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Research Article

Periodic Solutions of Some Impulsive Hamiltonian Systems with Convexity Potentials

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We study the existence of periodic solutions of some second-order Hamiltonian systems with impulses. We obtain some new existence theorems by variational methods.

1. Introduction

Consider the following systems:

$$\begin{gathered} \ddot{u}(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta\dot{u}(t_k) = g_k\big(u(t_k^-)\big), \quad k = 1, 2, \ldots, m, \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{1.1}$$

where $k \in \mathbb{Z}$, $u \in \mathbb{R}^n$, $\Delta\dot{u}(t_k) = \dot{u}(t_k^+) - \dot{u}(t_k^-)$ with $\dot{u}(t_k^\pm) = \lim_{t \to t_k^\pm} \dot{u}(t)$, $g_k(u) = \operatorname{grad}_u G_k(u)$, $G_k \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for each $k \in \mathbb{Z}$, there exists an $m \in \mathbb{Z}$ such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and we suppose that $f(t, u) = \operatorname{grad}_u F(t, u)$ satisfies the following assumption.

(A) $F(t, x)$ is measurable in t for $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| + \big|f(t, x)\big| \le a(|x|)b(t), \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and $t \in [0, T]$.

Many solvability conditions for problem (1.1) without impulsive effect are obtained, such as, the coercivity condition, the convexity conditions (see [1–4] and their references), the sublinear nonlinearity conditions, and the superlinear potential conditions. Recently, by using variational methods, many authors studied the existence of solutions of some second-order differential equations with impulses. More precisely, Nieto in [5, 6] considers linear conditions, [7–10] the sublinear conditions, and [11–16] the sublinear conditions and the other conditions. But to the best of our knowledge, except [7] there is no result about convexity conditions with impulsive effects. By using different techniques, we obtain different results from [7].

We recall some basic facts which will be used in the proofs of our main results. Let

$$H_T^1 = \left\{ u : [0,T] \longrightarrow \mathbb{R}^n \text{ absolutely continuous; } u(0) = u(T),\ \dot{u}(t) \in L^2(0,T;\mathbb{R}^n) \right\}, \tag{1.3}$$

with the inner product

$$\langle u, v \rangle = \int_0^T (u(t), v(t))dt + \int_0^T (\dot{u}(t), \dot{v}(t))dt, \quad \forall u, v \in H_T^1, \tag{1.4}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T (u(t), u(t))dt + \int_0^T (\dot{u}(t), \dot{u}(t))dt \right)^{1/2}, \quad \forall u \in H_T^1. \tag{1.5}$$

The space H_T^1 has some important properties. For $u \in H_T^1$, let $\overline{u} = (1/2T)\int_0^T u(t)dt$, and $\widetilde{u} = u(t) - \overline{u}$. Then one has Sobolev's inequality (see Proposition 1.3 in [1]):

$$\|\widetilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt. \tag{1.6}$$

Consider the corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t))dt + \sum_{k=1}^m G_k(u(t_k)). \tag{1.7}$$

It follows from assumption (A) and the continuity of g_k one has that φ is continuously differentiable and weakly lower semicontinuous on H_T^1. Moreover, we have

$$\langle \varphi\prime(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt + \int_0^T \big(f(t, u(t)), v(t)\big)dt + \sum_{k=1}^m \big(g_k(u(t_k)), v(t_k)\big), \tag{1.8}$$

for $u, v \in H_T^1$ and $\varphi\prime$ is weakly continuous and the weak solutions of problem (1.1) correspond to the critical points of φ (see [8]).

Theorem 1.1 ([2, Theorem 1.1]). *Suppose that V and W are reflexive Banach spaces, $\varphi \in C^1(V \times W, R)$, $\varphi(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$, and $\varphi(\cdot, w) : V \to R$ is convex for all $w \in W$ and φ' is weakly continuous. Assume that*

$$\varphi(0, w) \longrightarrow -\infty \tag{1.9}$$

as $\|w\| \to \infty$ and for every $M > 0$,

$$\varphi(v, w) \longrightarrow +\infty, \tag{1.10}$$

as $\|v\| \to \infty$ uniformly for $\|w\| \leq M$. Then φ has at least one critical point.

