \mathbb{C}^n$. It is well known that $H_z(\beta, \beta) = (1 - |z|^2 + |\langle \beta, z \rangle|^2)/(1 - |z|^2)^2$ in [9].

Let $F_c^{B_X}(z,\xi)$ be the infinitesimal form of Carathéodory metric of domain B_X . By the definition of the Carathéodory metric [11], we have for any $\xi \in X$,

$$F_c^{B_X}(z,\xi) = \sup\{|Df(z)\xi| : f \in H(B_X, B_X), f(z) = 0\},\tag{1.7}$$

where $H(B_X, B_X)$ denotes the family of holomorphic mappings which map B_X into B_X .

2. Some Lemmas

In order to prove the main results, we need the following lemmas. Let B_X be the unit ball in a complex Banach space X, and B_X is homogeneous.

Lemma 2.1 (see [11]). *If* $f \in H(B_X, B_X)$, *then*

$$F_c^{B_X}(f(z), Df(z)\xi) \le F_c^{B_X}(z,\xi), \quad z \in B_X, \ \xi \in X.$$
 (2.1)

In particular, when f is biholomorphic mapping, then $F_c^{B_X}(f(z), Df(z)\xi) = F_c^{B_X}(z, \xi)$.

Lemma 2.2 (see [12]). Consider the following:

$$F_c^{B_X}(0,\xi) = ||\xi||, \quad \xi \in X.$$
 (2.2)

Lemma 2.3. Let $f \in H(D, B_X)$. Then f can be written with the following n-variable power series given by

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in D.$$
 (2.3)

Then the following holds

$$F_c^{B_X}(a_0, a_k) \le 1 \tag{2.4}$$

for any integer $k \ge 0$.

Proof. For the fixed *k*, we define

$$f_k(z) = \sum_{j=1}^k \frac{f(e^{i(2\pi j/k)}z)}{k}.$$
 (2.5)

Then $f_k \in H(D, B_X)$. It is clear that

$$\frac{1}{k} \sum_{j=1}^{k} e^{i(2\pi j l/k)} = \begin{cases} 1, & \text{if } l \equiv 0 \pmod{k}, \\ 0, & \text{otherwise.} \end{cases}$$
 (2.6)

From the power series expansion of the holomorphic function f, we get

$$f_{k}(z) = \frac{1}{k} \left(\sum_{j=1}^{k} \left(a_{0} + \sum_{l=1}^{\infty} e^{i(2\pi jl/k)} \sum_{|\alpha|=l} a_{\alpha} z^{\alpha} \right) \right)$$

$$= a_{0} + \sum_{l=1}^{\infty} a_{lk} z^{lk}.$$
(2.7)

In terms of the homogeneity of B_X , we can take $\Psi \in \operatorname{Aut}(B_X)$ and $\Psi(a_0) = 0$, then $\Psi \circ f_k \in H(D, B_X)$. This implies that

$$\Psi \circ f_{k}(z) = \Psi \left(a_{0} + \sum_{l=1}^{\infty} a_{lk} z^{lk} \right)$$

$$= \Psi(a_{0}) + D\Psi(a_{0}) \left(\sum_{l=1}^{\infty} a_{lk} z^{lk} \right) + D^{2}\Psi(a_{0}) \left(\sum_{l=1}^{\infty} a_{lk} z^{lk} \right) + \cdots$$

$$= D\Psi(a_{0}) (a_{k}) z^{k} + D\Psi(a_{0}) (a_{2k}) z^{2k} + D\Psi(a_{0}) (a_{3k}) z^{3k} + \cdots$$

$$(2.8)$$

By making use of the orthogonality, we obtain

$$D\Psi(a_0)(a_\alpha)z^\alpha = \frac{1}{2\pi} \int_0^{2\pi} (\Psi \circ f_k) \Big(ze^{i\theta}\Big) e^{-i\alpha\theta} d\theta. \tag{2.9}$$

Hence,

$$\|D\Psi(a_0)(a_\alpha)z^\alpha\| \le \frac{1}{2\pi} \int_0^{2\pi} \|(\Psi \circ f_k)(ze^{i\theta})e^{-i\alpha\theta}\|d\theta \le 1. \tag{2.10}$$

This implies the following inequality

$$||D\Psi(a_0)(a_\alpha)|||z|^\alpha \le 1 \tag{2.11}$$

holds for any $z \in D$. Thus,

$$||D\Psi(a_0)(a_\alpha z^\alpha)|| \le 1 \tag{2.12}$$

holds for any $z \in \overline{D}$. It means that $||D\Psi(a_0)(a_\alpha)|| \le 1$. By Lemmas 2.1 and 2.2, we obtain

$$F_c^{B_X}(a_0, a_\alpha) = F_c^{B_X}(0, D\Psi(a_0)(a_\alpha)) = ||D\Psi(a_0)(a_\alpha)|| \le 1,$$
(2.13)

which is the desired result.

