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Review Article

Approximate Iteration Algorithm with Error Estimate for Fixed Point of Nonexpansive Mappings

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The purpose of this article is to present a general viscosity iteration process $\{x_n\}$ which defined by $x_n + 1 = (I - \alpha_n A)Tx_n + \beta_n \gamma f(x_n) + (\alpha_n - \beta_n)x_n$ and to study the convergence of $\{x_n\}$, where T is a nonexpansive mapping and A is a strongly positive linear operator, if $\{\alpha_n\}$, $\{\beta_n\}$ satisfy appropriate conditions, then iteration sequence $\{x_n\}$ converges strongly to the unique solution $x^* \in f(T)$ of variational inequality $\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0$, for all $x \in f(T)$. Meanwhile, a approximate iteration algorithm is presented which is used to calculate the fixed point of nonexpansive mapping and solution of variational inequality, the error estimate is also given. The results presented in this paper extend, generalize, and improve the results of Xu, G. Marino and Xu and some others.

1. Introduction

Iteration methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [1–4] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.1}$$

where $F(T)$ is the fixed points set of a nonexpansive mapping T on H, b is a given point in H, and $A : H \to H$ is strongly positive operator, that is, there exists a constant $\delta > 0$ with the property

$$\langle Ax, x \rangle \geq \delta \|x\|^2, \quad \forall x \in H. \tag{1.2}$$

Recall that $T : H \to H$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Throughout the rest of this paper, we denote by $F(T)$ the fixed points set of T and assume that $F(T)$ is nonempty. It is well known that $F(T)$ is closed convex (cf. [5]). In [4] (see also [2]), it is proved that the sequence $\{x_n\}$ defined by the iteration method below, with the initial guess x_0 chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0, \tag{1.3}$$

converges strongly to the unique solution of minimization problem (1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [6] introduced the viscosity approximation method for nonexpansive mappings (see [7] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial guess $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved [6, 7] that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution x^* in $F(T)$ of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in F(T). \tag{1.5}$$

Recently (2006), Marino and Xu [2] combine the iteration method (1.3) with the viscosity approximation method (1.4) and consider the following general iteration method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.6}$$

they have proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(T), \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x \rangle - h(x), \tag{1.8}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

The purpose of this paper is to present a general viscosity iteration process $\{x_n\}$ which is defined by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \beta_n \gamma f(x_n) + (\alpha_n - \beta_n)x_n \tag{1.9}$$

and to study the convergence of $\{x_n\}$, where T is a nonexpansive mapping and A is a strongly positive linear operator, if $\{\alpha_n\}$, $\{\beta_n\}$ satisfy appropriate conditions, then iteration sequence $\{x_n\}$ converges strongly to the unique solution $x^* \in F(T)$ of variational inequality (1.7). Meanwhile, an approximate iteration algorithm

$$x_{n+1} = (I - sA)Tx_n + t\gamma f(x_n) + (s - t)x_n \tag{1.10}$$

is presented which is used to calculate the fixed point of nonexpansive mapping and solution of variational inequality; the convergence rate estimate is also given. The results presented in this paper extend, generalize and improve the results of Xu [7], Marino and Xu [2], and some others.

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Lemma 2.1 (see [3]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \delta_n, \tag{2.1}$$

where $\{\lambda_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $(-\infty, +\infty)$ such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n / r_n \leq 0$, *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (see [5]). *Let H be a Hilbert space, K a closed convex subset of H, and $T : K \to K$ a nonexpansive mapping with nonempty fixed points set $F(T)$. If $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

The following lemma is not hard to prove.

Lemma 2.3. *Let H be a Hilbert space, K a closed convex subset of H, $f : H \to H$ a contraction with coefficient $0 < h < 1$, and A a strongly positive linear bounded operator with coefficient $\delta > 0$. Then, for $0 < \gamma < (\delta/h)$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\delta - \gamma h)\|x - y\|^2, \quad x, y \in H. \tag{2.2}$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\delta - \gamma h$.

Recall the metric (nearest point) projection P_K from a real Hilbert space H to a closed convex subset K of H is defined as follows: given $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \min_{y \in K} \|x - y\|. \tag{2.3}$$

P_K is characterized as follows.

Lemma 2.4. *Let K be a closed convex subset of a real Hilbert space H. Given that $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in K. \tag{2.4}$$

Lemma 2.5. *Assume that A is a strongly positive linear-bounded operator on a Hilbert space H with coefficient $\delta > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq (1 - \rho\delta)$.*

Proof. Recall that a standard result in functional analysis is that if V is linear bounded self-adjoint operator on H, then

$$\|V\| = \sup\{|\langle Vx, x\rangle| \ : x \in H, \|x\| = 1\}. \tag{2.5}$$

Now for $x \in H$ with $\|x\| = 1$, we see that

$$\langle (I - \rho A)x, x\rangle = 1 - \rho\langle Ax, x\rangle \geq 1 - \rho\|A\| \geq 0 \tag{2.6}$$

(i.e., $I - \rho A$ is positive). It follows that

$$\begin{aligned} \|I - \rho A\| &= \sup\{\langle (I - \rho A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \rho\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \rho\delta. \end{aligned} \tag{2.7}$$

□

The following lemma is also not hard to prove by induction.

Lemma 2.6. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + (\lambda_n + \mu_n)M, \tag{2.8}$$

where M is a nonnegative constant and $\{\lambda_n\}, \{\mu_n\}$ are sequences in $[0, +\infty)$ such that

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\sum_{n=0}^{\infty} \mu_n < \infty$.

Then $\{a_n\}$ is bounded.