2. Main Results

Theorem 2.1. *Assume that assumption (A) holds. If further*

- (H_1) *$F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, and*
- (H_2) *there exist η, $\theta > 0$ such that $G_k(x) \geq \eta|x| + \theta$, for all $x \in \mathbb{R}^n$, then* (1.1) *possesses at least one solution in H_T^1.*

Remark 2.2. (H_1) implies there exists a point $\overline{x}$ for which

$$\int_0^T \nabla F(t, \overline{x}) dt = 0. \tag{2.1}$$

Proof of Theorem 2.1. It follows Remark 2.2, (1.6), and (H_2) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T (F(t, u(t)) - F(t, \overline{x}))dt + \int_0^T F(t, \overline{x})dt + \sum_{k=1}^m G_k(u(t_k)) \\
&= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \overline{x})dt + \int_0^T (f(t, \overline{x}), u(t) - \overline{x})dt + \sum_{k=1}^m G_k(u(t_k)) \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \overline{x})dt + \int_0^T (f(t, \overline{x}), \tilde{u})dt + \sum_{k=1}^m \eta|\tilde{u} + \overline{u}| + m\theta \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \left(\int_0^T \big|f(t, \overline{x})\big| dt\right)\|\tilde{u}\|_\infty + m\eta|\overline{u}| - m\eta\|\tilde{u}\|_\infty + m\theta \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - C_0\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{1/2} + m\eta|\overline{u}| + m\theta,
\end{aligned} \tag{2.2}$$

for all $u \in H_T^1$ and some positive constant C_0. As $\|u\| \to \infty$ if and only if $(|u|^2 + \|\dot{u}\|_2^2)^{1/2} \to \infty$, we have $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. By Theorem 1.1 and Corollary 1.1 in [1], φ has a minimum point in H_T^1, which is a critical point of φ. Hence, problem (1.1) has at least one weak solution. □

Theorem 2.3. *Assume that assumption* (A) *and* (H_1) *hold. If further*

(H_3) *there exist* $\eta, \theta > 0$ *and* $\alpha \in (0,2)$ *such that* $G_k(x) \leq \eta|x|^\alpha + \theta$ *for all* $x \in \mathbb{R}^n$ *and*

(H_4) *there exist some* $\beta > \alpha$ *and* $\gamma > 0$ *such that*

$$|x|^{-\beta}\int_0^T F(t,x)dt \leq -\gamma, \tag{2.3}$$

for $|x| \geq M$ *and* $t \in [0,T]$, *where* M *is a constant, then* (1.1) *possesses at least one solution in* H_T^1.

Remark 2.4. We can find that our condition (H_4) is very different from condition (vii) in [7] since we prove this by the saddle point theorem substituted for the least action principle.

Proof of Theorem 2.3. We prove φ satisfies the (PS) condition at first. Suppose $\{u_n\}$ is such an sequence that $\{\varphi(u_n)\}$ is bounded and $\lim_{n\to\infty}\varphi'(u_n) = 0$. We will prove it has a convergent subsequence. By (H_3) and (1.6), we have

$$\begin{aligned}\sum_{k=1}^m G_k(u(t_k)) &\leq \sum_{k=1}^m \eta|\tilde{u}(t_k) + \overline{u}|^\alpha + m\theta \\ &\leq 4m\eta\left(|\tilde{u}(t_k)|^\alpha + |\overline{u}|^\alpha\right) + m\theta \\ &\leq C_1\|\dot{u}\|_2^\alpha + C_2|\overline{u}|^\alpha + C_3,\end{aligned} \tag{2.4}$$

for some positive constants C_1, C_2, C_3. By Remark 2.2, (1.6), and (2.4), we have

$$\begin{aligned}\varphi(u_n) &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt + \int_0^T (F(t,u_n(t)) - F(t,\overline{x}))dt + \int_0^T F(t,\overline{x})dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt + \int_0^T F(t,\overline{x})dt + \int_0^T \left(f(t,\overline{x}), u_n(t) - \overline{x}\right)dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt + \int_0^T F(t,\overline{x})dt + \int_0^T \left(f(t,\overline{x}), \tilde{u}_n\right)dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &\geq \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt - \left(\int_0^T \left|f(t,\overline{x})\right|dt\right)\|\tilde{u}_n\|_\infty - C_1\|\dot{u}_n\|_2^\alpha - C_2|\overline{u}_n|^\alpha - C_4 \\ &\geq \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt - C_5\left(\int_0^T |\dot{u}_n(t)|^2dt\right)^{1/2} - C_1\|\dot{u}_n\|_2^\alpha - C_2|\overline{u}_n|^\alpha - C_4,\end{aligned} \tag{2.5}$$

for some positive constants C_4, C_5, which implies that

$$C|\overline{u}_n|^{\alpha/2} \geq \left(\int_0^T |\dot{u}_n(t)|^2dt\right)^{1/2} - C_6, \tag{2.6}$$

for some positive constants C, C_6. By (1.6), the above inequality implies that

$$\|\widetilde{u}_n\|_\infty \le C_7\left(|\overline{u}_n|^{\alpha/2}+1\right), \tag{2.7}$$

for the positive constant C_7. The one has

$$|u_n(t)| \ge |\overline{u}_n| - |\widetilde{u}_n| \ge |\overline{u}_n| - \|\widetilde{u}_n\|_\infty \ge |\overline{u}_n| - C_7\left(|\overline{u}_n|^{\alpha/2}+1\right), \quad \forall t \in [0,T]. \tag{2.8}$$

If $\{|\overline{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\overline{u}_n| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{2.9}$$