3. Main Results

Theorem 3.1. Let $f: D \to B_X$ be a holomorphic mapping. Then the following inequality

$$F_c^{B_X}(f(z), f^{(k)}(z)) \le k! \frac{(1+|z|)^{k-1}}{(1-|z|^2)^k}$$
(3.1)

holds for $k \ge 1$ and $z \in D$.

Proof. Let $g(\xi)$ be a holomorphic function on D defined by

$$g(\xi) = f\left(\frac{z+\xi}{1+\overline{z}\xi}\right), \quad \xi \in D.$$
 (3.2)

Then g can be written as a power series as follows:

$$g(\xi) = \sum_{j=0}^{\infty} a_j \xi^j. \tag{3.3}$$

In order to obtain Theorem 3.1, we need to prove the following equality:

$$f^{(k)}(z) = \frac{k!}{1 - |z|^2} \sum_{i=0}^{k} {k-1 \choose i} a_{k-j} \overline{z}^{|j|}.$$
 (3.4)

Let 0 < r < 1 such that $D(z, r) \subset D$, the Cauchy integral formula shows that

$$f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w-z} dw.$$
 (3.5)

Thus,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w-z)^{k+1}} dw.$$
 (3.6)

Let $w = (z + \xi)/(1 + \overline{z}\xi)$. Then

$$\frac{dw}{d\xi} = \frac{1 - |z|^2}{(1 + \overline{z}\xi)^2}, \quad w - z = \xi \frac{1 - |z|^2}{(1 + \overline{z}\xi)^2}.$$
 (3.7)

Substituting (3.7) into (3.6), we get

$$f^{(k)}(z) = \frac{k!}{2\pi i \left(1 - |z|^2\right)^k} \int_{|(z+\xi)/(1+z\bar{\xi})|=r} \frac{g(\xi)(1+\bar{z}\xi)^{k-1}}{\xi^{k+1}} d\xi$$

$$= \frac{k!}{\left(1 - |z|^2\right)^k} \sum_{j=0}^{k-1} {k-1 \choose j} a_{k-j} \bar{z}^j,$$
(3.8)

which prove the equality (3.4).

From Lemma 2.3, we have for any integer $k \ge 1$,

$$F_c^{B_X}(a_0, a_k) \le 1.$$
 (3.9)

This implies that

$$F_{c}^{B_{X}}(f(z), f^{(k)}(z)) \leq F_{c}^{B_{X}}\left(a_{0}, \frac{k!}{\left(1 - |z|^{2}\right)^{k}} \sum_{j=0}^{k-1} {k-1 \choose j} a_{k-j} |z|^{j}\right)$$

$$\leq \frac{k!}{\left(1 - |z|^{2}\right)^{k}} (1 + |z|)^{k-1}$$
(3.10)

which completes the desired result.

Remark 3.2. If $B_X = D$, then the inequality (3.1) reduces to

$$\left| f^{(k)}(z) \right| \le k! \frac{1 - \left| f(z) \right|^2}{\left(1 - |z|^2 \right)^k} (1 + |z|)^{k-1}$$
 (3.11)

which coincides with the Theorem 1.1 of Dai and Pan [6] in one complex variable.

Theorem 3.3. Let $f: B^n \to B_X$ be a holomorphic mapping. Then the following inequality

$$F_{c}^{B_{X}}(f(z), D^{k}(f, z, \beta)) \leq k! \left(1 + \frac{|\langle \beta, z \rangle|}{((1 - |z|^{2})|\beta|^{2} + |\langle \beta, z \rangle|^{2})^{1/2}}\right)^{k-1} \left[F_{c}^{B^{n}}(z, \beta)\right]^{k}$$
(3.12)

holds for $k \ge 1$, $\beta \in \mathbb{C}^n \setminus \{0\}$ and $z \in B^n$.