Notation. We use $\rightarrow$ for strong convergence and $\rightharpoonup$ for weak convergence.

3. A general Iteration Algorithm with Bounded Linear Operator

Let H be a real Hilbert space, A be a bounded linear operator on H, and T be a nonexpansive mapping on H. Assume that the fixed point set $F(T) = \{x \in H : Tx = x\}$ of T is nonempty. Since $F(T)$ is closed convex, the nearest point projection from H onto $F(T)$ is well defined.

Throughout the rest of this paper, we always assume that A is strongly positive, that is, there exists a constant $\delta > 0$ such that

$$\langle Ax, x\rangle \geq \delta\|x\|^2, \quad \forall x \in H. \tag{3.1}$$

(Note: $\delta > 0$ is throughout reserved to be the constant such that (3.1) holds.)

Recall also that a contraction on H is a self-mapping f of H such that

$$\|f(x) - f(y)\| \leq h\|x - y\|, \quad \forall x, y \in H, \tag{3.2}$$

where $h \in (0,1)$ is a constant which is called contractive coefficient of f.

For given contraction f with contractive coefficient $0 < h < 1$, and $t \in [0,1), s \in (0,1), s \geq t$ such that $0 \leq t \leq s < \|A\|^{-1}$ and $0 < \gamma < \delta/h$, consider a mapping $S_{t,s}$ on H defined by

$$S_{t,s}x = (I - sA)Tx + t\gamma f(x) + (s-t)x, \quad x \in H. \tag{3.3}$$

Assume that

$$\frac{s-t}{s} \longrightarrow 0, \tag{3.4}$$

it is not hard to see that $S_{t,s}$ is a contraction for sufficiently small s, indeed, by Lemma 2.5 we have

$$\begin{aligned}\|S_{t,s}x - S_{t,s}y\| &\leq t\gamma\|f(x) - f(y)\| + \|(I - sA)(Tx - Ty)\| + \|(s-t)(x-y)\| \\ &\leq (t\gamma h + 1 - s\delta + s - t)\|x - y\| \\ &= (1 + t(\gamma h - 1) - s(\delta - 1))\|x - y\|.\end{aligned} \tag{3.5}$$

Hence, $S_{t,s}$ has a unique fixed point, denoted by $x_{t,s}$, which uniquely solves the fixed point equation:

$$x_{t,s} = (I - sA)Tx_{t,s} + t\gamma f(x_{t,s}) + (s-t)x_{t,s}. \tag{3.6}$$

Note that $x_{t,s}$ indeed depends on f as well, but we will suppress this dependence of $x_{t,s}$ on f for simplicity of notation throughout the rest of this paper. We will also always use γ to mean a number in $(0, \delta/h)$.

The next proposition summarizes the basic properties of $x_{t,s}$, $(t \le s)$.

Proposition 3.1. *Let $x_{t,s}$ be defined via* (3.6).

(i) $\{x_{t,s}\}$ *is bounded for* $t \in [0, \|A\|^{-1})$, $s \in (0, \|A\|^{-1})$.

(ii) $\lim_{s\to 0}\|x_{t,s} - Tx_{t,s}\| = 0$.

(iii) $\{x_{t,s}\}$ *defines a continuous surface for* $(t,s) \in [0, \|A\|^{-1}) \times (0, \|A\|^{-1})$, $t \le s$ *into* H.

Proof. Observe, for $s \in (0, \|A\|^{-1})$, that $\|I - sA\| \le 1 - s\delta$ by Lemma 2.5.

To show (i) pick $p \in F(T)$. We then have

$$\begin{aligned}\|x_{t,s} - p\| &= \|(I - sA)(Tx_{t,s} - p) + t(\gamma f(x_{t,s}) - Ap) - sAp + tAp\| \\ &\le (1 - s\delta)\|x_{t,s} - p\| + t\|\gamma f(x_{t,s}) - Ap\| + (s - t)\|Ap\| \\ &\le (1 - s\delta)\|x_{t,s} - p\| + s[\gamma h\|x_{t,s} - p\| + \|\gamma f(p) - Ap\|] + (s - t)\|Ap\| \\ &\le [1 - s(\delta - \gamma h)]\|x_{t,s} - p\| + s\|\gamma f(p) - Ap\| + (s - t)\|Ap\|.\end{aligned} \tag{3.7}$$

It follows that

$$\|x_{t,s} - p\| \le \frac{\|\gamma f(p) - Ap\|}{\delta - \gamma h} + \frac{s - t}{s}\frac{\|Ap\|}{\delta - \gamma h} < +\infty. \tag{3.8}$$

Hence $\{x_{t,s}\}$ is bounded.

(ii) Since the boundedness of $\{x_{t,s}\}$ implies that of $\{f(x_{t,s})\}$ and $\{ATx_{t,s}\}$, and observe that

$$\|x_{t,s} - Tx_{t,s}\| = \|tf(x_{t,s}) - sATx_{t,s} + (s - t)x_{t,s}\|, \tag{3.9}$$

we have

$$\lim_{s\to 0}\|x_{t,s} - Tx_{t,s}\| = 0. \tag{3.10}$$

To prove (iii) take $t, t_0 \in [0, \|A\|^{-1})$, s, $s_0 \in (0, \|A\|^{-1})$, $s \ge t, s_0 \ge t_0$ and calculate