By (2.8) and (2.9), we have

$$|u_n(t)| \ge \frac{1}{2}|\overline{u}_n|, \tag{2.10}$$

for all large n and every $t \in [0,T]$. By (2.10) and (H_4), one has $|u_n(t)| \ge M$ for all large n. It follows from (H_4), (2.4), (2.6), (2.7), and above inequality that

$$\begin{aligned}\varphi(u_n) &\le \left(C|\overline{u}_n|^{\alpha/2}+C_6\right)^2 - \int_0^T \gamma|u_n(t)|^\beta dt + C_2\|\widetilde{u}\|_\infty^\alpha + C_2|\overline{u}|^\alpha + C_3 \\ &\le \left(C|\overline{u}_n|^{\alpha/2}+C_6\right)^2 - 2^{-\beta}|\overline{u}_n|^\beta T\gamma + C_8\left(|\overline{u}_n|^{\alpha/2}+1\right)^\alpha + C_2|\overline{u}|^\alpha + C_3,\end{aligned} \tag{2.11}$$

for large n and the positive constant C_8, which contradicts the boundedness of $\varphi(u_n)$ since $\beta > \alpha$. Hence $(|\overline{u}_n|)$ is bounded. Furthermore, (u_n) is bounded by (2.6). A similar calculation to Lemma 3.1 in [9] shows that φ satisfies the (PS) condition. We now prove that φ satisfies the other conditions of the saddle point theorem. Assume that $\widetilde{H}_T^1 = \{u \in H_T^1 : \overline{u} = 0\}$, then $H_T^1 = \widetilde{H}_T^1 \oplus \mathbb{R}^n$. From above calculation, one has

$$\varphi(u) \ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - C_5\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{1/2} - C_1\|\dot{u}\|_2^\alpha - C_4, \tag{2.12}$$

for all $u \in \widetilde{H}_T^1$, which implies that

$$\varphi(u) \longrightarrow +\infty, \tag{2.13}$$

as $\|u\| \to \infty$ in $\widetilde{H}_T^1$. Moreover, by (H_3) and (H_4) we have

$$\begin{aligned}\varphi(x) &= \int_0^T F(t,x)dt + \sum_{k=1}^m G_k(x)\\ &\leq -T\gamma|x|^\beta + m\eta|x|^\alpha + m\theta,\end{aligned} \tag{2.14}$$

for $|x| > M$, which implies that

$$\varphi(x) \longrightarrow -\infty, \tag{2.15}$$

as $|x| \to \infty$ in $\mathbb{R}^n$ since $\beta > \alpha$. Now Theorem 2.3 is proved by (2.13), (2.15), and the saddle point theorem. □

Theorem 2.5. *Assume that assumption (A) holds. Suppose that $F(t,\cdot), G_k(x)$ are concave and satisfy*

(H_5) $G_k(x) \leq -\eta|x| + \theta$ *for some positive constant $\eta, \theta > 0$, then* (1.1) *possesses at least one solution in H_T^1.*

Proof of Theorem 2.5. Consider the corresponding functional φ on $\mathbb{R}^n \times \widetilde{H}_T^1$ given by

$$\varphi(u) = -\frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,u(t))dt - \sum_{k=1}^m G_k(u(t_k)), \tag{2.16}$$

which is continuously differentiable, bounded, and weakly upper semi-continuous on H_T^1. Similar to the proof of Lemma 3.1 in [2], one has that $\varphi(x+w)$ is convex in $x \in \mathbb{R}^n$ for every $w \in \widetilde{H}_T^1$. By the condition, we have $-G_k(x+w) \geq -2G_k((1/2)x) + G_k(-w)$. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\varphi(x+w) &= -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T F(t,x+w)dt - \sum_{k=1}^m G_k(x+w)\\ &\geq -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - \left(\int_0^T \left|f(t,\overline{x})\right|dt\right)\|w\|_\infty - \sum_{k=1}^m G_k(x+w) + C_9\\ &\geq -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - C_0\left(\int_0^T |\dot{w}|^2 dt\right)^{1/2} - 2G_k\left(\frac{1}{2}x\right) + G_k(-w) + C_9,\end{aligned} \tag{2.17}$$

which means $\varphi(x+w) \to +\infty$ as $|x| \to \infty$, uniformly for $w \in \widetilde{H}_T^1$ with $\|w\| \leq M$ by (H_5) and (1.6). On the other hand,

$$\begin{aligned}\varphi(w) &= -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T F(t,w)dt - \sum_{k=1}^{m} G_k(w) \\ &\leq -\frac{1}{2}\int_0^T |\dot{w}|^2 dt + C_0\left(\int_0^T |\dot{w}|^2 dt\right)^{1/2} + m\eta\|w\|_\infty + C_9,\end{aligned} \tag{2.18}$$

which implies that $\varphi(w) \to -\infty$ as $\|w\| \to \infty \in \widetilde{H}_T^1$ by (H_5) and (1.6). We complete our proof by Theorem 1.1. □

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