Proof. For any fixed $k \ge 1$, $\beta \in \partial B^n$, and $\xi \in B^n$. Define the following disk:

$$\Delta = \left\{ \lambda \in \mathbb{C} : \left| \xi + \lambda \beta \right|^2 < 1 \right\}. \tag{3.13}$$

Notice that $\langle \beta, \xi - \langle \xi, \beta \rangle \beta \rangle = 0$. Hence,

$$\begin{aligned} \left| \xi + \lambda \beta \right|^2 &= \left| \left(\lambda + \left\langle \xi, \beta \right\rangle \right) \beta + \xi - \left\langle \xi, \beta \right\rangle \beta \right|^2 \\ &= \left| \lambda + \left\langle \xi, \beta \right\rangle \right|^2 + \left| \xi - \left\langle \xi, \beta \right\rangle \beta \right|^2 < 1. \end{aligned} \tag{3.14}$$

That is,

$$\left|\lambda + \langle \xi, \beta \rangle\right| < \sqrt{1 - \left|\xi - \langle \xi, \beta \rangle \beta\right|^2} = \sqrt{1 - \left|\xi\right|^2 + \left|\langle \xi, \beta \rangle\right|^2}.$$
 (3.15)

Set $\sigma = \sqrt{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}$. For the fixed ξ and β , we define

$$g(\omega) = f(\xi + (\omega\sigma - \langle \xi, \beta \rangle)\beta), \quad \omega \in D.$$
 (3.16)

Then $g(\omega)$ is holomorphic mapping from the unit disk D to the homogeneous domain Ω . According to Theorem 3.1 to the functions g and $\omega' = (\langle \xi, \beta \rangle)/\sigma$, we have

$$F_c^{B_X}(g(\omega'), g^{(k)}(\omega')) \le k! \frac{(1+|\omega'|)^{k-1}}{(1-|\omega'|^2)^k},$$
 (3.17)

which holds for $k \ge 1$. Since $g(\omega') = f(\xi)$, and

$$|\omega'| = \frac{|\langle \beta, \xi \rangle|}{\sqrt{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}}, \qquad 1 - |\omega'|^2 = \frac{1 - |\xi|^2}{1 - |\xi|^2 + |\langle \xi, \beta \rangle|^2}.$$
 (3.18)

In terms of the chain rule, we have

$$g^{(k)}(\omega') = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial f^k(\xi)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_N^{\alpha_N}} (\sigma \beta)^{\alpha} = \sigma^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial f^k(\xi)}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_N^{\alpha_N}} \beta^{\alpha} = \sigma^k D^k(f, \xi, \beta).$$
(3.19)

Hence,

$$F_{c}^{B_{X}}\left(f(\xi), \sigma^{k} D^{k}(f, \xi, \beta)\right) \leq k! \left(1 + \frac{\left|\langle \beta, \xi \rangle\right|}{(1 - |\xi|^{2} + \left|\langle \beta, \xi \rangle\right|^{2})^{1/2}}\right)^{k-1} \left[\frac{(1 - |\xi|^{2}) + \left|\langle \beta, \xi \rangle\right|^{2}}{\left(1 - |\xi|^{2}\right)^{2}}\right]^{k} \sigma^{k}. \tag{3.20}$$

Note the definition of Carathéodory metric and $F_c^{B^n}(z,\beta) = (1-|z|^2+|\langle \beta,z\rangle|^2)/(1-|z|^2)^2$ in [11], we can get

$$F_c^{B_X}(f(z), D^k(f, z, \beta)) \le k! \left(1 + \frac{|\langle \beta, z \rangle|}{(1 - |z|^2 + |\langle \beta, z \rangle|^2)^{1/2}}\right)^{k-1} \left[F_c^{B^n}(z, \beta)\right]^k. \tag{3.21}$$

This gives the proof of the case $z = \xi$ and $\beta \in \partial B_n$. For general vector $\beta \in \mathbb{C}^n \setminus \{0\}$, we may substitute $\beta/\|\beta\|$ for β . By the homogeneous of β from the above inequality, we can obtain the same result, which completes the proof of the Theorem 3.3.

Remark 3.4. If $B_X = B^m$, then $H_{f(z)}(D^k(f,z,\beta),D^k(f,z,\beta)) = F_c^{B^m}(f(z),D^k(f,z,\beta))$ and $H_z(\beta,\beta) = F_c^{B^m}(z,\beta)$. Thus, the Theorem 3.3 reduces to Theorem A established by Dai et al. [9].

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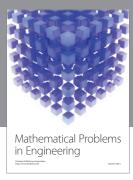
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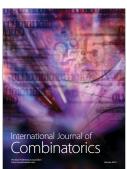








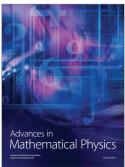




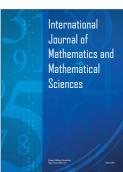


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