$$\begin{aligned}\|x_{t,s} - x_{t_0,s_0}\| &= \|(t - t_0)\gamma f(x_{t,s}) + t_0\gamma(f(x_{t,s}) - f(x_{t_0,s_0})) - (s - s_0)ATx_{t,s} \\ &\quad + (I - s_0A)(Tx_{t,s} - Tx_{t_0,s_0}) + (s - t)(x_{t,s} - x_{t_0,s_0}) + (s - s_0 + t_0 - t)x_{t_0,s_0}\| \\ &\le |t - t_0|\gamma\|f(x_{t,s})\| + t_0\gamma h\|x_{t,s} - x_{t_0,s_0}\| + |s - s_0|\|ATx_{t,s}\| \\ &\quad + (1 - s_0\delta)\|x_{t,s} - x_{t_0,s_0}\| + (s - t)\|x_{t,s} - x_{t_0,s_0}\| \\ &\quad + [|s - s_0| + |t - t_0|]\|x_{t_0,s_0}\|,\end{aligned} \tag{3.11}$$

which implies that

$$\begin{aligned}(s_0\delta - t_0\gamma h + t - s)\|x_{t,s} - x_{t_0,s_0}\| \le |t - t_0|\gamma\|f(x_{t,s})\| + |s - s_0|\|ATx_{t,s}\| \\ + [|s - s_0| + |t - t_0|]\|x_{t_0,s_0}\| \longrightarrow 0\end{aligned} \tag{3.12}$$

as $t \to t_0,\ s \to s_0$. Note that

$$\lim_{t \to t_0,\ s \to s_0}(s_0\delta - t_0\gamma h + t - s) = s_0(\delta - 1) - t_0(\gamma h - 1) > 0, \tag{3.13}$$

it is obvious that

$$\lim_{t \to t_0,\ s \to s_0}\|x_{t,s} - x_{t_0,s_0}\| = 0. \tag{3.14}$$

This completes the proof of Proposition 3.1. □

Our first main result below shows that $x_{t,s}$ converges strongly as $s \to 0$ to a fixed point of T which solves some variational inequality.

Theorem 3.2. *One has that $x_{t,s}$ converges strongly as $s \to 0\,(t \le s)$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \le 0, \quad z \in F(T). \tag{3.15}$$

Equivalently, One has $P_{F(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$, where $P_{F(T)}(\cdot)$ is the nearest point projection from H onto $F(T)$.

Proof. We first shows the uniqueness of a solution of the variational inequality (3.15), which is indeed a consequence of the strong monotonicity of $A-\gamma f$. Suppose $\tilde{x} \in F(T)$ and $\hat{x} \in F(T)$ both are solutions to (3.15), then

$$\begin{aligned}\langle (A - \gamma f)\tilde{x}, \tilde{x} - \hat{x}\rangle \le 0, \\ \langle (A - \gamma f)\hat{x}, \hat{x} - \tilde{x}\rangle \le 0.\end{aligned} \tag{3.16}$$

Adding up (3.16) gets

$$\langle (A - \gamma f)\tilde{x} - (A - \gamma f)\hat{x}, \tilde{x} - \hat{x}\rangle \le 0. \tag{3.17}$$

The strong monotonicity of $A - \gamma f$ implies that $\tilde{x} = \hat{x}$ and the uniqueness is proved. Below we use $\tilde{x} \in F(T)$ to denote the unique solution of (3.15).

To prove that $x_{t,s}$ converges strongly to $\tilde{x}$, we write, for a given $z \in F(T)$,

$$x_{t,s} - z = t\left(\gamma f(x_{t,s}) - \frac{s}{t}Az\right) + (I - sA)(Tx_{t,s} - z) + (s - t)x_{t,s} \tag{3.18}$$

to derive that

$$\begin{aligned}
\|x_{t,s}-z\|^2 &= \left\langle t\left(\gamma f(x_{t,s})-\frac{s}{t}Az\right)+(I-sA)(Tx_{t,s}-z)+(s-t)x_{t,s},x_{t,s}-z\right\rangle \\
&= t\langle \gamma f(x_{t,s})-Az,x_{t,s}-z\rangle+\langle (I-sA)(\mathrm{T}x_{t,s}-z),x_{t,s}-z\rangle \\
&\quad+(s-t)\langle x_{t,s}-Az,x_{t,s}-z\rangle \\
&\leq (1-s\delta)\|x_{t,s}-z\|^2+t\langle \gamma f(x_{t,s})-Az,x_{t,s}-z\rangle+(s-t)\langle x_{t,s}-Az,x_{t,s}-z\rangle.
\end{aligned} \tag{3.19}$$

It follows that

$$\begin{aligned}
\|x_{t,s}-z\|^2 &\leq \frac{t}{s\delta}\langle \gamma f(x_{t,s})-Az,x_{t,s}-z\rangle+\frac{s-t}{s\delta}\langle x_{t,s}-Az,x_{t,s}-z\rangle \\
&= \frac{t}{s\delta}\{\gamma\langle f(x_{t,s})-f(z),x_{t,s}-z\rangle+\langle \gamma f(z)-Az,x_{t,s}-z\rangle\} \\
&\quad+\frac{s-t}{s\delta}\langle x_{t,s}-Az,x_{t,s}-z\rangle \\
&\leq \frac{t}{s\delta}\left\{\gamma h\|x_{t,s}-z\|^2+\langle \gamma f(z)-Az,x_{t,s}-z\rangle\right\} \\
&+\frac{s-t}{s\delta}\langle x_{t,s}-Az,x_{t,s}-z\rangle,
\end{aligned} \tag{3.20}$$

which leads to

$$\begin{aligned}
\|x_{t,s}-z\|^2 \leq{}& \frac{t}{s\delta-t\gamma h}\langle \gamma f(z)-Az,x_{t,s}-z\rangle \\
&+\frac{s-t}{s\delta-t\gamma h}\langle x_{t,s}-Az,x_{t,s}-z\rangle.
\end{aligned} \tag{3.21}$$

Observe that condition (3.4) implies

$$\frac{s-t}{s\delta-t\gamma h}\longrightarrow 0, \tag{3.22}$$

as $s\to 0$. Since $x_{t,s}$ is bounded as $s\to 0$, $s\geq t$, then there exists real sequences $\{s_n\}$, $\{t_n\}$ in $[0,1]$ such that $s_n\to 0$, $s_n\geq t_n$ and $\{x_{t_n,s_n}\}$ converges weakly to a point $x^*\in H$. Using Proposition 3.1 and Lemma 2.2, we see that $x^*\in F(T)$, therefore by (3.21), we see $x_{t_n,s_n}\to x^*$. We next prove that x^* solves the variational inequality (3.15). Since

$$x_{t,s}=(I-sA)Tx_{t,s}+t\gamma f(x_{t,s})+(s-t)x_{t,s}, \tag{3.23}$$

we derive that

$$s(A-\gamma f)x_{t,s} = sAx_{t,s} - x_{t,s} + t\gamma f(x_{t,s}) - s\gamma f(x_{t,s}) + (I-sA)Tx_{t,s} + (s-t)x_{t,s}$$
$$= (I-sA)(Tx_{t,s}-x_{t,s}) + (s-t)(x_{t,s}-\gamma f(x_{t,s})), \tag{3.24}$$

so that

$$(A-\gamma f)x_{t,s} = \frac{1}{s}(I-sA)(Tx_{t,s}-x_{t,s}) + \frac{s-t}{s}(x_{t,s}-\gamma f(x_{t,s})). \tag{3.25}$$

It follows that, for $z \in F(T)$,

$$\begin{aligned}\langle (A-\gamma f)x_{t,s}, x_{t,s}-z\rangle &= \frac{1}{s}\langle (I-sA)(Tx_{t,s}-x_{t,s}), x_{t,s}-z\rangle \\ &\quad + \frac{s-t}{s}\langle x_{t,s}-\gamma f(x_{t,s}), x_{t,s}-z\rangle \\ &= \frac{-1}{s}\langle (I-T)x_{t,s}-(I-T)z, x_{t,s}-z\rangle \\ &\quad + \langle A(I-T)x_{t,s}, x_{t,s}-z\rangle \\ &\quad + \frac{s-t}{s}\langle x_{t,s}-\gamma f(x_{t,s}), x_{t,s}-z\rangle \\ &\le \langle A(I-T)x_{t,s}, x, s_t - z\rangle \\ &+ \frac{s-t}{s}\langle x_{t,s}-\gamma f(x_{t,s}), x_{t,s}-z\rangle,\end{aligned} \tag{3.26}$$

since $I-T$ is monotone (i.e., $\langle x-y, (I-T)x-(I-T)y\rangle \ge 0$ for $x, y \in H$). This is due to the nonexpansivity of T. Now replacing t, s in (3.26) with t_n, s_n and letting $n \to \infty$, we, noticing that $(I-T)x_{t_n,s_n} \to (I-T)x^* = 0$ for $x^* \in F(T)$, obtain

$$\langle (A-\gamma f)x^*, x^* - z\rangle \le 0. \tag{3.27}$$

That is, $x^* \in F(T)$ is a solution of (3.15), hence $x^* = \widetilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $x_{t,s}$ equals $\widetilde{x}$. Therefore, $x_{t,s} \to \widetilde{x}$ as $s \to 0$.

The variational inequality (3.15) can be rewritten as

$$\langle (I-A+\gamma f)\widetilde{x} - \widetilde{x}, \widetilde{x}-z\rangle \ge 0, \quad z \in F(T). \tag{3.28}$$

This, by Lemma 2.4, is equivalent to the fixed point equation

$$P_{F(T)}(I-A+\gamma f)\widetilde{x} = \widetilde{x}. \tag{3.29}$$

This complete the proof. □

Taking $t = s$ in Theorem 3.2, we get

Corollary 3.3 (see [7]). *One has that $x_t = x_{t,t}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, \quad z \in F(T). \tag{3.30}$$

Equivalently, One has $P_{F(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$, where $P_{F(T)}(\cdot)$ is the nearest point projection from H onto $F(T)$.

Next we study a general iteration method as follows. The initial guess x_0 is selected in H arbitrarily, and the $(n+1)$th iterate x_{n+1} is recursively defined by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \beta_n \gamma f(x_n) + (\alpha_n - \beta_n)x_n, \tag{3.31}$$

where $\{\alpha_n\} \subset (0,1)$, $\beta_n \in [0,1)$, $\beta_n \leq \alpha_n$ are sequences satisfying following conditions:

(C_1) $\alpha_n \to 0$;

(C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C_3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$;

(C_4) $\sum_{n=0}^{\infty}(\alpha_n - \beta_n) < \infty$.

Below is the second main result of this paper.

Theorem 3.4. *Let $\{x_n\}$ be general by Algorithm* (3.31) *with the sequences $\{\alpha_n\}, \{\beta_n\}$ of parameters satisfying conditions* (C_1)–(C_4). *Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Theorem 3.2.*

Proof. Since $\alpha_n \to 0$ by condition (C_1), we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all n.

We now observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(T)$ to obtain

$$\begin{aligned}\|x_{n+1} - p\| &= \|(I - \alpha_n A)(Tx_n - p) + \beta_n(\gamma f(x_n) - Ap) + (\alpha_n - \beta_n)(x_n - Ap)\| \\ &\leq \|I - \alpha_n A\|\|Tx_n - p\| + \beta_n\|\gamma f(x_n) - Ap\| + (\alpha_n - \beta_n)\|x_n - Ap\|\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n\delta)\|x_n - p\| + \beta_n\left[\gamma\|f(x_n) - f(p)\| + \|\gamma f(p) - Ap\|\right] \\
&\quad + (\alpha_n - \beta_n)\|x_n - Ap\| \\
&\le (1-\delta\alpha_n)\|x_n - p\| + \alpha_n\gamma h\|x_n - p\| - \alpha_n\gamma h\|x_n - p\| \\
&\quad + \beta_n\gamma h\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\| + (\alpha_n - \beta_n)\|x_n - Ap\| \\
&= \left[1 - (\delta - \gamma h)\alpha_n\right]\|x_n - p\| - (\alpha_n - \beta_n)\gamma h\|x_n - p\| \\
&\quad + \beta_n\|\gamma f(p) - Ap\| + (\alpha_n - \beta_n)(\|x_n - p\| + \|p - Ap\|) \\
&\le \left[1 - (\delta - \gamma h)\alpha_n\right]\|x_n - p\| + (\alpha_n - \beta_n)(1 - \gamma h)\|x_n - p\| \\
&\quad + \beta_n\|\gamma f(p) - Ap\| + (\alpha_n - \beta_n)\|p - Ap\| \\
&\le \left[1 - \left[(\delta - \gamma h)\alpha_n - (\alpha_n - \beta_n)|1 - \gamma h|\right]\right]\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\| \\
&\quad + (\alpha_n - \beta_n)\|p - Ap\| \\
&\le \left[1 - \left[(\delta - \gamma h)\alpha_n - (\alpha_n - \beta_n)|1 - \gamma h|\right]\right]\|x_n - p\| + (\delta - \gamma h)\alpha_n\frac{M}{\delta - \gamma h},
\end{aligned}
\tag{3.32}
$$

where $M \ge \|\gamma f(p) - Ap\| + \|p - Ap\|$ is a constant. By Lemma 2.6 we see that $\{x_n\}$ is bounded.

As a result, noticing

$$x_{n+1} - Tx_n = \alpha_n ATx_n + \beta_n\gamma f(x_n) + (\alpha_n - \beta_n)x_n \tag{3.33}$$

and $\alpha_n \to 0$, we obtain

$$x_{n+1} - Tx_n \longrightarrow 0. \tag{3.34}$$

But the key is to prove that

$$x_{n+1} - x_n \longrightarrow 0. \tag{3.35}$$

To see this, we calculate

$$
\begin{aligned}
\|x_{n+1} - x_n\| = \|&(I - \alpha_n A)(Tx_n - Tx_{n-1}) - (\alpha_n - \alpha_{n-1})ATx_{n-1} \\
&+\gamma\left[\alpha_n(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})f(x_{n-1})\right] + (\alpha_n - \beta_n)(x_n - x_{n-1})\|
\end{aligned}
$$

$$\begin{aligned}
&\le (1-(\delta-\gamma h)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|ATx_{n-1} - f(x_{n-1})\| \\
&\quad + (\alpha_n - \beta_n)\|x_n - x_{n-1}\| \\
&= (1+\alpha_n - \beta_n - (\delta-\gamma h)\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|ATx_{n-1} - f(x_{n-1})\| \\
&= [1-[(\delta-\gamma h - 1)\alpha_n + \beta_n]]\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|ATx_{n-1} - f(x_{n-1})\|.
\end{aligned} \tag{3.36}$$

Since

$$\sum_{n=0}^{\infty}[(\delta-\gamma h-1)\alpha_n + \beta_n] = \sum_{n=0}^{\infty}[(\delta-\gamma h)\alpha_n - (\alpha_n - \beta_n)] = \infty \tag{3.37}$$

and condition (C_3) holds, an application of Lemma 2.1 to (3.36) implies (3.35) which combined with (3.34), in turns, implies

$$x_n - Tx_n \longrightarrow 0. \tag{3.38}$$

Next we show that

$$\limsup_{n\to\infty}\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle \le 0, \tag{3.39}$$

where $\tilde{x}$ is obtained in Theorem 3.2.

To see this, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle = \lim_{k\to\infty}\langle x_{n_k} - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle. \tag{3.40}$$

We may also assume that $x_{n_k} \rightharpoonup z$. Note that $z \in F(T)$ in virtue of Lemma 2.2 and (3.38). It follows from the variational inequality (3.15) that

$$\limsup_{n\to\infty}\langle x_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle = \langle z - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle \le 0. \tag{3.41}$$

So (3.39) holds, thanks to (3.38).

Finally, we prove $x_n \to \tilde{x}$. To this end, we calculate

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \left\|(I-\alpha_n A)(Tx_n-\tilde{x})+\alpha_n(\gamma f(x_n)-A\tilde{x})-\alpha_n\gamma f(x_n)+\beta_n\gamma f(x_n)+(\alpha_n-\beta_n)x_n\right\|^2 \\
&= \left\|(I-\alpha_n A)(Tx_n-\tilde{x})+\alpha_n(\gamma f(x_n)-A\tilde{x})+(\alpha_n-\beta_n)(x_n-\gamma f(x_n))\right\|^2 \\
&= \|(I-\alpha_n A)(Tx_n-\tilde{x})\|^2+\left\|\alpha_n(\gamma f(x_n)-A\tilde{x})\right\|^2 \\
&\quad +\left\|(\alpha_n-\beta_n)(x_n-\gamma f(x_n))\right\|^2+2\left\langle (I-\alpha_n A)(Tx_n-\tilde{x}),\alpha_n(\gamma f(x_n)-A\tilde{x})\right\rangle \\
&\quad +2\left\langle (I-\alpha_n A)(Tx_n-\tilde{x}),(\alpha_n-\beta_n)(x_n-\gamma f(x_n))\right\rangle \\
&\quad +2\left\langle \alpha_n(\gamma f(x_n)-A\tilde{x}),(\alpha_n-\beta_n)(x_n-\gamma f(x_n))\right\rangle \\
&\le (1-\alpha_n\delta)^2\|x_n-\tilde{x}\|^2+\alpha_n^2\left\|\gamma f(x_n)-A\tilde{x}\right\|^2 \\
&\quad +(\alpha_n-\beta_n)^2\left\|x_n-\gamma f(x_n)\right\|^2 \\
&\quad +2\alpha_n\left\langle Tx_n-\tilde{x},\gamma f(x_n)-A\tilde{x}\right\rangle \\
&\quad -2\alpha_n^2\left\langle A(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\right\rangle \\
&\quad +2(\alpha_n-\beta_n)\left\langle Tx_n-\tilde{x},x_n-\gamma f(x_n)\right\rangle \\
&\quad -2\alpha_n(\alpha_n-\beta_n)\left\langle A(Tx_n-\tilde{x}),x_n-\gamma f(x_n)\right\rangle \\
&\quad +2\alpha_n(\alpha_n-\beta_n)\left\langle \gamma f(x_n)-A\tilde{x},x_n-\gamma f(x_n)\right\rangle \\
&\le (1-\alpha_n\delta)^2\|x_n-\tilde{x}\|^2+\alpha_n^2\left\|\gamma f(x_n)-A\tilde{x}\right\|^2+(\alpha_n-\beta_n)^2\left\|x_n-\gamma f(x_n)\right\|^2 \\
&\quad +2\alpha_n\left\langle Tx_n-\tilde{x},\gamma f(x_n)-\gamma f(\tilde{x})+\gamma f(\tilde{x})-A\tilde{x}\right\rangle \\
&\quad -2\alpha_n^2\left\langle A(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\right\rangle+2(\alpha_n-\beta_n)\left\langle Tx_n-\tilde{x},x_n-\gamma f(x_n)\right\rangle \\
&\quad -2\alpha_n(\alpha_n-\beta_n)\left\langle A(Tx_n-\tilde{x}),x_n-\gamma f(x_n)\right\rangle \\
&\quad +2\alpha_n(\alpha_n-\beta_n)\left\langle \gamma f(x_n)-A\tilde{x},x_n-\gamma f(x_n)\right\rangle \\
&\le (1-\alpha_n\delta)^2\|x_n-\tilde{x}\|^2+\alpha_n^2\left\|\gamma f(x_n)-A\tilde{x}\right\|^2+(\alpha_n-\beta_n)^2\left\|x_n-\gamma f(x_n)\right\|^2 \\
&\quad +2\alpha_n\left\langle Tx_n-\tilde{x},\gamma f(x_n)-\gamma f(\tilde{x})\right\rangle+2\alpha_n\left\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\right\rangle \\
&\quad -2\alpha_n^2\left\langle A(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\right\rangle+2(\alpha_n-\beta_n)\left\langle Tx_n-\tilde{x},x_n-\gamma f(x_n)\right\rangle \\
&\quad -2\alpha_n(\alpha_n-\beta_n)\left\langle A(Tx_n-\tilde{x}),x_n-\gamma f(x_n)\right\rangle \\
&\quad +2\alpha_n(\alpha_n-\beta_n)\left\langle \gamma f(x_n)-A\tilde{x},x_n-\gamma f(x_n)\right\rangle \\
&\le (1-\alpha_n\delta)^2\|x_n-\tilde{x}\|^2+\alpha_n^2\left\|\gamma f(x_n)-A\tilde{x}\right\|^2 \\
&\quad +(\alpha_n-\beta_n)^2\left\|x_n-\gamma f(x_n)\right\|^2+2\alpha_n\gamma h\|x_n-\tilde{x}\|^2 \\
&\quad +2\alpha_n\left\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\right\rangle-2\alpha_n^2\left\langle A(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\right\rangle \\
&\quad +2(\alpha_n-\beta_n)\left\langle Tx_n-\tilde{x},x_n-\gamma f(x_n)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- 2\alpha_n(\alpha_n - \beta_n)\langle A(Tx_n - \tilde{x}), x_n - \gamma f(x_n)\rangle \\
&+ 2\alpha_n(\alpha_n - \beta_n)\langle \gamma f(x_n) - A\tilde{x}, x_n - \gamma f(x_n)\rangle \\
\le& \left[(1 - \alpha_n\delta)^2 + 2\alpha_n\gamma h\right]\|x_n - \tilde{x}\|^2 + \alpha_n^2\|\gamma f(x_n) - A\tilde{x}\|^2 \\
&+ (\alpha_n - \beta_n)^2\|x_n - \gamma f(x_n)\|^2 + 2\alpha_n\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle \\
&- 2\alpha_n^2\langle A(Tx_n - \tilde{x}), \gamma f(x_n) - A\tilde{x}\rangle + 2(\alpha_n - \beta_n)\langle Tx_n - \tilde{x}, x_n - \gamma f(x_n)\rangle \\
&- 2\alpha_n(\alpha_n - \beta_n)\langle A(Tx_n - \tilde{x}), x_n - \gamma f(x_n)\rangle + 2\alpha_n(\alpha_n\gamma f(x_n) - A\tilde{x}, x_n - \gamma f(x_n)\rangle \\
\le& [1 - 2(\delta - \gamma h)\alpha_n]\|x_n - \tilde{x}\|^2 + \alpha_n^2\delta^2\|x_n - \tilde{x}\| + \alpha_n^2\|\gamma f(x_n) - A\tilde{x}\|^2 \\
&+ (\alpha_n - \beta_n)^2\|x_n - \gamma f(x_n)\|^2 + 2\alpha_n\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle \\
&- 2\alpha_n^2\langle A(Tx_n - \tilde{x}), \gamma f(x_n) - A\tilde{x}\rangle + 2(\alpha_n - \beta_n)\langle Tx_n - \tilde{x}, x_n - \gamma f(x_n)\rangle \\
&-2\alpha_n(\alpha_n - \beta_n)\langle A(Tx_n - \tilde{x}), x_n - \gamma \mathrm{f}(x_n)\rangle + 2\alpha_n(\alpha_n - \beta_n)\langle \gamma f(x_n) - A\tilde{x}, x_n - \gamma f(x_n)\rangle \\
\le& [1 - 2(\delta - \gamma h)\alpha_n]\|x_n - \tilde{x}\|^2 + \alpha_n^2\delta^2\|x_n - \tilde{x}\| + \alpha_n^2\|\gamma f(x_n) - A\tilde{x}\|^2 \\
&+ (\alpha_n - \beta_n)^2\|x_n - \gamma f(x_n)\|^2 + 2\alpha_n\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle \\
&- 2\alpha_n^2\langle A(Tx_n - \tilde{x}), \gamma f(x_n) - A\tilde{x}\rangle + 2(\alpha_n - \beta_n)\langle Tx_n - \tilde{x}, x_n - \gamma f(x_n)\rangle \\
&- 2\alpha_n(\alpha_n - \beta_n)\langle A(Tx_n - \tilde{x}), x_n - \gamma f(x_n)\rangle \\
&+ 2\alpha_n(\alpha_n - \beta_n)\langle \gamma f(x_n) - A\tilde{x}, x_n - \gamma f(x_n)\rangle \\
\le& [1 - 2(\delta - \gamma h)\alpha_n]\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle + \alpha_n M_n,
\end{aligned}
\tag{3.42}
$$

where

$$
\begin{aligned}
M_n = \alpha_n\|\gamma f(x_n) - A\tilde{x}\|^2 &+ \frac{(\alpha_n - \beta_n)^2}{\alpha_n}\|x_n - \gamma f(x_n)\|^2 \\
&- 2\alpha_n\langle A(Tx_n - \tilde{x}), \gamma f(x_n) - A\tilde{x}\rangle + 2\frac{\alpha_n - \beta_n}{\alpha_n}\langle Tx_n - \tilde{x}, x_n - \gamma f(x_n)\rangle \\
&- 2(\alpha_n - \beta_n)\langle A(Tx_n - \tilde{x}), x_n - \gamma f(x_n)\rangle \\
&+ 2(\alpha_n - \beta_n)\langle \gamma f(x_n) - A\tilde{x}, x_n - \gamma f(x_n)\rangle.
\end{aligned}
\tag{3.43}
$$

That is,

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \le& [1 - 2(\delta - \gamma h)\alpha_n]\|x_n - \tilde{x}\|^2 \\
&+ \alpha_n[2\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle + M_n].
\end{aligned}
\tag{3.44}
$$

Since $\{x_n\}$ is bounded, by the conditions of Theorem 3.4, we get $\lim_{n\to\infty} M_n = 0$ and $\sum_{n=0}^{\infty}(\delta - \gamma h)\alpha_n = \infty$, this together with (3.39) implies that

$$\limsup_{n\to\infty} \left[2\langle Tx_n - \tilde{x}, \gamma f(\tilde{x}) - A\tilde{x}\rangle + M_n\right] \le 0. \tag{3.45}$$

Now applying Lemma 2.1 to (3.44) concludes that $x_n \to \tilde{x}$. This complete the proof of Theorem 3.4. □

If pick $\alpha_n = \beta_n$, we obtain the result of Marino and Xu [2].

4. Approximate Iteration Algorithm and Error Estimate

In this section, we use the following approximate iteration algorithm:

$$y_{n+1} = (I - sA)Ty_n + t\gamma f(y_n) + (s - t)y_n, \tag{4.1}$$

for an arbitrary initial $y_0 \in H$, to calculate the fixed point of nonexpansive mapping and solution of variational inequality with bounded linear operator A, where A, T, γ, s, t and others δ, h as in the Section 3.

Meanwhile, the $\tilde{x} \in F(T)$ is obtained in Theorem 3.2 which is unique solution of variational inequality (3.15) and $\{x_n\} \to \tilde{x}$, as $n \to \infty$, $x_{t,s} \to \tilde{x}$ as $s \to 0$, where $\{x_n\}$ and $x_{t,s}$ are respectively defined by (3.31) and (3.6).

The following lemma will be useful for the establish of formula of convergence rate estimate.

Lemma 4.1 (Banach's Contractive Mapping Principle). *Let H be a Banach space and S be a contraction from H into self, that is,*

$$\|Sx - Sy\| \le \theta\|x - y\|, \quad \forall x, y \in H, \tag{4.2}$$

where $0 < \theta < 1$ is a constant. Then the Picard iterative sequence $x_{n+1} = Sx_n$, for arbitrary initial $x_0 \in H$, converges strongly to a unique fixed point x^* of S and

$$\|x_n - x^*\| \le \frac{\theta^n}{1 - \theta}\|x_0 - Sx_0\|. \tag{4.3}$$

For above T, A, f, γ, s, t, δ, we define the following contractive mapping:

$$S_{t,s}y = (I - sA)Ty + \gamma f(y) + (s - t)y \tag{4.4}$$

from H into self. In fact, it is not hard to see that $S_{t,s}$ is a contraction for sufficiently small s, indeed, by Lemma 2.5 we have, for any $x, y \in H$, that

$$\begin{aligned} \|S_{t,s}x - S_{t,s}y\| &\le t\gamma\|f(x) - f(y)\| + \|(I - sA)(Tx - Ty)\| + \|(s-t)(x-y)\| \\ &\le (t\gamma h + 1 - s\delta + s - t)\|x - y\| \\ &= (1 + t(\gamma h - 1) - s(\delta - 1))\|x - y\|. \end{aligned} \tag{4.5}$$

By using Lemma 4.1, then there exists unique fixed point $x_{t,s} \in H$ of $S_{t,s}$ and the iterative sequence

$$y_{n+1} = S_{t,s}y_n = (I - sA)Ty_n + \gamma f(y_n) + (s-t)y_n, \quad y_0 \in H, \tag{4.6}$$

converges strongly to this fixed point $x_{t,s}$. Meanwhile, from (4.3) and (4.5) we obtain

$$\|y_n - x_{t,s}\| \le \frac{(1 + t(\gamma h - 1) - s(\delta - 1))^n}{s(\delta - 1) - t(\gamma h - 1)}\|y_0 - S_{t,s}y_0\|. \tag{4.7}$$

On the other hand, from (3.21) we have

$$\begin{aligned} \|x_{t,s} - \tilde{x}\|^2 \le{}& \frac{t}{s\delta - t\gamma h}\langle \gamma f(\tilde{x}) - A\tilde{x}, x_{t,s} - \tilde{x}\rangle \\ &+ \frac{s-t}{s\delta - t\gamma h}(\langle x_{t,s} - \tilde{x}, x_{t,s} - \tilde{x}\rangle + \langle \tilde{x} - A\tilde{x}, x_{t,s} - \tilde{x}\rangle), \end{aligned} \tag{4.8}$$

which leads to

$$\begin{aligned} \left(1 - \frac{s-t}{s\delta - t\gamma h}\right)\|x_{t,s} - \tilde{x}\|^2 \le{}& \frac{t}{s\delta - t\gamma h}\langle \gamma f(\tilde{x}) - A\tilde{x}, x_{t,s} - \tilde{x}\rangle \\ &+ \frac{s-t}{s\delta - t\gamma h}\langle \tilde{x} - A\tilde{x}, x_{t,s} - \tilde{x}\rangle. \end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned} \left(1 - \frac{s-t}{s\delta - t\gamma h}\right)\|x_{t,s} - \tilde{x}\| &\le \frac{t}{s\delta - t\gamma h}\|\gamma f(\tilde{x}) - A\tilde{x}\| + \frac{s-t}{s\delta - t\gamma h}\|\tilde{x} - A\tilde{x}\|, \\ \|x_{t,s} - \tilde{x}\| &\le \frac{t}{s\delta - t\gamma h + t - s}\|\gamma f(\tilde{x}) - A\tilde{x}\| + \frac{s-t}{s\delta - t\gamma h + t - s}\|\tilde{x} - A\tilde{x}\|. \end{aligned} \tag{4.10}$$

Letting $D_1 = \|\gamma f(\tilde{x}) - A\tilde{x}\|$, $D_2 = \|\tilde{x} - A\tilde{x}\|$, it follows that

$$\|x_{t,s} - \tilde{x}\| \le \frac{t}{s\delta - t\gamma h + t - s}D_1 + \frac{s-t}{s\delta - t\gamma h + t - s}D_2. \tag{4.11}$$

From inequality (4.11) together with (4.7), and letting $D_3 = \|y_0 - S_{t,s}y_0\|$, we get

$$\|y_n - \tilde{x}\| \le \frac{t}{s\delta - t\gamma h + t - s}D_1 + \frac{s-t}{s\delta - t\gamma h + t - s}D_2 + \frac{\left(1 + t(\gamma h - 1) - s(\delta - 1)\right)^n}{s(\delta - 1) - t(\gamma h - 1)}D_3. \tag{4.12}$$

Inequality (4.12) is, namely, the error estimate for approximate fixed point y_n. Now, we give several special cases of inequality (4.12).

Error Estimate 1

Consider

$$\limsup_{n\to\infty}\|y_n - \tilde{x}\| \le \frac{t}{s\delta - t\gamma h + t - s}D_1 + \frac{s-t}{s\delta - t\gamma h + t - s}D_2. \tag{4.13}$$

Error Estimate 2

If $t = s$, then

$$\|y_n - \tilde{x}\| \le \frac{1}{(\delta - \gamma h)}D_1 + \frac{\left(1 - s(\delta - \gamma h)\right)^n}{s(\delta - \gamma h)}D_3, \tag{4.14}$$

which can be used to estimate error for iterative scheme

$$y_{n+1} = (I - sA)Ty_n + s\gamma f(y_n), \quad y_0 \in H. \tag{4.15}$$

Error Estimate 3

If $A = I$, then

$$\|y_n - \tilde{x}\| \le \frac{t}{s\delta - t\gamma h + t - s}D_1 + \frac{\left(1 + t(\gamma h - 1) - s(\delta - 1)\right)^n}{s(\delta - 1) - t(\gamma h - 1)}D_3, \tag{4.16}$$

which can be used to estimate error for iterative scheme

$$y_{n+1} = (1 - s)Ty_n + t\gamma f(y_n) + (s - t)x_n, \quad y_0 \in H. \tag{4.17}$$

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